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ALMOST PERIODIC SOLUTIONS OF A SINGLE SPECIES LOGARITHMIC POPULATION MODEL WITH FEEDBACK CONTROLS

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Abstract. *In this paper, a single species logarithmic population model is investigated. By applying the contraction mapping principle, a set of sufficient conditions is obtained for the existence and uniqueness of almost periodic solution of the single species logarithmic population model. The obtained sufficient conditions are given in terms of the algebraic inequalities which are easily checked. The results of this paper are completely new and generalize those of the previous studies.*

Keywords: Single species logarithmic population model, Almost periodic solution, Exponential dichotomy, Existence and uniqueness

1. **Introduction.** It is well known that a periodically varying environment plays an important role in the evolutionary theory as the selective forces on models in a fluctuating environment differing from those in a stable environment. Thus, the assumption of periodicity on the parameters is a way of incorporating the periodicity of the environment [1]. However, in real life, the periodic parameters often undergo certain perturbations; thus we think that almost periodic oscillatory parameters are more accordant with reality. In recent years, numerous scholars pay much attention on the almost periodic oscillatory behavior of population models. For example, Lu and Ge [2] considered the almost periodic solutions of the following single species neutral logarithmic model

$$\frac{dN(t)}{dt} = N(t)\left[r(t) - \sum_{j=1}^{n} a_j(t)\ln N(t-\sigma_j(t)) - \sum_{j=1}^{n} b_j(t)\frac{d\ln N(t-\tau_j(t))}{dt}\right]. \tag{1}$$

In 2010, Alzabut et al. [1] studied the almost periodic solutions for a delay logarithmic population model. In 2006, Chen [3] investigated the periodic solution and almost periodic solutions of a neutral multi-species logarithmic population model. Recently, Alzabut et al. [4] considered the almost periodic solution for the following delay logarithmic population model

$$\dot{x}(t) = x(t)[\gamma(t) - a_1(t)\ln x(t) - a_2(t)\ln x(t-\tau(t))] \tag{2}$$

where x denotes the size of population, $\gamma(t)$ denotes the growth rate while there are plenty of resources and there is no intra-specific competition for these resources, $a_1(t)$ stands for the measure of the competition among the individuals, $a_2(t)$ is added to generalize the model with the same interpretation of competitive effects and $\tau(t)$ denotes a maturation delay in the sense that competition involves adults who have matured by an age of $\tau(t)$ units. In details, one can see [4]. By applying the continuation theorem of coincidence degree theory, the authors proved that system (2) has at least one positive almost periodic solution.

It shall be pointed out that ecosystem in the real world is continuously distributed by unpredictable forces which can result in changes in the biological parameters such as survival rates. Of practical interest in ecology is the question whether or not an ecosystem can withstand those unpredictable disturbances which persist for a finite period of time [5-10]. In the language of control variables, we call the disturbance functions as control variables. Inspired by the discussion above, we will study the following single species logarithmic population model with feedback controls as follows

$$\begin{cases} \dot{x}(t) = x(t)[\gamma(t) - a_1(t)\ln x(t) - a_2(t)\ln x(t-\tau(t)) - b_1(t)u(t) - b_2(t)u(t-\sigma(t))], \\ \dot{u}(t) = -\alpha(t)u(t) + \beta(t)\ln x(t) + \varrho(t)\ln x(t-\delta(t)), \end{cases} \tag{3}$$

where u denotes indirect feedback control variable.

The main aim of this article is to establish some sufficient conditions for the existence and uniqueness of almost periodic solutions of (3). Our results are new and complement those of the previous studies in [4]. To the best of our knowledge, it is the first time to investigate the single species logarithmic population model with feedback controls by contraction mapping principle. Let

$$m(f) = \lim_{T\to+\infty}\frac{1}{T}\int_0^T f(t)dt = 0,$$

where $f(t)$ is almost periodic function. Throughout this paper, we make the following assumptions.

(H1) $\gamma(t)$, $a_1(t)$, $a_2(t)$, $b_1(t)$, $b_2(t)$, $\alpha(t)$, $\beta(t)$, $\varrho(t)$ are continuous real-valued nonnegative almost periodic functions on $\mathbb{R}$.

(H2) $\tau(t), \sigma(t)$ and $\delta(t)$ are nonnegative, continuously differentiable and almost periodic functions on $t\in\mathbb{R}$. Moreover, $\dot{\tau}(t), \dot{\sigma}(t)$ and $\dot{\delta}(t)$ are all uniformly continuous on $\mathbb{R}$ with

$$\inf_{t\in\mathbb{R}}\{1-\dot{\tau}(t)\} > 0, \quad \inf_{t\in\mathbb{R}}\{1-\dot{\sigma}(t)\} > 0, \quad \inf_{t\in\mathbb{R}}\{1-\dot{\delta}(t)\} > 0.$$

System (3) is supplemented with the initial value conditions

$$\begin{aligned} &x(s) = \varphi_x(s) \ge 0,\ s\in(-\theta,0],\ \varphi_x(0) > 0,\ \sup_{s\in(-\theta,0]}\varphi_x(s) < +\infty, \\ &u(s) = \varphi_u(s) \ge 0,\ \varphi_u(0) > 0,\ s\in(-\theta,0], \end{aligned} \tag{4}$$

where $\theta = \max_{t\in\mathbb{R}}\{\tau(t), \delta(t), \sigma(t)\}$. It is easy to see that there exists a positive solution $y(t) = (x(t), u(t))$ of system (3) satisfying the initial value condition (4).

The remainder of the paper is organized as follows. In Section 2, we present some sufficient conditions for the existence and uniqueness of almost periodic solution of (3). An example is given in Section 3. A brief conclusion is drawn in Section 4.

2. Existence and Uniqueness of Almost Periodic Solution.

In this section, we will establish sufficient conditions on the existence and uniqueness of almost periodic solutions of (3). For convenience, we introduce some definitions and lemmas which will be used in what follows.

Definition 2.1. [11,12] *Let $f(t):\mathbb{R}\to\mathbb{R}^n$ be continuous in t. $f(t)$ is said to be almost periodic on $\mathbb{R}$, and if for any $\varepsilon > 0$, the set $T(f,\varepsilon) = \{\delta : |f(t+\delta) - f(t)| < \varepsilon, \forall t\in\mathbb{R}\}$ is relatively dense, i.e., for $\forall\varepsilon > 0$, it is possible to find a real number $l = l(\varepsilon) > 0$, for any interval with length $l(\varepsilon)$, there exists a number $\delta = \delta(\varepsilon)$ in this interval such that $|f(t+\delta) - f(t)| < \varepsilon$, for $\forall\in\mathbb{R}$.*

Definition 2.2. *Let $z\in\mathbb{R}^n$ and $Q(t)$ be an $n\times n$ continuous matrix defined on $\mathbb{R}$. The linear system*

$$\frac{dz}{dt} = Q(t)z(t) \tag{5}$$

is said to admit an exponential dichotomy on $\mathbb{R}$ *if there exist constants* $k, \lambda > 0$, *projection* P *and the fundamental matrix* $Z(t)$ *of (5) satisfying*

$$||Z(t)PZ^{-1}(s)|| \leq ke^{-\lambda(t-s)},\ for\ t \geq s, \quad ||Z(t)(I-P)Z^{-1}(s)|| \leq ke^{-\lambda(t-s)},\ for\ \leq s.$$

Lemma 2.1. [11,12] *If the linear system (5) admits an exponential dichotomy, then almost periodic system*

$$\frac{dz}{dt} = Q(t)z(t) + g(t)\ (g(t)\ is\ an\ n \times 1\ continuous\ matrix\ defined\ on\ \mathbb{R}) \tag{6}$$

has a unique almost periodic solution $z(t)$ *and*

$$z(t) = \int_{-\infty}^{t} Z(t)PZ^{-1}(s)g(s)ds - \int_{t}^{+\infty} Z(t)(I-P)Z^{-1}(s)g(s)ds.$$

Lemma 2.2. [12,13] *Let* $a_i(t)$ *be an almost periodic function on* $\mathbb{R}$ *and* $a_i(t) > 0$. *Then the system*

$$\frac{dz}{dt} = diag(-a_1(t), -a_2(t), \cdots, -a_n(t))z(t) \tag{7}$$

admits an exponential dichotomy.

Remark 2.1. *It follows from Lemma 2.2 that system (7) has a unique almost periodic solution* $z(t)$ *which takes the form*

$$z(t) = \int_{-\infty}^{t} Z(t)Z^{-1}(s)g(s)ds = \left(\int_{-\infty}^{t} e^{-\int_s^t a_1(u)du}g_1(s)ds, \cdots, \int_{-\infty}^{t} e^{-\int_s^t a_n(u)du}g_n(s)ds\right).$$

Lemma 2.3. *Let* m *be a positive integer and* B *be a Banach space. If the mapping* $\Gamma : B \to B$ *is a contraction mapping, then* $\Gamma : B \to B$ *has exactly one fixed point in* B, *where* $\Gamma^m = \Gamma(\Gamma^{m-1})$.

By (H1), $m(\alpha) > 0$. In view of Lemma 2.1, we have the following result.

Lemma 2.4. $(x(t), u(t))^T$ *is an almost periodic solution of system (3) if and only if it is an almost periodic solution of*

$$\begin{cases} \dot{x}(t) = x(t)[\gamma(t) - a_1(t)\ln x(t) - a_2(t)\ln x(t-\tau(t)) - b_1(t)u(t) - b_2(t)u(t-\sigma(t))], \\ u(t) = \int_{-\infty}^{t} e^{\int_s^t \alpha(\zeta)d\zeta}[\beta(s)\ln x(s) + \varrho(s)\ln x(s-\delta(s))]ds. \end{cases} \tag{8}$$

Obviously, (8) is equivalent to the following system

$$\begin{aligned} \dot{x}(t) \ =\ x(t)\Bigg[&\gamma(t) - a_1(t)\ln x(t) - a_2(t)\ln x(t-\tau(t)) \\ &-b_1(t)\int_{-\infty}^{t} e^{\int_s^t \alpha(\zeta)d\zeta}(\beta(s)\ln x(s) + \varrho(s)\ln x(s-\delta(s)))ds \\ &-b_2(t)\int_{-\infty}^{t-\sigma(t)} e^{\int_s^{t-\sigma(t)} \alpha(\zeta)d\zeta}(\beta(s)\ln x(s) + \varrho(s)\ln x(s-\delta(s)))ds\Bigg]. \end{aligned} \tag{9}$$

Now we are in a position to state our main results on the existence and uniqueness of almost periodic solution for system (3).

Theorem 2.1. *In addition to (H1) and (H2), if the following condition*
(H3) $\int_{-\infty}^{t} e^{-\int_s^t a_1(\zeta)d\zeta}\Theta < 1$, *where*

$$\Theta = \left[a_2(s) + (b_1(s) + b_2(s))\int_{-\infty}^{s} e^{\int_\xi^s \alpha(\zeta)d\zeta}(\beta(\xi) + \varrho(\xi))d\xi\right],$$

then system (3) has a unique positive almost periodic solution.

Proof: Let $x(t)=e^{y(t)}$, and then (9) takes the form

$$\begin{aligned}\dot{y}(t) &= -a_1(t)y(t)-a_2(t)y(t-\tau(t))+\gamma(t)\\ &\quad -b_1(t)\int_{-\infty}^{t} e^{\int_s^t \alpha(\zeta)d\zeta}(\beta(s)y(s)+\varrho(s)y(s-\delta(s)))ds\\ &\quad -b_2(t)\int_{-\infty}^{t-\sigma(t)} e^{\int_s^{t-\sigma(t)} \alpha(\zeta)d\zeta}(\beta(s)y(s)+\varrho(s)y(s-\delta(s)))ds. \qquad (10)\end{aligned}$$

Clearly, if system (10) has an almost periodic solution $y^*(t)$, then $x^*(t)=e^{y^*(t)}$ is an almost periodic solution of (9). In view of Lemma 2.1, we can conclude that $\left(e^{y^*(t)},u^*(t)\right)^T$ is an almost periodic solution of (4), where

$$u^*(t)=\int_{-\infty}^{t} e^{\int_s^t \alpha_i(\zeta)d\zeta}[\beta(s)x(s)+\varrho(s)x^*(s-\delta(s))]ds.$$

Now we will show that (10) has a unique almost periodic solution. Firstly, we define $B=\{\psi(t)|\psi(t)$ is a continuous almost periodic function$\}$. Obviously, B is a Banach space with the norm $||\psi||=\max\sup_{t\in\mathbb{R}}|y(t)|$.

For any $\psi(t)\in B$, we consider the following almost periodic system

$$\begin{aligned}\dot{y}(t) &= -a_1(t)y(t)-a_2(t)\psi(t-\tau(t))+\gamma(t)\\ &\quad -b_1(t)\int_{-\infty}^{t} e^{\int_s^t \alpha(\zeta)d\zeta}(\beta(s)\psi(s)+\varrho(s)\psi(s-\delta(s)))ds\\ &\quad -b_2(t)\int_{-\infty}^{t-\sigma(t)} e^{\int_s^{t-\sigma(t)} \alpha(\zeta)d\zeta}(\beta(s)\psi(s)+\varrho(s)\psi(s-\delta(s)))ds. \qquad (11)\end{aligned}$$

By (H1), we know that $m(a_1)>0$. In view of Lemma 2.2, the linear system

$$\dot{y}(t)=-a_1(t)y(t) \qquad (12)$$

admits an exponential dichotomy on $\mathbb{T}$. Then system (12) has exactly one almost periodic solution as follows

$$y^{\psi}(t)=\int_{-\infty}^{t} e^{-\int_s^t a_1(\zeta)d\zeta}h^{\psi}(s)ds, \qquad (13)$$

where

$$\begin{aligned}h^{\psi}(s) &= -a_2(t)\psi(t-\tau(t))+\gamma(t)\\ &\quad -b_1(t)\int_{-\infty}^{t} e^{\int_s^t \alpha(\zeta)d\zeta}(\beta(s)\psi(s)+\varrho(s)\psi(s-\delta(s)))ds\\ &\quad -b_2(t)\int_{-\infty}^{t-\sigma(t)} e^{\int_s^{t-\sigma(t)} \alpha(\zeta)d\zeta}(\beta(s)\psi(s)+\varrho(s)\psi(s-\delta(s)))ds. \qquad (14)\end{aligned}$$

Define a mapping $F:B\to B$ as follows

$$F\psi(t)=Z^{\psi}(t),\ \text{for any } \psi\in B. \qquad (15)$$

For any $\phi,\psi\in B$, we have

$$|(F(\phi)-F(\psi))|\le\int_{-\infty}^{t} e^{-\int_s^t a_1(\zeta)d\zeta}|h^{\phi}(s)-h^{\psi}(s)|ds. \qquad (16)$$

On the other hand, by (14), we get

$$\begin{aligned}
&|h^{\phi}(s)-h^{\psi}(s)| \\
=\ & a_2(s)|\phi(s-\tau(s))-\psi(s-\tau(s))| \\
&+b_1(s)\int_{-\infty}^{s} e^{\int_{\xi}^{s}\alpha(\zeta)d\zeta}(\beta(\xi)|\phi(\xi)-\psi(\xi)|+\varrho(\xi)|\phi(\xi-\delta(\xi))-\psi(\xi-\delta(\xi))|)d\xi \\
&+b_2(s)\int_{-\infty}^{s-\sigma(s)} e^{\int_{\xi}^{s-\sigma(s)}\alpha(\zeta)d\zeta}(\beta(\xi)|\phi(\xi)-\psi(\xi)|+\varrho(\xi)|\phi(\xi-\delta(\xi))-\psi(\xi-\delta(\xi))|)d\xi \\
\le\ & a_2(s)\sup_{t\in\mathbb{R}}|\phi(t)-\psi(t)|+b_1(s)\int_{-\infty}^{s} e^{\int_{\xi}^{s}\alpha(\zeta)d\zeta}(\beta(\xi)\sup_{t\in\mathbb{R}}|\phi(t)-\psi(t)| \\
&+\varrho(\xi)\sup_{t\in\mathbb{R}}|\phi(t)-\psi(t)|)d\xi+b_2(s)\int_{-\infty}^{s-\sigma(s)} e^{\int_{\xi}^{s-\sigma(s)}\alpha(\zeta)d\zeta}(\beta(\xi)\sup_{t\in\mathbb{R}}|\phi(t)-\psi(t)| \\
&+\varrho(\xi)\sup_{t\in\mathbb{R}}|\phi(t)-\psi(t)|)d\xi \\
\le\ & \left[a_2(s)+(b_1(s)+b_2(s))\int_{-\infty}^{s} e^{\int_{\xi}^{s}\alpha(\zeta)d\zeta}(\beta(\xi)+\varrho(\xi))d\xi\right] \\
&\times\sup_{t\in\mathbb{R}}|\phi(t)-\psi(t)|=\Theta\sup_{t\in\mathbb{R}}|\phi(t)-\psi(t)|, \qquad (17)
\end{aligned}$$

where

$$\Theta=\left[a_2(s)+(b_1(s)+b_2(s))\int_{-\infty}^{s} e^{\int_{\xi}^{s}\alpha(\zeta)d\zeta}(\beta(\xi)+\varrho(\xi))d\xi\right].$$

It follows from (16) and (17) that

$$|(F(\phi)-F(\psi))|\le\int_{-\infty}^{t} e^{-\int_{s}^{t}a_1(\zeta)d\zeta}\Theta\sup_{t\in\mathbb{R}}|\phi(t)-\psi(t)|ds. \qquad (18)$$

Then we get $\sup_{t\in\mathbb{R}}|(F(\phi(t))-F(\psi(t)))|\le\Lambda\sup_{t\in\mathbb{R}}|(\phi(t)-\psi(t))|$. For any positive integer m, we have

$$\begin{aligned}
&\sup_{t\in\mathbb{R}}|(F^m(\phi(t))-F^m(\psi(t)))| \\
=\ &\sup_{t\in\mathbb{R}}|(F(F^{m-1}(\phi(t)))-F(F^{m-1}(\psi(t))))| \\
\le\ &\Lambda\sup_{t\in\mathbb{R}}|(F^{m-1}(\phi(t))-F^{m-1}(\psi(t)))|\le\cdots\le\Lambda^m\sup_{t\in\mathbb{R}}|(\phi(t)-\psi(t))|. \qquad (19)
\end{aligned}$$

By (H3), we get $\lim_{m\to+\infty}\Lambda^m=0$, which implies that there exists a positive integer N^* and a positive constant $\mu_0<1$ such that $\Lambda^{N^*}=\kappa\le\mu_0$. It follows that $\left|\left(F^{N^*}(\phi)-F^{N^*}(\psi)\right)\right|\le\kappa\sup_{t\in\mathbb{R}}|\phi(t)-\psi(t)|\le\mu_0||\phi-\psi||$, which implies that the mapping $F^{N^*}:B\to B$ is a contraction mapping. In view of Lemma 2.3, F has a unique fixed point $y^*(t)$ in B. Thus system (10) has a unique almost periodic solution $y^*(t)$, and then $x^*(t)=e^{y^*(t)}$ is the unique almost periodic solution of (9). Thus, by Lemma 2.4, $\left(e^{y^*(t)},u^*(t)\right)^T$ is the unique almost periodic solution of (3). The proof of Theorem 2.1 is completed.

3. **An Example.** In this section, to illustrate the feasibility of our theoretical findings obtained in previous sections, we give an example. Consider the following single species logarithmic population model

$$\begin{cases}\dot{x}(t)=x(t)[\gamma(t)-a_1(t)\ln x(t)-a_2(t)\ln x(t-\tau(t))-b_1(t)u(t)-b_2(t)u(t-\sigma(t))],\\ \dot{u}(t)=-\alpha(t)u(t)+\beta(t)\ln x(t)+\varrho(t)\ln x(t-\delta(t)),\end{cases} \qquad (20)$$

where $\gamma(t) = 0.2$, $a_1(t) = 0.2\sin t$, $a_2(t) = 0.4\cos t$, $\tau(t) = 0.3$, $\sigma(t) = 0.1$, $\alpha(t) = 0.4$, $\beta(t) = 0.5\sin t$, $\varrho(t) = 0.2\cos t$, and $\delta(t) = 0.02$. Then all the conditions (H1)–(H3) hold. Thus system (20) has exactly one unique almost periodic solution. The fact is shown by the computer simulations in Figure 1.

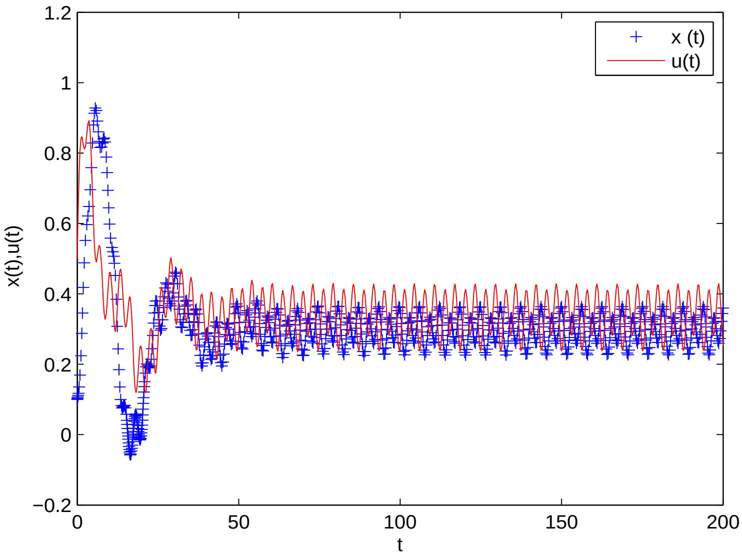

FIGURE 1. Transient response of state variables $x(t)$ and $u(t)$

4. **Conclusions.** In this paper, we study a single species logarithmic population model. By applying the contraction mapping principle, we establish some sufficient conditions for the existence and uniqueness of almost periodic solution of the single species logarithmic population model. The obtained sufficient conditions are given in terms of the algebraic inequalities which are easily checked. The obtained results in this paper are completely new and generalize those of the previous studies in [4]. Recently, the global exponential stability of the single species logarithmic population model has received much attention, and we leave this interesting problem for our future work.

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