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LEVEL HITTING ANALYSIS OF BROWNIAN MODELS APPLIED FOR OPTIMIZATION OF TAKE-PROFIT LEVEL SETTING FOR TRADING

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Abstract. *This paper aims to average the number of level hitting in a time-discrete model based on geometric Brownian motion. A closed-form solution is obtained by stochastic calculation. By the computational results of both expected upward and downward barrier crossing times, trading strategies can be formed with an optimal take-profit level. Real market trading test is conducted accordingly and the approaches developed in this paper are testified to be profitable.*

Keywords: Brownian motion, Running maximum, Level hitting, Optimization

1. **Introduction.** This research studies the probability of asset price hitting certain predetermined level by estimating the number of upward (downward) crossing in a period of time. The result can be significant for pricing the options based on multiple time intervals. Typical example is barrier option, of which topic there has been extensive literature dealing with pricing and hedging. Regarding the hitting behaviors, one approach is to study the law of the maximum (or minimum) by T. Guillaume [1, 2], which gives formulae for standard step-up and step-down barrier options, as well as partial and outside step-up and step-down barrier options. Another recent research about hitting time of a drifted Brownian motion by A. Dassios and Y. Y. Zhang [3] applies double Laplace transform, and E. Renault et al. [4] investigate the transaction prices based on a hitting-time model.

Motivation of this research comes from some trading strategies adopted by traders. In practice, the actual meaning of this level considered for hitting is a kind of suppressive line, and the trader buys in when the price is under this suppressive level (denoted as L) at the beginning of each trading day in an up-trend movement and sets this L as a take-profit level. Applying this strategy, the trader expects the deal closed by hitting the take-profit level within the same day, so it is tempted to compute the expected frequency of such an event. Suppose the frequency is high, it is deemed as a worthy deal; otherwise, it is better not to take a long position in this situation. For stochastic analysis and computation in this paper, we refer to [6, 7].

This research pioneers a method to measure the hitting frequency of the Brownian motion, which is useful but has not yet been concluded by existing research, mostly because of the inconveniences caused by the infinite hitting of Brownian motion locally, cf. Lemma 3.1. Our paper is proceeded as follows. We formulate the problem in Section 2 and consider the most direct measure of level crossing problem in Section 3, where we also analyze the limit of our defined measure U_n, whose expression for fixed n is given by Section 4. We apply these results for trading strategy design and optimize the take-profit level in Section 5. We conclude the paper in Section 6.

2. **Modeling and Formulation.** The asset price $(Y_t)_{t\in\mathbb{R}_+}$ is modeled by a geometric Brownian motion

$$dY_t = \mu Y_t dt + \sigma Y_t dB_t, \qquad Y_0 > 0,\ t \in [0, T], \tag{1}$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion, $\mu \in \mathbb{R}$ is a constant drift factor and $\sigma \in (0, \infty)$ is a constant volatility. Consider the observation points $\mathcal{T}_n := \{T_0, T_1, \ldots, T_n\}$, where $T_0 = 0$, $T_n = T$ and $T_{k+1} - T_k = \delta_n := T/n$ for $k = 0, \ldots, n-1$. In practice, we can view δ_n as one trading day and consider the consecutive n days, and it is not necessary to pass n to infinity for some asymptotic results. Letting $L \in \mathbb{R}_+$ be the upper barrier in our setting, we aim to compute the average number of subintervals where the upper crossing happens, and specifically, it is defined as

$$U_n := \mathbb{E}\left[\sum_{i=1}^{n} \mathbf{1}_{\left\{Y_{(i-1)\delta_n} < L \ \&\ \hat{Y}_{(i-1)\delta_n, i\delta_n} \geq L\right\}}\right], \tag{2}$$

where $\hat{Y}_{s,t} := \sup\limits_{s \leq r \leq t} Y_r$ for $0 \leq s \leq t \leq T$. Equivalently, we have

$$U_n = \sum_{i=1}^{n} P\left(Y_{(i-1)\delta_n} < L \ \&\ \hat{Y}_{(i-1)\delta_n, i\delta_n} \geq L\right), \quad n \in \mathbb{N}_+, \tag{3}$$

and U_n will be computed by the following Proposition 4.1.

3. **Limit Analysis of U_n.** Given the process $(Y_t)_{t\in\mathbb{R}_+}$ and the level $L \in \mathbb{R}_+$, it is tempted to expect the crossing frequency during the whole time horizon $[0, T]$; however, the following lemma illuminates that the result is trivial.

Lemma 3.1. *Given a standard Brownian motion $(B_t)_{t\in\mathbb{R}_+}$, for any function f on $\mathbb{R}$ and $L \in \mathbb{R}$, $T > 0$, the set*

$$\{t \in [0, T];\ f(B_t) = L\}$$

is almost surely either empty or has infinitely many points.

This lemma is an obvious corollary of a well known property about zeros of Brownian motion that, the set of zeros of Brownian motion is a.s. closed, without any isolated points, and has zero Lebsgue measure, see, e.g., Proposition (3.12) of [8], and the proof applies the Blumenthal's 0-1 law, see, e.g., [5]. Therefore, the expectation of total times of level crossing through the level L is infinity although the averaged times of level crossing by the observed price curves are supposed to be finite within a limited time span. This is a defect of the Brownian diffusion modelling, especially when we consider the frequency of level crossing. Although we expect the limit of U_n when n goes to infinity to be a meaningful measure of level crossing behaviors over the whole time horizon $[0, T]$, the following proposition denies this conception.

Proposition 3.1. *For $\{U_n\}_{n\geq 1}$ defined by (2), we have $\lim\limits_{n\to\infty} U_n = \infty$.*

Proof: First we define $V_n := \mathbb{E}\left[\sum_{i=1}^{n} \mathbf{1}_{\{\exists t \in ((i-1)\delta_n, i\delta_n], Y_t = L\}}\right]$, it follows that $U_n > V_n$. Given $Y_{t_0} = L$ and $t_0 \in [0,T]$, by Lemma 3.1 we see that there almost surely exists a decreasing sequence $\{x_k\}_{k\in\mathbb{N}_+}$ approaching to t_0 from the righthand side, such that $Y_{x_k} = L$, $x_{k+1} < x_k$ for any $k \in \mathbb{N}_+$ and $\lim_{k\to\infty} x_k = t_0$. Since $\cup_{k\in\mathbb{N}_+}\mathcal{T}_{2^k}$ is dense in $[0,T]$, there also exists a decreasing sequence $\{y_k\}_{k\in\mathbb{N}_+} \in \cup_{k\in\mathbb{N}_+}\mathcal{T}_{2^k}$ approaching to t_0 from the righthand side; furthermore, a subsequence $\{y_{k_i}\}_{k_i\in\{k\}}$ can be picked from $\{y_k\}$, s.t., there a.s. exists an x_j inside $(y_{k_{i+1}}, y_{k_i})$ for each $i > N$ and some $N > 0$. By the continuity of $(Y_t)_{t\in[0,T]}$, we have $\sum_{i=1}^{\infty} \mathbf{1}_{\{\exists t \in ((i-1)\delta_n, i\delta_n], Y_t = L\}} = \infty$, a.s., given $U_{t_0} = L$ for some $t_0 \in [0,T]$. By Fatou's Lemma, we have $\liminf_{n\to\infty} V_n \geq \infty$.

4. Computation of Level Crossings.

Proposition 4.1. *For any $i \in \{2, 3, \ldots, n\}$, we have*

$$P\left(Y_{(i-1)\delta_n} < L \;\&\; \hat{Y}_{(i-1)\delta_n, i\delta_n} \geq L\right)$$
$$= \int_0^L \frac{1}{\sigma y\sqrt{2\pi(i-1)\delta_n}} e^{-\frac{\left(\log(\frac{y}{Y_0}) - \left(\mu - \frac{1}{2}\sigma^2\right)(i-1)\delta_n\right)^2}{2\sigma^2(i-1)\delta_n}} \left(1 - \Phi\left(d_1(y)\right) + \left(\frac{L}{y}\right)^{\frac{2\mu}{\sigma^2}-1} \Phi\left(d_2(y)\right)\right) dy,$$

and $P\left(Y_0 < L \;\&\; \hat{Y}_{0,\delta_n} \geq L\right) = \mathbf{1}_{\{Y_0 < L\}}\left[1 - \Phi(d_1(Y_0)) + \left(\frac{L}{Y_0}\right)^{\frac{2\mu}{\sigma^2}-1} \Phi(d_2(Y_0))\right]$ *for the case $i = 1$, where for any $y > 0$, $d_1(y)$, $d_2(y)$ are defined by*

$$d_1(y) := \frac{\log\left(\frac{L}{y}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)\delta_n}{\sigma\sqrt{\delta_n}}, \quad d_2(y) := \frac{-\log\left(\frac{L}{y}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)\delta_n}{\sigma\sqrt{\delta_n}}. \tag{4}$$

Proof: Note that the solution to SDE (1) is $Y_t = Y_0 \exp\left(\left(\mu - \frac{1}{2}\sigma\right)t + \sigma B_t\right)$, for $t \in [0,T]$. By the Markov property of $(Y_t)_{t\in[0,T]}$ we have

$$\begin{aligned} &P\left(Y_{(i-1)\delta_n} < L \;\&\; \hat{Y}_{(i-1)\delta_n, i\delta_n} \geq L\right) \\ &= \mathbb{E}\left[\mathbf{1}_{\{Y_{(i-1)\delta_n} < L\}}\, \mathbb{E}\left[\mathbf{1}_{\{\hat{Y}_{0,\delta_n} \geq L\}} \mid Y_0 = y\right]\Big|_{y = Y_{(i-1)\delta_n}}\right]. \end{aligned} \tag{5}$$

Applying Girsanov theorem, we have the following expression, for any $L \in \mathbb{R}_+$,

$$P\left(\hat{Y}_{0,\delta_n} \leq L\right) = \Phi(d_1(Y_0)) - \left(\frac{L}{Y_0}\right)^{\frac{2\mu}{\sigma^2}-1} \Phi(d_2(Y_0)). \tag{6}$$

Plugging (6) into (5), we obtain that for any $i \in \{2, \cdots, n\}$,

$$\begin{aligned} &P\left(Y_{(i-1)\delta_n} < L \;\&\; \hat{Y}_{(i-1)\delta_n, i\delta_n} \geq L\right) \\ &= \int_0^L \left(1 - \Phi(d_1(y)) + \left(\frac{L}{y}\right)^{\frac{2\mu}{\sigma^2}-1} \Phi(d_2(y))\right) f_Y(y, (i-1)\delta_n) dy, \end{aligned} \tag{7}$$

where $f_Y(y,t) := \frac{1}{\sigma y\sqrt{2\pi t}} \exp\left(-\frac{\left(\log\frac{y}{Y_0} - \left(\mu - \frac{1}{2}\sigma^2\right)t\right)^2}{2\sigma^2 t}\right)$, is the probability density of Y_t. In particular, we consider the case when $i = 1$. If $Y_0 > L$, we have $P\left(Y_0 < L \;\&\; \hat{Y}_{0,\delta_n} \geq L\right) = 0$; on the other hand, by a similar approach as (7), we have

$$P\left(Y_0 < L \;\&\; \hat{Y}_{0,\delta_n} \geq L\right) = 1 - \Phi\left(d_1(Y_0)\right) + \left(\frac{L}{Y_0}\right)^{\frac{2\mu}{\sigma^2}-1} \Phi\left(d_2(Y_0)\right). \tag{8}$$

Hence we complete the proof by combining (7) and (8).

Similarly, we consider the problem of downwards crossing through the lower barrier by defining

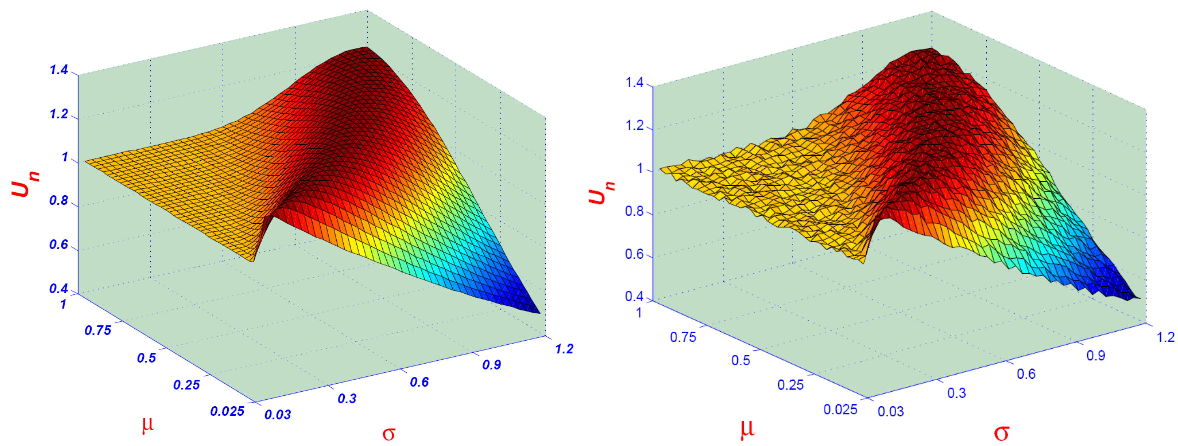
$$\begin{aligned} D_n &:= \mathbb{E}\left[\sum_{i=1}^{n} \mathbf{1}_{\left\{Y_{(i-1)\delta_n} > L \ \& \ \check{Y}_{(i-1)\delta_n, i\delta_n} \le L\right\}}\right] \\ &= \sum_{i=1}^{n} P\left(Y_{(i-1)\delta_n} > L \ \& \ \check{Y}_{(i-1)\delta_n, i\delta_n} \le L\right), \quad n \in \mathbb{N}_+, \end{aligned} \tag{9}$$

where $\check{Y}_{s,t} := \inf_{s \le r \le t} Y_r$ for $0 \le s \le t \le T$ and D_n are calculated by Proposition 4.2.

Proposition 4.2. *For any $i \in \{2, 3, \dots, n\}$, we have*

$$\begin{aligned} &P\left(Y_{(i-1)\delta_n} > L \ \& \ \check{Y}_{(i-1)\delta_n, i\delta_n} \le L\right) \\ &= \int_0^L \left(1 - \Phi\left(-d_1(y)\right) + \left(\frac{L}{y}\right)^{\frac{2\mu}{\sigma^2} - 1} \Phi\left(-d_2(y)\right)\right) f_Y\left(y, (i-1)\delta_n\right) dy. \end{aligned}$$

To conclude this section, we testify our expressions of expected number of barrier crossings by numerical implements and compare it with Monte Carlo simulation. Set $Y_0 = 100$, $L = 101$, $T = 10$, $\delta_n = 1$ and vary both the drift μ and the volatility σ, the computational result of U_n according to (3) and Proposition 4.1 is plotted in the following Figure 1(a), which consumes 12659 seconds while Monte Carlo simulation consumes 34227 seconds to achieve the data in the same level of accuracy. Restriction on running time budget within 20000 seconds will result in the following Figure 1(b) by Monte Carlo method, which appears much rougher than the left one.



(a) (b)

FIGURE 1. Expected times of barrier crossings through different parameters (μ, σ)

5. **Application to Take-Profit Level Setting.** In this section, we apply the results in Section 2 and Section 4 to designing a strategy for trading on derivatives. Simulating the price process with $(Y_t)_{t \in [0,T]}$ in (1), we investigate an optimization problem of setting take-profit level. Let Y_0 denote the current price and L be the take-profit level $L > 0$, we try to find the optimal L according to a utility function based on the expected number of upper barrier crossing U_n for given $n \in \mathbb{N}_+$ and $T > 0$. Defined by (2), U_n is a function of L, so we define $U_n(L) := \mathbb{E}\left[\sum_{i=1}^{n} \mathbf{1}_{\left\{Y_{(i-1)\delta_n} < L \ \& \ \hat{Y}_{(i-1)\delta_n, i\delta_n} \ge L\right\}}\right]$ for $L \ge 0$, and based on the

results of Propositions 4.1 and 4.2, we have the utility function $\Lambda(L)$ of $L > 0$ for fixed $n \in \{2, 3, \ldots\}$, $\Lambda(L) := \max(\log(L - Y_0)U_n(L), \log(Y_0 - L)D_n(L))$, and proceed to find the optimal take-profit level, denoted as $\hat{L}$, which satisfies

$$\Lambda\left(\hat{L}\right) = \max_{L>0} \Lambda(L). \tag{10}$$

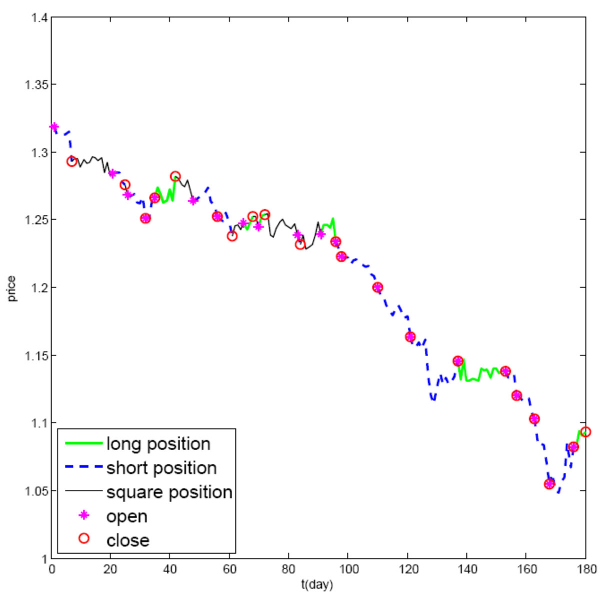
Define $\Lambda_1(L) := \log(L - Y_0)U_n(L)$ and $\Lambda_2(L) := \log(Y_0 - L)D_n(L)$, we have $\max_{L>0} \Lambda(L) = \max\left(\max_{L>0} \Lambda_1(L), \max_{L>0} \Lambda_2(L)\right)$, where $\max_{L>0} \Lambda_1(L)$ and $\max_{L>0} \Lambda_2(L)$ are numerically computable with (3), Proposition 4.1 and Proposition (9), 4.2 respectively. We implement it with a parameters set $\{\mu = 0.01, \sigma = 0.01, Y_0 = 100, \delta_n = 1\}$ as follows.

i) Determine $\hat{L}_1$ by computing $\Lambda_1(L)$, the mapping $L \to \Lambda_1(L)$ is reflected in a belly curve, and the maximum value $\Lambda_1(L)$ is achieved when $\hat{L}_1 = 106.1$ and $\Lambda_1\left(\hat{L}\right) = 1.6384$.

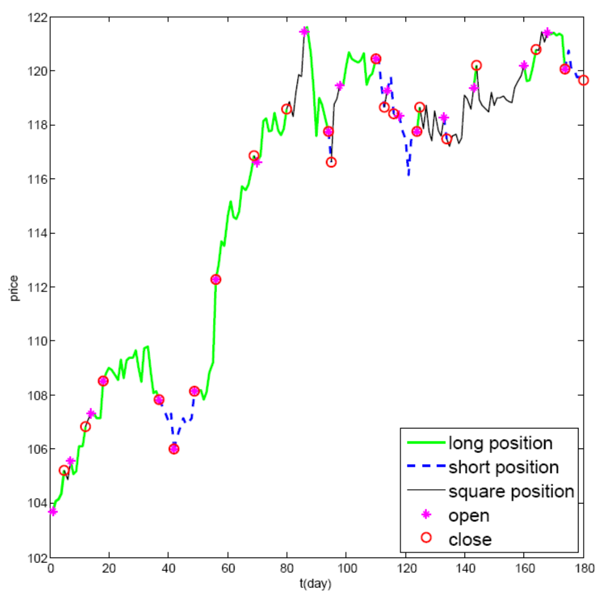
ii) Determine $\hat{L}_2$ by computing $\Lambda_2(L)$. Similarly we obtain that $\hat{L}_2 = 99.9$ and $\Lambda_2\left(\hat{L}\right) = 1.07982$.

iii) Compare $\Lambda_1\left(\hat{L}_1\right)$ and $\Lambda_2\left(\hat{L}_2\right)$ to determine $\Lambda\left(\hat{L}\right)$ and $\hat{L}$. In this case, $\Lambda_2\left(\hat{L}\right) > \Lambda_1\left(\hat{L}\right)$, and hence we conclude that the global optimal level $\hat{L} = \hat{L}_1 = 106.1$. Intuitively, we only consider the upward crossings since μ is positive, and $\hat{L}_2 = 99.9$ means that it is suggested to close the position as soon as possible as we opened it in a direction against the trend we detected.

To show the real profitability in the trading practice, we conduct a testing in the FX market. We choose two pairs, one is EURUSD, the other is USDJPY for they are the most significant currency pairs. This experiment lasted 180 market days from 28-th Aug 2014 to 25-th Mar 2015, the exact day we started to write this section. On each day, we update the parameter μ and σ based on the closing prices of the previous 10 days, and let $T = 10$, $\delta_n = 1$, then solve the maximization problem (10) to achieve the optimal take-profit level $\hat{L}$ and the corresponding value of $\Lambda\left(\hat{L}\right)$. Once the computational result



(a)



(b)

FIGURE 2. Trading on EURUSD and USDJPY

shows $\Lambda\left(\hat{L}\right) \geq 1.1$ and position is flat, we open a position, specifically, long position is taken when $\Lambda_1\left(\hat{L}_1\right) > \Lambda_2\left(\hat{L}_2\right)$, and we short for the case $\Lambda_1\left(\hat{L}_1\right) < \Lambda_2\left(\hat{L}_2\right)$. All trades are recorded in Figure 2 below, where thick lines denote the state in a long position, blue dash lines stand for a short position, and thin lines indicate that there are no unclosed trades. Besides, starting points of each trade are marked with a red circle, while the terminal points are marked with a star. Through the trading practice shown in Figure 2(a) for EURUSD, the total profit is 2398 pips, obtained by summing up all deals. For the another pair - USDJPY, performed with more sways, based on the trading practice shown in Figure 2(b), the total profit is 1752.6 pips and taking one long position through all the 180 market days will win 1592.7 pips. Our trading strategy outperforms the absolute increase of the index by 10 percent.

6. **Conclusions and Further Research.** By this paper, we showed that the expected number of level hitting of geometric Brownian motion is computable by discretizing the time horizon, although Browian motion has infinite hitting locally. Our method is able to be extended to other diffusion processes, and its application to barrier option is also available.

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