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LINEAR QUADRATIC OPTIMIZATION MODELS OF UNCERTAIN SWITCHED SYSTEMS

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Abstract. *This paper studies a quadratic optimal control problem for uncertain switched linear systems with subsystems perturbed by human uncertainty which is neither like randomness nor like fuzziness. The goal is to jointly design a deterministic switching law and a continuous feedback to minimize the expectation of a quadratic cost function. A two-stage algorithm is applied to handle such model. In the first stage, the minimum value of the cost function and the optimal control are obtained under fixed switching instants, and in the second stage, the mutation ant colony optimization (MACO) algorithm is used to get the optimal switching instants. An example is shown to validate the method.*
Keywords: Optimal control, Uncertain switched system, Equation of optimality, Linear quadratic, MACO algorithm

1. Introduction.

Optimal control of switched systems is a challenging problem that has received much research attention in recent years [1-3]. Linear quadratic (LQ) control is one of the most fundamental and widely used tools in many fields of modern real life. In [3,4], the LQ model of switched systems is discussed. However, a majority of these methods are based on deterministic models for subsystem dynamics.

However, indeterminacy is ubiquitous in realistic system models and the complexity of the world makes the control systems we face uncertain in various forms such as randomness and fuzziness. Nevertheless, lots of human uncertainty behaves neither like randomness nor like fuzziness, such as oil field reserves, bridge strength, and enemy force. In order to deal with these human uncertainty, an uncertainty theory was founded by Liu [5] in 2007, and refined in 2010 [6]. So far, the content of uncertainty theory has been developed to a fairly complete system for modeling human uncertainty. Based on uncertain theory, Zhu [7] introduced and dealt with the expected value model of uncertain optimal control problem by using dynamic programming in 2010. Recently, Yan and Zhu [8, 9] studied bang-bang control models with expected value criterion and optimistic value criterion respectively for uncertain switched systems.

Also many practical LQ problems contain the human uncertainty which may affect the running of systems. So to study LQ optimization models of uncertain switched system is necessary. In this paper, we just introduce such uncertain model and focus on the solution approaches of the problems where the order of the sequence of the active subsystems is known. Our aim is to establish an optimal linear quadratic optimization model of uncertain switched system and provide an efficient computational algorithm to solve this optimal switching problem.

The rest of the paper is organized as follows. In Section 2, some basic concepts are reviewed. In Section 3, the LQ model of uncertain switched system with fixed switching instants is formulated. In Section 4, two-stage algorithm is applied to handle such model. In Section 5, the effectiveness of the proposed method is tested with a numerical example. In Section 6, the conclusion is given.

2. Preliminary.

For convenience, we give some useful concepts in uncertainty theory [5, 6, 10]. Let Γ be a nonempty set, and $\mathcal{L}$ a σ-algebra over Γ. Each element $A \in \mathcal{L}$ is called an event. A set function $\mathcal{M}$ defined on the σ-algebra $\mathcal{L}$ is called an uncertain measure if it satisfies (i) $\mathcal{M}(\Gamma) = 1$; (ii) $\mathcal{M}(A) + \mathcal{M}(A^c) = 1$ for any event A; (iii) $\mathcal{M}(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mathcal{M}(A_i)$ for every countable sequence of events A_i. The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space. An uncertain variable is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set R of real numbers, and an uncertain vector is a measurable function from an uncertainty space to R^n. The uncertainty distribution $\Phi : R \to [0, 1]$ of an uncertain variable ξ is defined by $\Phi(x) = \mathcal{M}\{\xi \leq x\}$ for any real number x. The expected value of an uncertain variable ξ is defined by $E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq r\}\mathrm{d}r - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq r\}\mathrm{d}r$ provided that at least one of the two integrals is finite.

An uncertain process C_t is said to be a canonical process if (i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous; (ii) C_t has stationary and independent increments; (iii) every increment $C_{s+t} - C_s$ is a normal uncertain variable with expected value 0 and variance t^2, denoted by $C_{s+t} - C_s \sim \mathcal{N}(0, t)$ whose uncertainty distribution is $\Phi(x) = \left(1 + \exp\left(\frac{-\pi x}{\sqrt{3}t}\right)\right)^{-1}$ for $x \in R$. Suppose C_t is a canonical process, f and g are two given functions. Then

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t \tag{1}$$

is called an uncertain differential equation. A solution of (1) is an uncertain process X_t that satisfies $X_t = X_0 + \int_0^t f(s, X_s)\mathrm{d}s + \int_0^t g(s, X_s)\mathrm{d}C_s$.

3. Problem Formulation.

3.1. Uncertain switched system.

Considering an uncertain switched system consisting of the following subsystems:

$$\begin{array}{l} \mathrm{d}X_s = (A_i(s)X_s + B_i(s)u_s)\mathrm{d}s + \sigma(s, u_s, X_s)\mathrm{d}C_s, \quad s \in [0, T] \\ X_0 = x_0, i \in I = \{1, 2, \cdots, M\} \end{array} \tag{2}$$

where $X_s \in R^n$ is the state vector and $u_s \in R^r$ is the decision vector in a domain U, $A_i : [0, T] \to R^{n\times n}$, $B_i : [0, T] \to R^{n\times r}$ are some twice continuously differentiable functions for $i \in I$, $C_s = (C_{s1}, C_{s2}, \cdots, C_{sk})^{\tau}$, where $C_{s1}, C_{s2}, \cdots, C_{sk}$ are independent canonical processes.

An optimal control problem of such a system involves finding an optimal control u_t^* and an optimal switching law such that a given cost function is minimized. A switching law in $[0, T]$ for system (2) is defined as $\Lambda = ((t_0, i_0), (t_1, i_1), \cdots, (t_K, i_K))$, where t_k $(k = 0, 1, \cdots, K)$ satisfying $0 = t_0 \leq t_1 \leq \cdots \leq t_K \leq t_{K+1} = T$ are the switching instants and $i_k \in I$ for $k = 0, 1, \cdots, K$. Here (t_k, i_k) indicates that at instants t_k, the system switches from subsystem i_{k-1} to i_k. During the time interval $[t_k, t_{k+1})$ ($[t_K, T]$ if $k = K$), subsystem i_k is active. Since many practical problems only involve optimizations in which a prespecified order of active subsystems is given, for convenience, we assume subsystem i is active in $[t_{i-1}, t_i)$.

3.2. **LQ optimal control problem.** In this paper, we consider a kind of special model of uncertain switched systems with a quadratic objective function subject to some linear uncertain differential equations. Then the following uncertain expected value LQ model of uncertain switched systems is considered.

$$\left\{\begin{array}{l} J(t,x) = \min\limits_{u_t} E\left[\int_t^T \left(\frac{1}{2}X_s^\tau Q(t)X_s + X_s^\tau V(t)u_s + \frac{1}{2}u_s^\tau R(t)u_s + M(t)X_s\right.\right. \\ \qquad\qquad \left.\left. +N(t)u_s + W(t)\right)\mathrm{d}s + \frac{1}{2}X_T^\tau Q_T X_T + M_T X_T + L_T\right] \\ \text{s.t. } \ \mathrm{d}X_s = (A_i(s)X_s + B_i(s)u_s)\mathrm{d}s + \sigma(s,u_s,X_s)\mathrm{d}C_s \\ \qquad s \in [t_{i-1},t_i),\ i = 1,2,\cdots,K+1 \\ \qquad X_t = x \end{array}\right. \tag{3}$$

where T, x_0 are given, $Q(t) \in R^{n\times n}$, $V(t) \in R^{n\times r}$, $R(t) \in R^{r\times r}$, $M(t) \in R^n$, $N(t) \in R^r$, $W(t) \in R$ are functions of time t and $Q_T, Q(t) \geq 0$, $R(t) > 0$. $J(t,x)$ denotes the optimal value obtained in $[t,T]$ with the condition that at time t we are in state $X_t = x$. The aim to discuss this model is to find not only an optimal control u_t^* but also an optimal switching law.

By the equation of optimality [11] to deal with the model (3), the following conclusion can be obtained.

Theorem 3.1. *Assume that $J(t,x)$ be twice differentiable on $[t_{i-1},t_i)\times R^n$, then we have*

$$\begin{array}{rl} -J_t(t,x) = \min\limits_{u_t} & \left[\frac{1}{2}x^\tau Q(t)x + x^\tau V(t)u_t + \frac{1}{2}u_t^\tau R(t)u_t + M(t)x + N(t)u_t + W(t)\right. \\ & \left. +(A_i(t)x + B_i(t)u_t)^\tau \nabla_x J(t,x)\right] \end{array} \tag{4}$$

where $J_t(t,x)$ is the partial derivative of the function $J(t,x)$ in t, and $\nabla_x J(t,x)$ is the gradient of $J(t,x)$ in x.

4. **Two Stage Approach.** In order to solve problem (3), we decompose it into two stages. Stage (a) deals with conventional uncertain LQ problems which seek the minimum value of J with respect to the switching instants. Stage (b) solves an optimization problem.

4.1. **Stage (a).** In this stage, we fix the switching instants $t_1, t_2, \cdots, t_K$ and handle the following model to find the optimal value.

$$\left\{\begin{array}{l} \quad J(0,x_0,t_1,\cdots,t_K) \\ = \min\limits_{u_s} E\left[\int_0^T \left(\frac{1}{2}X_s^\tau Q(t)X_s + X_s^\tau V(t)u_s + \frac{1}{2}u_s^\tau R(t)u_s + M(t)X_s\right.\right. \\ \quad \left.\left. +N(t)u_s + W(t)\right)\mathrm{d}s + \frac{1}{2}X_T^\tau Q_T X_T + M_T X_T + L_T\right] \\ \text{s.t. } \ \mathrm{d}X_s = (A_i(s)X_s + B_i(s)u_s)\mathrm{d}s + \sigma(s,u_s,X_s)\mathrm{d}C_s \\ \qquad s \in [t_{i-1},t_i),\ i = 1,2,\cdots,K+1 \\ \qquad X_0 = x_0 \end{array}\right. \tag{5}$$

Applying Equation (4) to model (5), we have the following conclusion.

Theorem 4.1. *Assume that $J(t,x)$ be twice differentiable on $[t_{i-1},t_i)\times R^n$ $(i = 1,2,\cdots, K+1)$. Let $Q(t)$, $V(t)$, $R(t)$, $M(t)$, $N(t)$, $W(t)$, $A_i(t)$, $B_i(t)$, $R^{-1}(t)$ be continuous bounded functions of t, and $Q(t), Q_T \geq 0$, $R(t) > 0$. The optimal control of model (5) when $t \in [t_{i-1},t_i)$ is that*

$$u_t^{(i)*} = -R^{-1}(B_i^\tau P_i(t) + V^\tau)x - R^{-1}\left(B_i^\tau S_i^\tau(t) + N^\tau\right) \tag{6}$$

for $i = 1,2,\cdots,K+1$, where $P_i(t) = P_i^\tau(t)$, $S_i(t)$ satisfy

$$\left\{\begin{array}{l} -\dot{P}_i(t) = Q + P_i(t)A_i + A_i^\tau P_i(t) - (P_i(t)B_i + V)R^{-1}(B_i^\tau P_i(t) + V^\tau) \\ P_{K+1}(T) = Q_T \text{ and } P_i(t_i) = P_{i+1}(t_i) \text{ for } i \leq K \end{array}\right. \tag{7}$$

and

$$\begin{cases} -\dot{S}_i(t) = M + S_i(t)A_i - (N + S_i(t)B_i)R^{-1}(B_i^\tau P_i(t) + V^\tau) \\ S_{K+1}(T) = M_{_T} \text{ and } S_i(t_i) = S_{i+1}(t_i) \text{ for } i \le K \end{cases} \tag{8}$$

The optimal value of model (5) is

$$J(0, x_0, t_1, \cdots, t_K) = \frac{1}{2}x_0^\tau P_1(0)x_0 + S_1(0)x_0 + L_1(0) \tag{9}$$

where $L_i(t)$*,* $t \in [t_{i-1}, t_i)$ *satisfies*

$$\begin{cases} -\dot{L}_i(t) = W - \frac{1}{2}(S_i(t)B_i + N)R^{-1}(B_i^\tau S_i^\tau(t) + N^\tau) \\ L_{K+1}(T) = L_{_T} \text{ and } L_i(t_i) = L_{i+1}(t_i) \text{ for } i \le K \end{cases} \tag{10}$$

Proof: It follows from the equation of optimality (4) that

$$\begin{aligned} -J_t(t,x) = \min_{u_t} \Big[& \frac{1}{2}x^\tau Qx + x^\tau V u_t + \frac{1}{2}u_t^\tau R u_t + Mx + Nu_t + W \\ & + (A_i x + B_i u_t)^\tau \nabla_x J(t,x) \Big] \end{aligned}$$

Let

$$\begin{aligned} L\left(u_t^{(i)}\right) = & \frac{1}{2}x^\tau Qx + x^\tau V u_t^{(i)} + \frac{1}{2}u_t^{(i)\tau} R u_t^{(i)} + Mx + N u_t^{(i)} + W \\ & + \left(A_i x + B_i u_t^{(i)}\right)^\tau \nabla_x J(t,x) \end{aligned}$$

The optimal control $u_t^{(i)*}$ satisfies $\frac{\partial L\left(u_t^{(i)}\right)}{\partial u_t^{(i)}} = V^\tau x + R u_t^{(i)} + N^\tau + B_i^\tau \nabla_x J(t,x) = 0$.

Since $\frac{\partial^2 L(u_t^{(i)})}{\partial^2 u_t^{(i)}} = R > 0$, we have

$$u_t^{(i)*} = -R^{-1}(V^\tau x + N^\tau + B_i^\tau \nabla_x J(t,x)), \quad t \in [t_{i-1}, t_i) \tag{11}$$

When $t \in [t_K, T]$, according to (4), we have

$$\begin{aligned} -J_t(t,x) = \min_{u_t} \Big[& \frac{1}{2}x^\tau Qx + x^\tau V u_t^{(K+1)} + \frac{1}{2}u_t^{(K+1)\tau} R u_t^{(K+1)} + Mx + N u_t^{(K+1)} + W \\ & + \left(A_i x + B_i u_t^{(K+1)}\right)^\tau \nabla_x J(t,x) \Big] \end{aligned} \tag{12}$$

Since $J(T, x_T) = \frac{1}{2}X_T^\tau Q_T X_T + M_T X_T + L_T$, we guess

$$J(t,x) = \frac{1}{2}x^\tau P_{K+1}(t)x + S_{K+1}(t)x + L_{K+1}(t), \quad t \in [t_K, T] \tag{13}$$

and

$$P_{K+1}(t) = P_{K+1}^\tau(t), \quad P_{K+1}(T) = Q_T, \quad S_{K+1}(T) = M_T, \quad L_{K+1}(T) = L_T$$

So

$$\frac{\partial J}{\partial t} = \frac{1}{2}x^\tau \dot{P}_{K+1}(t)x + \dot{S}_{K+1}(t)x + \dot{L}_{K+1}(t), \quad \nabla_x J(t,x) = P_{K+1}(t)x + S_{K+1}^\tau(t) \tag{14}$$

Thus, it follows from (11) that

$$u_t^{(K+1)*} = -R^{-1}(B_{K+1}^\tau P_{K+1}(t) + V^\tau)x - R^{-1}(B_{K+1}^\tau S_{K+1}^\tau(t) + N^\tau) \tag{15}$$

Substituting (14) and (15) into (12) we have that

$$\begin{aligned}
&-\frac{1}{2}x^\tau \dot{P}_{K+1}(t)x - \dot{S}_{K+1}(t)x - \dot{L}_{K+1}(t) \\
=&\frac{1}{2}x^\tau \big[Q + P_{K+1}(t)A_{K+1} + A^\tau_{K+1}P_{K+1}(t) \\
&-(P_{K+1}(t)B_{K+1} + V)R^{-1}(B^\tau_{K+1}P_{K+1}(t) + V^\tau)\big]\, x \\
&+ \big[S_{K+1}(t)A_{K+1} - (N + S_{K+1}(t)B_{K+1})R^{-1}(B^\tau_{K+1}P_{K+1}(t) + V^\tau) + M\big]\, x \\
&+ \left[W - \frac{1}{2}(S_{K+1}(t)B_{K+1} + N)R^{-1}\left(B^\tau_{K+1}S^\tau_{K+1}(t) + N^\tau\right)\right]
\end{aligned}$$

Therefore, we have

$$\begin{cases} -\dot{P}_{K+1}(t) = Q + P_{K+1}(t)A_{K+1} + A^\tau_{K+1}P_{K+1}(t) \\ \qquad\qquad -(P_{K+1}(t)B_{K+1} + V)R^{-1}(B^\tau_{K+1}P_{K+1}(t) + V^\tau) \\ P_{K+1}(T) = Q_T \end{cases}$$

$$\begin{cases} -\dot{S}_{K+1}(t) = M + S_{K+1}(t)A_{K+1} - (N + S_{K+1}(t)B_{K+1})R^{-1}(B^\tau_{K+1}P_{K+1}(t) + V^\tau) \\ S_{K+1}(T) = M_T \end{cases}$$

and

$$\begin{cases} -\dot{L}_{K+1}(t) = W - \dfrac{1}{2}(S_{K+1}(t)B_{K+1} + N)R^{-1}(B^\tau_{K+1}S^\tau_{K+1}(t) + N^\tau) \\ L_{K+1}(T) = L_T \end{cases}$$

By the same method as above procedure, we can get the conclusion.

4.2. **Stage (b).** According to Theorem 4.1, $2(K+1)$ Riccati matrix differential equations have to be solved in order to solve the model (5). Then the optimal cost $J(0, x_0, t_1, \cdots, t_K)$ can be obtained by (9). Denote $\tilde{J}(t_1, \cdots, t_K) = J(0, x_0, t_1, \cdots, t_K)$. The next stage is to solve an optimization problem $\min\limits_{0\le t_1\le t_2\cdots\le t_K\le T} \tilde{J}(t_1, \cdots, t_K)$.

For model (5), we cannot obtain the analytical expressions of solutions according to Theorem 4.1. However, most optimization algorithms need explicit forms of the first order derivative of the objective functions. Being presented with such difficulties, evolutionary meta heuristic algorithms may be a good choice to solve stage (b). A new intelligent algorithm combining a mutation ant colony optimization algorithm and a simulated annealing method (MACO) was designed by Zhu [12] to solve continuous optimization models. Since the MACO was shown its superiority to some other evolutionary meta heuristic algorithms in [12], it is employed to solve the optimization problem above.

5. **An Example.** Consider the following example of LQ models for uncertain switched systems

$$\begin{cases} J(0, x_0) = \min\limits_{u(s)} E\left[\displaystyle\int_0^1 \left(-X(s) - u(s) + \frac{1}{2}u^2(s) + 1\right) \mathrm{d}s - X^2(1)\right] \\ \textit{subsystem } 1 : \mathrm{d}X(s) = [u(s) - \alpha_1 X(s)]\mathrm{d}s + \sigma X(s)\mathrm{d}C_s, \quad s \in [0, t_1) \\ \textit{subsystem } 2 : \mathrm{d}X(s) = [u(s) - \alpha_2 X(s)]\mathrm{d}s + \sigma X(s)\mathrm{d}C_s, \quad s \in [t_1, t_2) \\ \textit{subsystem } 3 : \mathrm{d}X(s) = [u(s) - \alpha_3 X(s)]\mathrm{d}s + \sigma X(s)\mathrm{d}C_s, \quad s \in [t_2, 1] \\ \quad X(0) = 1 \end{cases}$$

Stage (a): Fix t_1, t_2 and formulate $\tilde{J}(t_1, t_2)$ according to Theorem 4.1.

The solutions of Riccati equations of (7) and (8) of this example are

$$P_3(t) = \frac{m_3 e^{m_3 t}}{-e^{m_3 t} + n_3}, \quad S_3(t) = \frac{-2(m_3+1)e^{m_3 t} - 2n_3 + c_3 m_3 e^{\frac{1}{2}m_3 t}}{m_3(n_3 - e^{m_3 t})}$$

for $i = 3$, where $m_3 = 2\alpha_3$, $S_{t_3} = -1$, $n_3 = (-S_{t_3} - \alpha_3)e^{m_3}$, $c_3 = \left(\frac{4}{m_3} + 1\right) e^{\frac{1}{2}m_3}$.

In addition, we have

$$P_2(t) = \frac{-m_2 S_{t_2} e^{m_2 t}}{S_{t_2} e^{m_2 t} + n_2}, \quad S_2(t) = \frac{2S_{t_2}(m_2+1)e^{m_2 t} - 2n_2 + c_2 m_2 e^{\frac{1}{2}m_2 t}}{m_2(n_2 + S_{t_2} e^{m_2 t})}$$

for $i = 2$, where $m_2 = 2\alpha_2$, $S_{t_2} = \frac{1}{2}P_3(t_2)$, $n_2 = (-S_{t_2} - \alpha_2)e^{m_2 t_2}$, $S'_{t_2} = S_3(t_2)$, $c_2 = \left(-\frac{m_2}{2}S'_{t_2} - 2S_{t_2} - 1 - \frac{4S_{t_2}}{m_2}\right) e^{\frac{1}{2}m_2 t_2}$, and

$$P_1(t) = \frac{-m_1 S_{t_1} e^{m_1 t}}{S_{t_1} e^{m_1 t} + n_1}, \quad S_1(t) = \frac{2S_{t_1}(m_1+1)e^{m_1 t} - 2n_1 + c_1 m_1 e^{\frac{1}{2}m_1 t}}{m_1(n_1 + S_{t_1} e^{m_1 t})}$$

for $i = 1$, where $m_1 = 2\alpha_1$, $S_{t_1} = \frac{1}{2}P_2(t_1)$, $n_1 = (-S_{t_1} - \alpha_1)e^{m_1 t_1}$, $S'_{t_1} = S_2(t_1)$, $c_1 = (-\frac{m_1}{2}S'_{t_1} - 2S_{t_1} - 1 - \frac{4S_{t_1}}{m_1})e^{\frac{1}{2}m_1 t_1}$.

According to Theorem 4.1, the optimal value is $\tilde{J}(t_1, t_2) = \frac{1}{2}P_1(0) + S_1(0) + L_1(0)$, where

$$\begin{aligned} L_1(0) = &\int_0^{t_1} \left[-\frac{1}{2}S_1^2(t) + S_1(t) + \frac{1}{2}\right] \mathrm{d}t + \int_{t_1}^{t_2} \left[-\frac{1}{2}S_2^2(t) + S_2(t) + \frac{1}{2}\right] \mathrm{d}t \\ &+ \int_{t_2}^{1} \left[-\frac{1}{2}S_3^2(t) + S_3(t) + \frac{1}{2}\right] \mathrm{d}t \end{aligned}$$

Stage (b): Find the optimal switching instant t_1^*, t_2^* according to MACO algorithm.

Choose $\alpha_1 = \frac{1}{3}$, $\alpha_2 = \frac{1}{4}$, $\alpha_3 = \frac{1}{2}$. By applying MACO algorithm, we find the optimal switching instant $t_1^* = 0.303$, $t_2^* = 0.462$. The optimal control is

$$u_t^* = \begin{cases} 1 - \dfrac{675.14e^{0.667t} - 1197.62e^{0.333t} - 502.78}{167.68 + 135.07e^{0.667t}} + \dfrac{135.17e^{0.667t}x(t)}{251.39 + 202.5e^{0.667t}}, & t \in [0, 0.303) \\ 1 + \dfrac{10.44e^{0.5t} - 19.79e^{0.25t} + 8.14}{2.04 - 1.74e^{0.5t}} - \dfrac{1.74e^{0.5t}x(t)}{4.07 - 3.48e^{0.5t}}, & t \in [0.303, 0.462) \\ 1 + \dfrac{4e^t - 8.24e^{0.5t} + 2.718}{1.359 - e^t} - \dfrac{e^t x(t)}{1.359 - e^t}, & t \in [0.462, 1] \end{cases}$$

6. **Conclusion.** An LQ optimization model for uncertain switched systems with subsystems perturbed by human uncertainty has been presented, together with a method to design a control strategy and a switching law. Divide-and-conquer manner was borrowed to deal with such model. In stage (a), we found the minimum value of the cost function and the optimal control under fixed switching instants. In stage (b), MACO algorithm was used to find the optimal switching instants. The example validated the method well. LQ optimization models of uncertain switched systems for continuous time were considered in this paper. However, many real-world switched systems are discrete-time. Also, sometimes we need to discretize continuous-time switched systems in order to solve our problems. Therefore, the research on problems of optimal control for uncertain discrete-time switched systems is our future work.

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