ICIC Express Letters ICIC International ©2016 ISSN 1881-803X
Volume **10**, Number **10**, October **2016** pp. 2377–2382

SEMIDEFINITE PROGRAMMING ALGORITHM FOR TOA-BASED MULTIPLE SOURCE LOCALIZATION

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Received February 2016; accepted May 2016

Abstract. *When the transmission time is considered as unavailable, a semidefinite programming (SDP) algorithm is proposed for time of arrival (TOA) based multiple source localization by relaxing the model into the SDP convex optimization problem. The proposed SDP problem can be solved with interior point methods which are self initialized and require no initialization compared with the conventional maximum likelihood (ML) estimator. The Cramér-Rao Lower Bound (CRLB) for this problem is then given for comparison. The simulations show that the SDP algorithm provides robust solutions for the joint estimates of source locations and transmission time.*
Keywords: Wireless sensor networks, Localization, Semidefinite programming, Time of arrival, Ship model self-propulsion

1. **Introduction.** Wireless sensor networks (WSNs) have been emerging as attractive technologies with interesting applications such as medical, environmental and military monitoring. Usually, WSNs use a large number of sensor nodes able to communicate together in wireless mode and collaborate to provide information for common missions including the test of ship model self-propulsion. To make the data collected from sensor nodes meaningful, it often requires related locations of sensor nodes [1]. The accuracy performance of ship model location awareness is also the key technology problem in the test. It is difficult to locate the ship due to the small scale and high velocity model. Global positioning system (GPS) is the most important technology to provide location awareness around the globe through a constellation of at least 24 satellites. However, the effectiveness of GPS is limited at each low-cost and tiny sensor node for its huge volume, energy consumption and hardware cost.

Generally, in WSNs, there are some anchor nodes with known position, while the positions of some source nodes are unknown. Then the locations of source nodes are estimated by using the anchor nodes and the corresponding distance measurements between the sensor nodes. These measurements may include time of arrival (TOA) [2], time difference of arrival (TDOA) [3], angle of arrival (AOA) and received signal strength (RSS) [4]. Among the different types of measurements, TOA measurements are relatively common in modern sensor networks under the collaborative nature of the sensor nodes.

There are a number of estimation algorithms which have been proposed for source localization over the past years. The well-known maximum likelihood (ML) estimator can achieve excellent accuracy performance at sufficiently small noise which is assumed to be Gaussian. However, the numerical solution of ML estimator needs the initial points or may suffer from local minima and even divergence problems. To overcome the shortcomings of the ML estimator, linear estimator and convex optimization method are proposed by using some approximations and relaxations [5]. However, the performance of the linear estimator is worse due to the singularity especially when the noises are enough large. Another alternative method for the ML convergence problem is convex optimization techniques, including semidefinite programming (SDP) [6, 7] and second order cone programming (SOCP) [8]. By relaxing the localization model into convex optimization problem, the SDP and SOCP algorithms provide robust solutions for source location estimates.

In the TOA-based localization model, source node transmits a packet including known preamble and transmission time. Each sensor only needs to identify the known preamble to record its arrival time. The propagation time is obtained by subtracting the transmission time from the arrival time and directly used to estimate the source locations. However, the transmission time is not always available due to the network attack or clock offset. One way to tackle this problem is to exploit the time difference of arrival (TDOA) which eliminates the transmission time. Although the dependence on the initial transmission time is eliminated by TDOA, the measurement subtraction for computing TDOA strengthens the noise and usually leads to degraded performance. Assuming the start transmission time as unavailable, the authors in [9] proposed second order cone relaxation for conventional single source localization. Concurrently active multiple source nodes substantially complicate the problem, so the joint estimation of the transmission time and multiple source locations is proposed by using the linear programming (LP) relaxation and sensible approximations [10]. However, the solutions of the LP technique also need the initial points and cannot ensure the global convergence.

To overcome the shortcoming of LP technique, an SDP algorithm is proposed for TOA-based multiple source localization in this paper. The transmission time and source locations are jointly estimated by relaxing the localization model into SDP convex optimization problem. For arbitrary symmetric matrices $\mathbf{A}$, $\mathbf{A} \succeq 0$ means that $\mathbf{A}$ is positive semidefinite. $\|*\|$ denotes ℓ_2 norm. $[\mathbf{A}]_{i,j}$ denotes the the element at the ith row and jth column of matrix $\mathbf{A}$.

2. **Problem Specification.**

Assume that $M+N$ sensor nodes including M anchor nodes and N source nodes are deployed in a 2-dimensional geographical region. Denote the positions of all source nodes by $\mathbf{x}_i = [x_i \ \ y_i]^T$, $i = 1, 2, \ldots, M$. The positions of anchor nodes are denoted by $\mathbf{a}_j = [x_j \ \ y_j]^T$, $j = M+1, M+2, \ldots, M+N$ and known as these nodes may be positioned or the nodes may have GPS. The localization goal is to determine the location of the other M source nodes (i.e., estimate $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_M$). The sensor nodes receive the signal transmitted by the source nodes and detect the time of arrival. The time of arrival measurement $b_{i,j}$ at sensor node j can be easily modeled as

$$b_{i,j} = t_{i,j} + \tau_i + n_{i,j} \quad i \in \mathcal{S}, \ j \in \mathcal{E}_i \tag{1}$$

where $\mathcal{S} = \{i | i = 1, 2, \ldots, M\}$, $\mathcal{E}_i$ represents the set of all sensor nodes connected to source node i, $j \in \mathcal{E}_i$ means that sensor node j can be connected to source node i, τ_i is the unknown time instant at which source node i transmits the signal to be measured, $n_{i,j}$ is the additive measurement noise with zero mean and variance $\delta_{i,j}^2$, $t_{i,j}$ is the true propagation time which can also be formulated as

$$\begin{cases} t_{i,j} = \frac{\|\mathbf{x}_i - \mathbf{a}_j\|}{c} & i \in \mathcal{S}, \ j \in \mathcal{A}_i \\ t_{i,j} = \frac{\|\mathbf{x}_i - \mathbf{x}_j\|}{c} & i \in \mathcal{S}, \ j \in \mathcal{B}_i \end{cases} \tag{2}$$

where $\mathcal{A}_i$ denotes the set of all anchor nodes connected to the source node i, $\mathcal{B}_i$ denotes the set of all source nodes connected to the source node i, $\mathcal{E}_i = \mathcal{A}_i \cup \mathcal{B}_i$, c is the speed of light, and $\|\cdot\|$ denotes the Euclidean norm. In the proposed localization model, the start transmission time τ_i is assumed to be unknown and required to be estimated along with the source locations $\mathbf{x}_i$, so there are in total $3M$ unknown parameters, which are defined as

$$\tau = [\tau_1 \ \tau_2 \ \ldots \ \tau_M]^T \tag{3}$$

$$\mathbf{X} = [\mathbf{x}_1 \ \mathbf{x}_2 \ \ldots \ \mathbf{x}_M] \tag{4}$$

where $\tau \in \mathbb{R}^{M\times 1}$, $\mathbf{X} \in \mathbb{R}^{2\times M}$. Since the noise $n_{i,j}$ is Gaussian, the well known ML estimator of the proposed model is simply obtained by the following minimization problem

$$\begin{aligned} &\min_{\mathbf{X},\tau} \sum_{i\in\mathcal{S}} \sum_{j\in\mathcal{E}_i} \frac{1}{\delta_{i,j}^2} (b_{i,j} - t_{i,j} - \tau_i)^2 \\ &\text{s.t.} \quad t_{i,j} = \frac{\|\mathbf{x}_i - \mathbf{a}_j\|}{c} \quad i\in\mathcal{S},\ j\in\mathcal{A}_i \\ &\qquad\ \ t_{i,j} = \frac{\|\mathbf{x}_i - \mathbf{x}_j\|}{c} \quad i\in\mathcal{S},\ j\in\mathcal{B}_i \end{aligned} \tag{5}$$

The optimization problem in (5) is highly nonlinear and nonconvex and solved by the iterative numerical methods, which rely on the initial point. If the initial point is not sufficiently close to the global minimum, the numerical methods may converge to a local minimum or a saddle point causing a large estimation error. To ensure the global convergence, an SDP algorithm is proposed for TOA-based multiple source localization in the following.

3. **Semidefinite Programming Algorithm.** $K = |\mathcal{E}_i,\ i\in\mathcal{S}|$ denotes the total link number of all source nodes in the network. To obtain an SDP form for the proposed model, the cost function of the ML estimator is firstly formulated as a linear function. By stacking (1) in an ascending order of i and j, we produce the new TOA measurement vector $\mathbf{b} = [b_{i,j}|i\in\mathcal{S},\ j\in\mathcal{E}_i]^T$, the propagation time vector $\mathbf{t} = [t_{i,j}|i\in\mathcal{S},\ j\in\mathcal{E}_i]^T$, the noise vector $\mathbf{n} = [n_{i,j}|i\in\mathcal{S},\ j\in\mathcal{E}_i]^T$ and the transmit time vector $\tau = [\tau_i|i = 1,2,\ldots,M]^T$. $\mathbf{b}\in\mathbb{R}^{K\times 1}$, $\mathbf{t}\in\mathbb{R}^{K\times 1}$, $\mathbf{n}\in\mathbb{R}^{K\times 1}$ and $\tau\in\mathbb{R}^{M\times 1}$. So the matrix form of (1) is rewritten as

$$\mathbf{b} = \mathbf{I}_K\mathbf{t} + \mathbf{F}\tau + \mathbf{n} \tag{6}$$

where $\mathbf{I}_K$ denotes the $K\times K$ identity matrix, $\mathbf{F}$ is a $K\times M$ with 1 at the ith column, the row of corresponding measurement $T_{i,j}$ and 0's elsewhere. Defining a new unknown vector $\mathbf{h} = \left[\mathbf{t}^T \ \ \tau^T\right]^T$, (6) is rewritten as

$$\mathbf{b} = \mathbf{A}\mathbf{h} + \mathbf{n} \tag{7}$$

where $\mathbf{A} = [\mathbf{I}_K;\mathbf{F}]^T$, $\mathbf{A}\in\mathbb{R}^{K\times(K+M)}$, $\mathbf{h}\in\mathbb{R}^{(K+M)\times 1}$. So the cost function of the ML estimator in (5) can be alternatively written as

$$\mathrm{Tr}\left\{\mathbf{\Sigma}(\mathbf{A}\mathbf{h}-\mathbf{b})^T(\mathbf{A}\varphi-\mathbf{b})\right\} = \mathrm{Tr}\left\{\mathbf{\Sigma}\left(\mathbf{A}^T\mathbf{H}\mathbf{A} - 2\mathbf{A}^T\mathbf{h}\mathbf{b} + \mathbf{b}^T\mathbf{b}\right)\right\} \tag{8}$$

where $\mathbf{\Sigma} = \mathrm{diag}\left\{\frac{1}{\delta_{i,j}^2}|i\in\mathcal{S}, j\in\mathcal{E}_i\right\}$, $\mathbf{H} = \mathbf{h}\mathbf{h}^T$. The diagonal elements of the matrix $\mathbf{H}$ are denoted as $[\mathbf{H}]_{p,p}$, which is rewritten as

$$\begin{cases} [\mathbf{H}]_{p,p} = \begin{bmatrix} \mathbf{a}_j \\ -\mathbf{e}_i \end{bmatrix}^T \mathbf{Z} \begin{bmatrix} \mathbf{a}_j \\ -\mathbf{e}_i \end{bmatrix} & i\in\mathcal{S},\ j\in\mathcal{A}_i \\ [\mathbf{H}]_{p,p} = \begin{bmatrix} \mathbf{0}_2 \\ \mathbf{e}_i - \mathbf{e}_j \end{bmatrix}^T \mathbf{Z} \begin{bmatrix} \mathbf{0}_2 \\ \mathbf{e}_i - \mathbf{e}_j \end{bmatrix} & i\in\mathcal{S},\ j\in\mathcal{B}_i \end{cases} \tag{9}$$

where $p = 1,2,\ldots,K$, $\mathbf{e}_i$ is an $M\times 1$ column vector with 1 at the ith entry and 0's elsewhere, $\mathbf{0}_2$ is a 2×1 zero column vector

$$\mathbf{Z} = \begin{bmatrix} \mathbf{I}_2 & \mathbf{X} \\ \mathbf{X}^T & \mathbf{Y} \end{bmatrix} \tag{10}$$

where $\mathbf{Y} = \mathbf{X}^T\mathbf{X}$. Then the cost function of the ML estimator in (5) is rewritten as

$$\begin{aligned}
&\min_{\mathbf{X},\mathbf{h},\mathbf{H},\tau} \mathrm{Tr}\left\{\mathbf{\Sigma}\left(\mathbf{A}^T\mathbf{H}\mathbf{A} - 2\mathbf{A}^T\mathbf{h}\mathbf{b}\right)\right\} \\
&\text{s.t.} \quad [\mathbf{H}]_{p,p} = \begin{bmatrix} \mathbf{a}_j \\ -\mathbf{e}_i \end{bmatrix}^T \mathbf{Z} \begin{bmatrix} \mathbf{a}_j \\ -\mathbf{e}_i \end{bmatrix} \quad i \in \mathcal{S},\ j \in \mathcal{A}_i \\
&\qquad [\mathbf{H}]_{p,p} = \begin{bmatrix} \mathbf{0}_2 \\ \mathbf{e}_i - \mathbf{e}_j \end{bmatrix}^T \mathbf{Z} \begin{bmatrix} \mathbf{0}_2 \\ \mathbf{e}_i - \mathbf{e}_j \end{bmatrix} \quad i \in \mathcal{S},\ j \in \mathcal{B}_i \\
&\qquad \mathbf{Z} = \begin{bmatrix} \mathbf{I}_2 & \mathbf{X} \\ \mathbf{X}^T & \mathbf{Y} \end{bmatrix} \\
&\qquad \mathbf{Y} = \mathbf{X}^T\mathbf{X}, \quad \mathbf{H} = \mathbf{h}\mathbf{h}^T
\end{aligned} \tag{11}$$

where the constant term $\mathbf{b}^T\mathbf{b}$ is removed from the cost function. It is noted that the cost function of (11) is linear with the variables of $\mathbf{H}$ and $\mathbf{h}$. However, the constraints in (11) make the problem nonconvex. To obtain the convex SDP form, we relax $\mathbf{Y} = \mathbf{X}^T\mathbf{X}$ as $\mathbf{Y} \succeq \mathbf{X}^T\mathbf{X}$. So (10) is reformulated as

$$\mathbf{Z} = \begin{bmatrix} \mathbf{I}_2 & \mathbf{X} \\ \mathbf{X}^T & \mathbf{Y} \end{bmatrix} \succeq \mathbf{0}_{M+2} \tag{12}$$

where $\mathbf{0}_{M+2}$ denotes the $M+2$ by $M+2$ zero matrix. Similarly $\mathbf{H} = \mathbf{h}\mathbf{h}^T$ is relaxed as the following

$$\begin{bmatrix} \mathbf{H} & \mathbf{h} \\ \mathbf{h}^T & 1 \end{bmatrix} \succeq \mathbf{0}_{K+M+1} \tag{13}$$

The above relaxations of the constraints degrade the performance of the source location estimates, but the convex SDP form is obtained. Using the relaxations, we rewrite the optimization problem of (11) as the SDP form

$$\begin{aligned}
&\min_{\mathbf{X},\mathbf{h},\mathbf{H},\tau} \mathrm{Tr}\left\{\mathbf{\Sigma}\left(\mathbf{A}^T\mathbf{H}\mathbf{A} - 2\mathbf{A}^T\mathbf{h}\mathbf{b}\right)\right\} \\
&\text{s.t.} \quad [\mathbf{H}]_{p,p} = \begin{bmatrix} \mathbf{a}_j \\ -\mathbf{e}_i \end{bmatrix}^T \mathbf{Z} \begin{bmatrix} \mathbf{a}_j \\ -\mathbf{e}_i \end{bmatrix} \quad i \in \mathcal{S},\ j \in \mathcal{A}_i \\
&\qquad [\mathbf{H}]_{p,p} = \begin{bmatrix} \mathbf{0}_2 \\ \mathbf{e}_i - \mathbf{e}_j \end{bmatrix}^T \mathbf{Z} \begin{bmatrix} \mathbf{0}_2 \\ \mathbf{e}_i - \mathbf{e}_j \end{bmatrix} \quad i \in \mathcal{S},\ j \in \mathcal{B}_i \\
&\qquad \mathbf{Z} \succeq \mathbf{0}_{M+2} \\
&\qquad \begin{bmatrix} \mathbf{H} & \mathbf{h} \\ \mathbf{h}^T & 1 \end{bmatrix} \succeq \mathbf{0}_{K+M+1}
\end{aligned} \tag{14}$$

The SDP optimization problem of (14) is convex and can be solved with well known algorithms such as interior point methods which are self initialized and requires no initialization from the user. Unlike the ML estimator, the cost function of the SDP problem in (14) is linear which ensures that there is only one minimum point. In MATLAB simulations, standard SDP solvers such as SeDuMi and SDPT3 are employed to solve SDP optimization problems. Extracting from $\mathbf{Z}$ we can obtain the source location estimates $\mathbf{X}$.

4. **Evaluation.** To test the performance of the proposed SDP algorithm, we conduct a group of simulations with 5 anchor nodes and 15 source nodes deployed in a 20 m $\times$ 20 m square region. The geographic locations of the source nodes and anchor nodes are shown as Figure 1. All noise variances $\delta_{i,j}^2$ are all set to δ^2. The proposed SDP convex optimization algorithm was implemented by the CVX toolbox using SeDuMi as the solver. The performance is evaluated in terms of the root mean square error (RMSE) with 500 Monte Carlo (MC) runs.

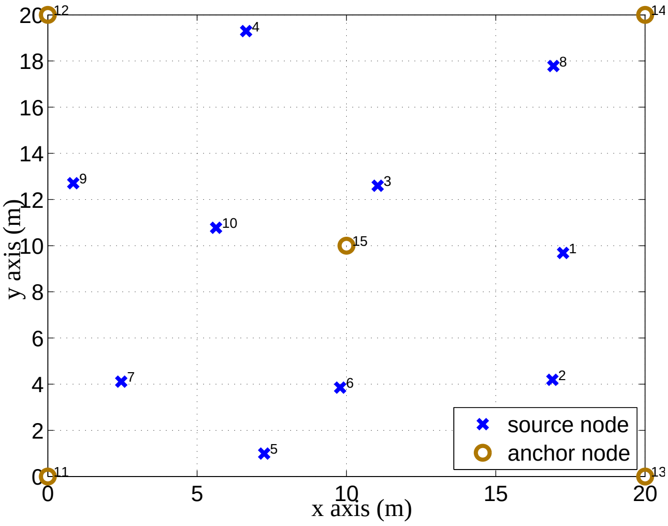
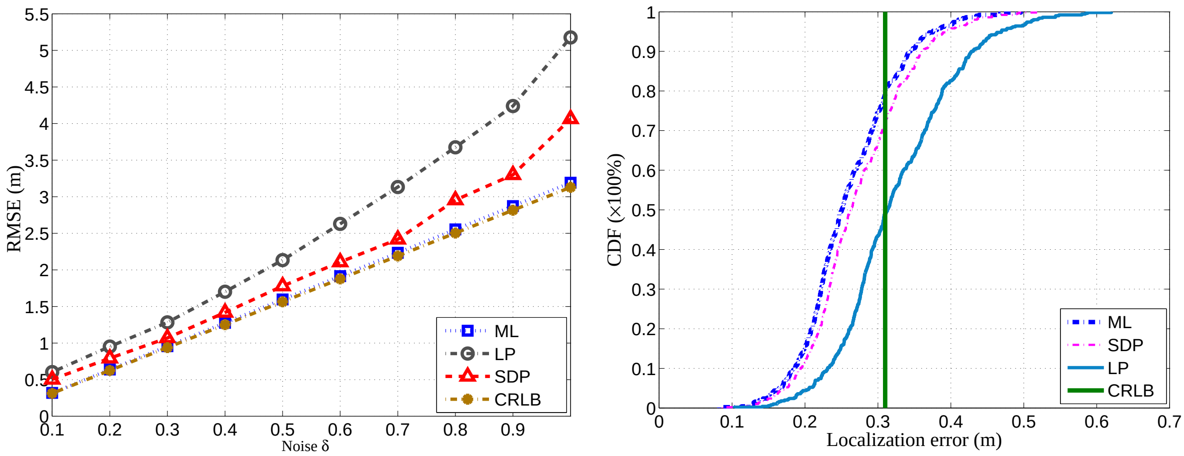


FIGURE 1. Geometry of deployed 20 sensor nodes



(a) Performance comparison with different noises

(b) CDF of localization error, $\delta^2 = 0.1^2$

FIGURE 2. Performance comparison of different methods

Since the source nodes are connected with different anchor nodes, the RMSE performance is diverse in the different source nodes. So the RMSE of the proposed algorithm is calculated by averaging over all estimated source locations in the network. When the transmission times of source nodes are all set to 1 s and assumed to be unknown, the CRLB of source location estimates is derived in [10]. The ML estimator is solved by the Levenberg-Marquardt algorithm using the true source location as the initial point. Considering the full connectivity of all sensor nodes in Figure 1, we perform Monte Carlo simulations with 500 ensemble runs to evaluate the RMSE of the location estimation. The performance comparison is plotted in Figure 2 with ML estimator, the LP technique proposed in [10], the proposed SDP algorithm and the CRLB of source location estimation. When the noise variance δ^2 is varied from 0.1^2 to 1^2, the RMSE performance of four different methods is plotted in Figure 2(a). It is observed that the RMSE performance degrades as the noise increases. The ML estimator is close to the CRLB and provides much better accuracy performance due to the reasonable initialization. Compared with the LP, the RMSE of the proposed SDP algorithm is reduced. Careful examinations indicate that only the anchor-source measurements are employed into the LP technique, so the accuracy performance of the LP is worse than that of the SDP algorithm.

To further compare the performance of the different methods, we plot the cumulative distribution function (CDF) in Figure 2(b) when δ^2 is set to 1^2. It can be seen that the performance order shown in Figure 2(b) is the same as Figure 2(a). 90% of the simulated runs are less than 0.43 m in the LP, 0.36 m in the proposed SDP and 0.34 m in the ML estimator. The proposed SDP algorithm provides better accuracy performance than that of the LP algorithm. Compared with the ML estimator and the method of the LP, the proposed SDP algorithm does not rely on the initialization and provides more robust solutions for the source location estimates when the noise variance is increased from 0.1^2 to 1^2. Of course, the complexity of SDP algorithm is larger than the ML and LP algorithms due to plenty of variables and equality constraints.

5. **Conclusions.** When the transmission time is unavailable, we address the problem of TOA-based multiple source localization and build the optimization function of the corresponding ML estimator. The numerical solution of ML estimator needs the initial guess to ensure the global convergence. Without a good initial guess, however, local convergence may occur. So the SDP algorithm is proposed for estimating the source locations along with the transmission time. The accuracy performance of the SDP is better than that of LP technique, for not only the anchor-source but also the source-source measurements are employed into the optimization model. The SDP algorithm provides robust solutions for multiple source location estimates. However, the computational complexity of SDP algorithm is high due to a large number of variables and equality constraints.

Acknowledgment. This work is partially supported by Zhejiang Provincial Natural Science Foundation LY16F020036 and ZAFU Scientific Research Development Foundation Project 2013FR086.

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