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A LOWER BOUND ON DATA RATES FOR STATE ESTIMATION IN THE PRESENCE OF MULTIPLE EIGENVALUES

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Abstract. *This paper investigates the state estimation problem for discrete-time linear control systems, where sensors and controllers are connected via a stationary memoryless uncertain digital channel. On the basis of a fixed-length coding scheme, a fixed data rate is employed to transmit information of the plant state in order to ensure observability. In particular, we deal with the case where system matrix has distinct eigenvalues with algebraic multiplicity larger than one, and present necessary and sufficient conditions on the data rate of the channel for observability. It is shown in our results that, there exists a lower bound on the data rate above which the system is observable.*
Keywords: State estimation, Multiple eigenvalues, Fixed data rate, Lower bound, Discrete-time linear systems

1. **Introduction.** In recent years, the field of quantized control has developed rapidly in many applications where sensors and controllers communicate over channels with limited data rate [1-3]. The data rate limitations in these applications have a significant effect on state estimation of networked control systems [4,5].

The research on Gaussian linear systems was addressed in [6]. Information theory was employed in control systems as a powerful conceptual aid, which extended existing fundamental limitations of feedback systems, and was used to derive necessary and sufficient conditions for robust stabilization of uncertain linear systems, Markov jump linear systems and unstructured uncertain systems [7-9]. Control under communication constraints inevitably suffers signal transmission delay, data packet dropout and measurement quantization which might be potential sources of instability and poor performance of control systems [10]. [11] investigated the quantized feedback control problem for stochastic time-invariant linear control systems. A predictive control policy under data-rate constraints was proposed to stabilize the unstable plant in the mean square sense. [12] addressed LQ (linear quadratic) control of MIMO (multi-input multi-output), discrete-time linear systems, and gave the inherent tradeoffs between LQ cost and data rates. In [13], a quantized-observer based encoding-decoding scheme was designed, which integrated the state observation with encoding-decoding. [14] addressed some of the challenging issues on moving horizon state estimation for networked control systems in the presence of multiple packet dropouts.

In this paper, we consider the state estimation problem for discrete-time linear control systems, deal with the case where system matrix has distinct eigenvalues with algebraic multiplicity larger than one, and present necessary and sufficient conditions on the data rate of the channel for observability. In particular, on the basis of a fixed-length coding scheme, a fixed data rate is employed to transmit information of the plant state in order to ensure observability. Here, we stress that, the fixed data rate is not viewed as a special case of the time-varying data rate. The case with a time-varying data rate often employs the variable-length encoding scheme, and ensures control performance in an average or expected sense. However, for the case with a fixed data rate, control performances can be guaranteed at any time for control systems. Our work here differs in that we present a lower bound on the fixed data rate for state estimation in the presence of multiple eigenvalues.

The remainder of this paper is organized as follows: Section 2 introduces problem formulation; Section 3 deals with the state estimation problem for discrete-time linear control systems; Conclusions are stated in Section 4.

2. Problem Formulation.

Consider the discrete-time linear system

$$\begin{array}{c} X(k+1) = AX(k), \\ Y(k) = CX(k) \end{array} \tag{1}$$

where $X(k) \in \mathbb{R}^n$ is the state process, and $Y(k) \in \mathbb{R}^m$ is the measured output. A and C are known constant matrices with appropriate dimensions. Here, it is assumed that the pair (A, C) is observable. Let $B_l(z)$ denote the set $\{x : |x - z| \le l\}$ centered at z. The initial state $X(0)$ is a bounded, uncertain variable satisfying $\|X(0)\| \in B_{\phi_0}(0)$, where ϕ_0 is a known constant.

In the literature, it is often assumed that there exists a nonsingular real matrix H that diagonalizes $A = H'\Lambda H$, where $\Lambda = diag[a_1, a_2, \cdots, a_n]$. In such case, it is convenient to transform the system (1) so as to decouple its dynamical modes. However, it is impossible to find such a matrix H that diagonalizes A in many cases.

Without loss of generality, we put the system matrix A into Jordan canonical form in this paper. As stated in [15], there exists a real similarity matrix $H \in \mathbb{R}^{n\times n}$ such that

$$G = HAH' = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_d \end{bmatrix} \tag{2}$$

holds. Let $a_1, a_2, \cdots, a_d$ denote the distinct eigenvalues of system matrix A, and let the algebraic multiplicity of each a_i be n_i $(i = 1, 2, \cdots, d)$. Clearly, $n_1 + n_2 + \cdots + n_d = n$. Then, the block J_i can be written as

$$J_i = \begin{bmatrix} a_i & 1 & \cdots & 0 & 0 \\ 0 & a_i & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_i & 1 \\ 0 & 0 & \cdots & 0 & a_i \end{bmatrix} \in \mathbb{R}^{n_i\times n_i}, \tag{3}$$

where each a_i is larger than 1 $(i = 1, 2, \cdots, d)$ since the stable part does not play any key role on observability of the system (1).

Furthermore, the transformed state is defined as

$$\bar{X}(k) := HX(k).$$

Then, the system (1) can be rewritten as

$$\begin{array}{c} \bar{X}(k+1) = G\bar{X}(k), \\ Y(k) = CH'\bar{X}(k). \end{array} \tag{4}$$

Here, we partition the transformed state vector $\bar{X}(k)$ into the vectors $\bar{X}(k,1), \bar{X}(k,2), \cdots,$ $\bar{X}(k,d)$ corresponding to each subsystem. Then, the ith subsystem can be written as

$$\bar{X}(k+1,i) = J_i\bar{X}(k,i), \quad i = 1, 2, \cdots, d. \tag{5}$$

Namely, the system (4) is decomposed into d subsystems. We define $\tilde{X}(k) := H\hat{X}(k)$ and $\bar{E}(k) := HE(k)$. Then, we also partition $\tilde{X}(k)$ and $\bar{E}(k)$ into the vectors $\tilde{X}(k,1), \tilde{X}(k,2),$

$\cdots, \tilde{X}(k,d)$ and the vectors $\bar{E}(k,1), \bar{E}(k,2), \cdots, \bar{E}(k,d)$, respectively. For the ith subsystem, we set

$$\bar{X}(k,i) := [\bar{x}_1(k,i) \ \ \bar{x}_2(k,i) \ \ \cdots \ \ \bar{x}_{n_i}(k,i)]',$$
$$\tilde{X}(k,i) := [\tilde{x}_1(k,i) \ \ \tilde{x}_2(k,i) \ \ \cdots \ \ \tilde{x}_{n_i}(k,i)]',$$
$$\bar{E}(k,i) := [\bar{e}_1(k,i) \ \ \bar{e}_2(k,i) \ \ \cdots \ \ \bar{e}_{n_i}(k,i)]'.$$

In this paper, we consider the case where the sensors and controllers are geographically separated and connected by a stationary memoryless uncertain digital channel without data dropout and time delay. At each time step the channel can transmit without error R bits of the information on the plant states that is provided by the sensors. Specifically, we deal with the case where the data rate R provided by such a channel is an invariant constant. Namely, the plant state is quantized, and encoded by the fixed-length coding scheme [16].

Let $\hat{X}(k)$ and $E(k)$ denote the state estimate and the estimation error, respectively. We define the estimation error as

$$E(k) := X(k) - \hat{X}(k).$$

$X(k)$ is causally encoded via an operator Θ as

$$\alpha(k) = \Theta(k, X(0), X(1), \cdots, X(k)), \tag{6}$$

where the codeword $\alpha(k)$ is transmitted over such a channel, and decoded via an operator Υ as

$$\hat{X}(k) = \Upsilon\left(k, \hat{\alpha}(0), \hat{\alpha}(1), \cdots, \hat{\alpha}(k)\right), \tag{7}$$

where $\hat{\alpha}(k)$ denotes the received symbol at the decoder.

The system (1) is observable if there exists an encoder and decoder such that the following holds:

$$\limsup_{k\to\infty} \|E(k)\| < \infty. \tag{8}$$

In this paper, we consider the system (1) with multiple eigenvalues, and argue the state estimation problem under the data rate limitation. The main task here is to present a lower bound on the data rate for observability of the system (1).

3. **A Lower Bound on the Data Rate for Observability.** In this section, we deal with the state estimation problem for discrete-time linear system over a digital communication channel, and present a lower bound on the data rate for observability. Here, we will derive necessary and sufficient conditions on the data rate for observability in the presence of multiple eigenvalues. Namely, we investigate the minimum data rate for observability of the system (1).

The main result of this section is given below.

Theorem 3.1. *Consider the system (1) with multiple eigenvalues in the sense (2). The plant state is coded and estimated by the schemes (6) and (7), respectively. Then, the necessary and sufficient condition on the data rate R for observability is that*

$$R \geq \left\lceil \sum_{i=1}^{d} n_i \log_2 a_i \right\rceil \ \textit{(bits/sample)},$$

where $\lceil\cdot\rceil$ represents the ceil function, and is defined as $\lceil x\rceil := \min\{k \in \mathbb{Z} : k > x\}$.

Proof: Notice that

$$\|\bar{X}(0)\| = \|HX(0)\| \leq \|H\| \cdot \|X(0)\| \in B_{\|H\|\phi_0}(0).$$

Clearly, we have

$$\bar{x}_j(0,i) \in B_{\|H\|\phi_0}(0), \ \ j = 1, 2, \cdots, n_i; \ \ i = 1, 2, \cdots, d. \tag{9}$$

Furthermore, we set

$$\tilde{X}(0) = H\hat{X}(0) = 0.$$

Then, we obtain

$$\bar{e}_j(0,i) = \bar{x}_j(0,i) - \tilde{x}_j(0,i) \in B_{\|H\|\phi_0}(0).$$

For any time k, we assume that

$$\bar{x}_j(k,i) \in B_{l_j(k,i)}(C_j(k,i))$$

holds. Then, we have

$$\tilde{x}_j(k,i) = C_j(k,i),$$
$$\bar{e}_j(k,i) = \bar{x}_j(k,i) - \tilde{x}_j(k,i) \in B_{l_j(k,i)}(0).$$

Here, we divide the range $B_{l_j(k,i)}(C_j(k,i))$ into $m(i,j) \in \mathbb{N}$ equal intervals. It follows from [16] that the corresponding data rate $R(i,j)$ must satisfy the following inequality:

$$R(i,j) \geq \lceil \log_2 m(i,j) \rceil \text{ (bits/sample)}. \tag{10}$$

Furthermore, it follows from (5) that

$$\bar{x}_{n_i}(k+1,i) = a_i \bar{x}_{n_i}(k,i), \quad i = 1, 2, \cdots, d.$$

At time $k+1$, we have

$$\bar{x}_{n_i}(k+1,i) \in B_{l_{n_i}(k+1,i)}(C_{n_i}(k+1,i)),$$
$$\tilde{x}_{n_i}(k+1,i) = C_{n_i}(k+1,i),$$
$$\bar{e}_{n_i}(k+1,i) = \bar{x}_{n_i}(k+1,i) - \tilde{x}_{n_i}(k+1,i) \in B_{l_{n_i}(k+1,i)}(0),$$

where

$$l_{n_i}(k+1,i) = \tfrac{a_i}{m(i,n_i)} l_{n_i}(k,i). \tag{11}$$

Combined with the equalities (9) and (11), this implies that

$$l_{n_i}(k,i) = \left(\tfrac{a_i}{m(i,n_i)}\right)^k \|H\|\phi_0. \tag{12}$$

Then, we have

$$\lim_{k\to\infty} l_{n_i}(k,i) = 0,$$

if the following inequality holds:

$$m(i,n_i) > a_i. \tag{13}$$

Thus, we obtain

$$\limsup_{k\to\infty} |\bar{e}_{n_i}(k,i)| = 0. \tag{14}$$

Similarly, it follows from (5) that

$$\bar{x}_j(k+1,i) = a_i \bar{x}_j(k,i) + \bar{x}_{j+1}(k,i), \quad i = 1, 2, \cdots, d; \quad j = 1, 2, \cdots, n_i - 1.$$

At time $k+1$, we have

$$\bar{x}_j(k+1,i) \in B_{l_j(k+1,i)}(C_j(k+1,i)),$$
$$\tilde{x}_j(k+1,i) = C_j(k+1,i),$$
$$\bar{e}_j(k+1,i) = \bar{x}_j(k+1,i) - \tilde{x}_j(k+1,i) \in B_{l_j(k+1,i)}(0),$$

where

$$l_j(k+1,i) = \tfrac{a_i}{m(i,j)} l_j(k,i) + l_{j+1}(k,i) \qquad i = 1, 2, \cdots, d; \quad j = 1, 2, \cdots, n_i - 1. \tag{15}$$

Combined with the equalities (9) and (15), this implies that

$$l_j(k,i) = \left[1 + \left(\tfrac{a_i}{m(i,j)}\right)^2 + \cdots + \left(\tfrac{a_i}{m(i,j)}\right)^k\right] \|H\|\phi_0. \tag{16}$$

Then, we have

$$\lim_{k\to\infty} l_j(k,i) = \tfrac{1}{1-\frac{a_i}{m(i,j)}} \|H\|\phi_0,$$

if the following inequality holds:

$$m(i,j) > a_i. \tag{17}$$

Thus, we obtain

$$\limsup_{k\to\infty} |\bar{e}_j(k,i)| = \frac{1}{1-\frac{a_i}{m(i,j)}}\|H\|\phi_0, \quad i=1,2,\cdots,d; \quad j=1,2,\cdots,n_i-1. \tag{18}$$

Combined with the equalities (14) and (18), this implies that

$$\lim_{k\to\infty} \|E(k)\| < \frac{1}{1-\frac{a_i}{m(i,j)}}\|H\|\phi_0.$$

Thus, the system (1) is observable if the inequalities (13) and (17) hold. This implies that

$$R \geq \left\lceil \sum_{i=1}^{d} n_i \log_2 a_i \right\rceil \text{ (bits/sample)}. \tag{19}$$

The proof of sufficiency is complete.

Conversely, if the system (1) is observable, we have

$$\lim_{k\to\infty} \|E(k)\| < \infty.$$

It means that

$$\limsup_{k\to\infty} |\bar{e}_j(k,i)| < \infty, \quad i=1,2,\cdots,d; \quad j=1,2,\cdots,n_i.$$

It follows from (12) and (16) that the following inequality must hold:

$$m(i,j) > a_i, \quad i=1,2,\cdots,d; \quad j=1,2,\cdots,n_i. \tag{20}$$

Combined with the equalities (10) and (20), this implies that the inequality (19) must hold. The proof of necessity is complete. □

Clearly, it is shown in Theorem 3.1 that, the system (1) is observable if the fixed data rate is larger than the lower bound given, which is different from ones in the literature. Furthermore, the data rate in our results is fixed, but ones in the literature are time varying. Using the fixed data rate may lead to better control performances.

Furthermore, we also consider the case where there exists a nonsingular real matrix that can diagonalize the system matrix, and give the following result.

Corollary 3.1. *Consider the system (1). The plant state is coded and estimated by the schemes (6) and (7), respectively. Assume that there exists a nonsingular real matrix H that diagonalizes $A = H'GH$, where $G = diag[a_1, a_2, \cdots, a_n]$. Then, the necessary and sufficient condition on the data rate R for observability is that*

$$R \geq \left\lceil \sum_{i=1}^{n} \log_2 a_i \right\rceil \textit{ (bits/sample)}.$$

Proof: By the assumption, we know that $a_1, a_2, \cdots, a_n$ are the distinct eigenvalues of system matrix A. Namely, we have

$$n_1 = n_2 = \cdots = n_n = 1.$$

Thus, it follows from (19) that

$$R \geq \left\lceil \sum_{i=1}^{n} \log_2 a_i \right\rceil \text{ (bits/sample)}.$$

Thus, the proof is complete. □

4. **Conclusions.** In this paper, we addressed the state estimation problem in the presence of multiple eigenvalues, and derived the necessary and sufficient conditions on the data rate for observability. Clearly, the data rate of the channel has important effects on observability of networked control systems. However, communication bandwidth is not large enough to ensure the real-time data transmission between sensors and controllers in many applications. Thus, it is necessary and significant to present a lower bound on the data rate for observability of linear control systems. The study of nonlinear system with limited data rate will be our future work.

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