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A NOVEL GENERALIZED ITEM RELATIONAL STRUCTURE THEORY BASED ON LIU'S NORMALIZATION AND CONSISTENCY CRITERIA

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Abstract. *In this study, based on weighted mean, we can extend any item ordering theory from dichotomous scoring to polytomous scoring. However, before this study, there is no validity criterion to detect any item ordering theory for polytomous scoring whether is valid or not; this paper defines finite correlation coefficient and item difficulty for polytomous scoring corresponding to dichotomous scoring; based on these new definitions, we propose the generalized criteria of completeness, normalization and consistency for polytomous scoring corresponding to the original ones for dichotomous scoring, respectively. Two well-known item ordering theories: Takeya's IRS and Liu et al.'s LIRS, can be extended from dichotomous scoring to polytomous scoring, denoted as GIRS and GLIRS. And then, several important properties of them and counter examples are provided. This paper points out that not only does IRS not satisfy the three above-mentioned original criteria, but also its generalization, GIRS, does not satisfy the generalized criteria of them, and only the new theory, GLIRS, can satisfy both of the generalized and the original criteria of completeness, normalization and strict consistency.*

Keywords: Item relational structure theory, Liu's item relational structure theory, Generalized Liu's item relational structure theory

1. **Introduction.** Liu [1] pointed out that Airasian & Bart's Ordering Theory (OT) [2] does not exclude the independence case, and Takeya's Item Relational Structure (IRS) Theory [3] may have negative ordering coefficient, and the value of the ordering coefficient from easier item to more difficult item will be less than that from more difficult item to easier item, if the correlation of two items is negative. Liu and Ju [4] proposed an Improved Item Relational Structure Theory, IIRS, of which the above conditions will not exist, but IIRS does not exclude the unreasonable condition of the same order coefficient value in varied difficulty items. Because the ordering coefficient value of easier item to

more difficult item must be greater than ordering coefficient value from more difficult item to easier item, it is just the strict consistency criterion.

To cover the drawbacks of the above theories, if the correlation of the items is negative, Liu et al. [5] applied Liu's generalized inverse function rectified the direction of ordering coefficient to proposing their improved theory, Liu's Item Relational Structure Theory (LIRS) [1]. However, all of above-mentioned item relational structure theories can only be used for dichotomous scoring.

In this study, based on weighted mean, all of item ordering theories for dichotomous scoring can be extended to polytomous scoring. For comparing the validity of GIRS and GLIRS, the important definitions of finite correlation coefficient and item difficulty, and the criteria of completeness, normalization and consistency are also extended to polytomous scoring. And then, we show that the generalized IRS still does not satisfy the generalized completeness, normalization and consistency criteria, only generalized LIRS, GLIRS, can satisfy the three generalized criteria.

This paper is organized as follows. Section 2 proposes the new important criteria of any generalized item ordering theory for polytomous scoring. Section 3 introduces how to construct the generalized model of any ordering theory. Section 4 constructs the generalized Liu's item relational structure theory. Section 5 discusses the important properties of the proposed generalized new model of Liu's item relational structure theory. Finally, Section 6 concludes the paper.

2. New Important Criteria of Generalized Item Ordering Theories.

Before now, there is no criterion that can be used to detect any generalized item ordering theories whether is valid or not for polytomous scoring; in this study, we will propose one to correspond dichotomous scoring as follows.

Definition 2.1. *Item relational structure theories for polytomous scoring.*

$I_i = x_s \in D_i = \{x_s | s = 1, 2, \ldots, m_i\}$, $\ni 0 = x_1 < x_2 < \cdots < x_{m_i} = m_i$, $i = 1, 2, \ldots, n$, $P(I_i = x_s)$, $P(I_j = y_t)$, $P(I_i = x_s, I_j = y_t)$, $i, j = 1, 2, \ldots, n$ *are the marginal and joint probability of* $(I_i = x_s)$, $(I_j = y_t)$ *and* $(I_i = x_s, I_j = y_t)$, *respectively.*

The difficulty of item i, d_i, *and the finite correlation coefficient of* I_i *and* I_j, ρ_{ij}^L, *for polytomous scoring, respectively are defined as*

$$d_i = \frac{1}{m_i} E(I_i), \quad i, j = 1, 2, \ldots, n \tag{1}$$

$$\rho_{ij}^L = \frac{1}{m_i \times m_j} [E(I_i \times I_j) - E(I_i)E(I_j)], \quad -1 \leq \rho_{ij}^L \leq 1 \tag{2}$$

where $E(I_i) = \sum_{t=1}^{m_j} x_s P(I_i = x_s)$, $E(I_i \times I_j) = \sum_{s=1}^{m_i} \sum_{t=1}^{m_j} x_s y_t P(I_i = x_s, I_j = y_t)$.

Let $\gamma_{ij}^{(X)}$ *be ordering coefficient of the item relational structure theory* X *from* I_i *to* I_j, *such that*

$$\gamma_{ij}^{(X)} : D_i \times D_j \to (-\infty, 1] \tag{3}$$

If $\gamma_{ij}^{(X)} > C_{ij}$, *then* I_i *is a precondition of* I_j, *denoted* $I_i \to_X I_j$; *otherwise,* $I_i \nrightarrow_X I_j$, *where* C_{ij} *is a constant,* $C_{ij} \in (0, 1)$.

Definition 2.2. *The important criteria of generalized item relational structure theories.*

Let $\gamma_{ij}^{(X)}$ *be ordering coefficient of the ordering theory* X *from* I_i *to* I_j,

(i) $\forall (x_s, y_t) \in D_i \times D_j$, $\exists b \in [0, 1]$ *s.t.* $b = \gamma_{ij}^{(X)}(x_s, y_t)$ *(Completeness criterion)* (4)

(ii) $\gamma_{ij}^{(X)} \in [0, 1]$ *(Normalization criterion)* (5)

(iii) $d_i \leq d_j \Rightarrow \gamma_{ij}^{(X)} \leq \gamma_{ji}^{(X)}$ *(Consistency criterion)* (6)

3. **Constructing Generalized Item Relational Structure Theory by Weighted Mean.** Based on weighted mean method, for any given item relational structure theory of dichotomous scoring, we can construct a corresponded generalized item relational structure theory of polytomous scoring as follows.

Definition 3.1. *Generalized Item Relational Structure theory (GIRS).*

If $T = \{I_1, I_2, \ldots, I_n\}$, $P(I_i = x_s)$, $P(I_j = y_t)$, $P(I_i = x_s, I_j = y_t)$ *are defined as Definition 2.1, then* $\gamma_{ij}^{(GIRS)}$ *is the ordering coefficient of the item relational structure theory GIRS from* I_i *to* I_j, *such that*

$$\gamma_{ij}^{(GIRS)} : D_i \times D_j \to (-\infty, 1], \quad \gamma_{ij}^{(GIRS)} = 1 - \frac{\sum_{s=1}^{m_i} \sum_{t=1}^{m_j} [y_t - x_s]^+ u_{ij}(s,t)}{\sum_{s=1}^{m_i} \sum_{t=1}^{m_j} [y_t - x_s]^+} \tag{7}$$

where

$$[z]^+ = \begin{cases} z & \text{if } z > 0 \\ 0 & \text{if } z \le 0 \end{cases}, \quad u_{ij}(s,t) = \frac{P(I_i = x_s, I_j = y_t)}{P(I_i = x_s)P(I_j = y_t)} \tag{8}$$

If $\gamma_{ij}^{(GIRS)} > 0.5$, *then* I_i *is a precondition of* I_j, *denoted* $I_i \to_{GIRS} I_j$; *otherwise,* $I_i \nrightarrow_{GIRS} I_j$.

Theorem 3.1. *Important properties of GIRS.*

(i) $D_i = D_j = \{0, 1\} \Rightarrow \gamma_{ij}^{(GIRS)} = \gamma_{ij}^{(IRS)}$ (9)

(ii) $\gamma_{ij}^{(GIRS)} \in (-\infty, 1]$ (10)

(iii) $P(I_i = x_s)P(I_j = y_t) = 0 \Rightarrow \dfrac{P(I_i = x_s, I_j = y_t)}{P(I_i = x_s)P(I_j = y_t)}$ *is undefined*

(iv) $d_i \ge d_j \nRightarrow \gamma_{ij}^{(GIRS)} \ge \gamma_{ji}^{(GIRS)}$ (11)

Proof: (i), (iii) and (iv) are trivial.

$$\begin{aligned} &\text{(ii) } \gamma_{ij}^{(IRS)} = 1 - u_{ij}(s,t) \in (-\infty, 1] \\ &\Rightarrow \gamma_{ij}^{(GIRS)} = 1 - \frac{\sum_{s=1}^{m_i} \sum_{t=1}^{m_j} [y_t - x_s]^+ u_{ij}(s,t)}{\sum_{s=1}^{m_i} \sum_{t=1}^{m_j} [y_t - x_s]^+} \in (-\infty, 1] \end{aligned} \tag{12}$$

From (i), if $D_i = D_j = \{0, 1\}$, then $\gamma_{ij}^{(GIRS)}$ is reduced to $\gamma_{ij}^{(IRS)}$, the GIRS theory is reduced to Takeya's IRS theory. Since Liu [1] and Liu and Ju [4] proved that IRS satisfies no criterion as Definition 2.2, we can view $\gamma_{ij}^{(IRS)}$ as a special case and a counter example of $\gamma_{ij}^{(GIRS)}$; therefore, GIRS also satisfies no criterion as Definition 2.2.

Example 3.1. *The joint probability of item* I_1 *and* I_2 *is listed in Table 1.*

$$d_1 = \frac{1}{2}E(I_1) = 0.8 > d_2 = \frac{1}{2}E(I_2) = 0.75, \ \frac{1}{4}E(I_1 I_2) = 0.575 \Rightarrow \rho_{12}^{(L)} = -0.075 < 0$$

$$\begin{aligned} \gamma_{12}^{(GIRS)} &= 1 - \frac{1 \times \frac{0.1}{0.2 \times 0.6} + 2 \times \frac{0.1}{0.1 \times 0.6}}{1+2} = -\frac{7}{18} \\ &< \gamma_{21}^{(GIRS)} = 1 - \frac{1 \times \frac{0.2}{0.7 \times 0.3} + 2 \times \frac{0.1}{0.7 \times 0.1}}{1+2} = -\frac{17}{63} \\ &< 0 \end{aligned}$$

This example shows that GIRS does not satisfy the three criteria of completeness, normalization and consistency.

Table 1. The joint probability of item I_1 and I_2

$P(I_1=x, I_2=y)$	$I_2=2$	$I_2=1$	$I_2=0$	$P(I_1=x)$
$I_1=2$	0.40	0.20	0.10	0.70
$I_1=1$	0.10	0.10	0.00	0.20
$I_1=0$	0.10	0.00	0.00	0.10
$P(I_2=y)$	0.60	0.30	0.10	1

4. Constructing Generalized Liu's Item Relational Structure Theory.

Definition 4.1. *Liu's Item Relational Structure theory, LIRS* [5].

If $T=\{I_1, I_2, \ldots, I_n\}$, $I_i=x$, $I_j=y \in \{0,1\}$ *is a test including n item with dichotomous scoring and* $P(I_i=x)$, $P(I_j=y)$, *and* $P(I_i=x, I_j=y)$ *are marginal and joint probability of* $(I_i=x)$, $(I_j=y)$, $(I_i=x, I_j=y)$ *respectively, then* $\gamma_{ij}^{(LIRS)}$ *is the ordering coefficient of the completed item relational structure theory, LIRS, from* I_i *to* I_j, *such that*

$$\gamma_{ij}^{(LIRS)} = \begin{cases} 0.1 & \text{if } P(I_i=0)P(I_j=1)=0 \\ 0.1+0.9\gamma_{ij}^{(IRS)} & \text{if } \rho_{ij}^{L} \geq 0 \\ 0.1\left[1-\left(1-\gamma_{ij}^{(IRS)}\right)^{-1}\right] & \text{if } \rho_{ij}^{L} < 0 \end{cases} \tag{13}$$

If $\gamma_{ij}^{(LIRS)} > 0.55$, *then* $I_i \rightarrow_{LIRS} I_j$; *otherwise,* $I_i \nrightarrow_{LIRS} I_j$.

Theorem 4.1. *Important properties of LIRS* [5].

(i) $P(I_i=0)P(I_j=1)=0 \Rightarrow \dfrac{P(I_i=0, I_j=1)}{P(I_i=0)P(I_j=1)}=1$ *(Completeness criterion)* (14)

(ii) $\gamma_{ij}^{(LIRS)} \in [0,1]$ *(Normalization criterion)* (15)

(iii) $P(I_i=x) \leq P(I_j=y) \Rightarrow \gamma_{ij}^{(LIRS)} \leq \gamma_{ji}^{(LIRS)}$ *(Consistency criterion)* (16)

Definition 4.2. *Generalized Liu's Item Relational Structure theory, GLIRS.*

If $T=\{I_1, I_2, \ldots, I_n\}$, $P(I_i=x_s)$, $P(I_j=y_t)$, $P(I_i=x_s, I_j=y_t)$ *are defined as Definition 2.1, then* $\gamma_{ij}^{(GLIRS)}$ *is the ordering coefficient of the item relational structure theory GLIRS from* I_i *to* I_j, *such that*

$$\gamma_{ij}^{(GLIRS)} : D_i \times D_j \to [0,1], \quad \gamma_{ij}^{(GLIRS)} = 1 - \frac{\sum_{s=1}^{m_i}\sum_{t=1}^{m_j}[y_t-x_s]^+ f\left(\frac{P(I_i=x_s, I_j=y_t)}{P(I_i=x_s)P(I_j=y_t)}\right)}{\sum_{s=1}^{m_i}\sum_{t=1}^{m_j}[y_t-x_s]^+} \tag{17}$$

where

$$[z]^+ = \begin{cases} 0 & \text{if } z<0 \\ z & \text{if } z \geq 0 \end{cases}, \quad f(u) = \begin{cases} u & \text{if } 0 \leq u \leq 1 \\ 1 & \text{if } u \geq 1 \text{ or } P(I_i=x_s)P(I_j=y_t)=0 \end{cases} \tag{18}$$

If $\gamma_{ij}^{(GLIRS)} > 0.5$, *then* $I_i \rightarrow_{GLIRS} I_j$; *otherwise,* $I_i \nrightarrow_{GLIRS} I_j$.

5. Important Properties of Generalized Liu's Item Relational Structure Theory.

Theorem 5.1. *Important properties of GLIRS.*

(i) $D_i=D_j=\{0,1\} \Rightarrow \gamma_{ij}^{(GLIRS)} = \gamma_{ij}^{(CIRS)}$ (19)

(ii) $P(I_i=x_s)P(I_j=y_t)=0 \Rightarrow \dfrac{P(I_i=x_s, I_j=y_t)}{P(I_i=x_s)P(I_j=y_t)}=1$ *(Completeness criterion)* (20)

(iii) $\gamma_{ij}^{(GLIRS)} \in [0,1]$ *(Normalization criterion)* (21)

(iv) $d_i \geq d_j \Rightarrow \gamma_{ij}^{(GLIRS)} \geq \gamma_{ji}^{(GLIRS)}$ *(Consistency criterion)* (22)

Proof: (i) and (ii) are trivial.

(iii) $$0 \leq f\left(\frac{P(I_i = x_s, I_j = y_t)}{P(I_i = x_s)P(I_j = y_t)}\right) \leq 1$$

$$\Rightarrow \gamma_{ij}^{(GLIRS)} = 1 - \frac{\sum_{s=1}^{m_i}\sum_{t=1}^{m_j}[y_t - x_s]^+ f\left(\frac{P(I_i=x_s,I_j=y_t)}{P(I_i=x_s)P(I_j=y_t)}\right)}{\sum_{s=1}^{m_i}\sum_{t=1}^{m_j}[y_t - x_s]^+} \in [0,1] \tag{23}$$

(iv) $(x_s - y_t) = [x_s - y_t]^+ - [x_s - y_t]^+$ and $0 \leq P(I_i = x_s)P(I_j = y_t) \leq 1$,

$$P(I_i = x_s, I_j = y_t) \leq f\left(\frac{P(I_i = x_s, I_j = y_t)}{P(I_i = x_s)P(I_j = y_t)}\right) \leq 1 \tag{24}$$

If $d_i \geq d_j > 0$, let $b = \sum_{s=1}^{m_i}\sum_{t=1}^{m_j}[x_s - y_t]^+ = \sum_{s=1}^{m_i}\sum_{t=1}^{m_j}[y_t - x_s]^+ > 0$, then we can obtain

$$\begin{aligned} 0 \ &\leq\ d_i - d_j = \sum_{s=1}^{m_i}\sum_{t=1}^{m_j}(x_s - y_t)P(I_i = x_s, I_j = y_t) \\ &=\ \sum_{s=1}^{m_i}\sum_{t=1}^{m_j}([x_s - y_t]^+ - [y_t - x_s]^+)P(I_i = x_s, I_j = y_t) \\ &\leq\ \sum_{s=1}^{m_i}\sum_{t=1}^{m_j}([x_s - y_t]^+ - [y_t - x_s]^+) f\left(\frac{P(I_i=x_s,I_j=y_t)}{P(I_i=x_s)P(I_j=y_t)}\right) \\ &=\ b^{-1}\left(\gamma_{ij}^{(GIRS)} - \gamma_{ji}^{(GIRS)}\right) \end{aligned} \tag{25}$$

Therefore, $d_i \geq d_j \Rightarrow \gamma_{ij}^{(GIRS)} \geq \gamma_{ji}^{(GIRS)}$.

Example 5.1. *The joint probability of item I_1 and I_2 is listed in Table 1.*

From Example 3.1 we know that

$$d_1 = 0.8 > d_2 = 0.75,\ \rho_{12}^{(L)} = -0.075 < 0,\ \gamma_{12}^{(GIRS)} = -\frac{7}{18} < \gamma_{21}^{(GIRS)} = -\frac{17}{63} < 0 \tag{26}$$

However,

$$\begin{aligned} \gamma_{12}^{(GIIRS)} \ &=\ 1 - \frac{1 \times f\left(\frac{5}{6}\right) + 2 \times f\left(\frac{5}{3}\right)}{1+2} = \frac{1}{18} \\ &>\ \gamma_{21}^{(GIIRS)} = 1 - \frac{1 \times f\left(\frac{20}{21}\right) + 2 \times f\left(\frac{10}{7}\right)}{1+2} = \frac{1}{62} \end{aligned} \tag{27}$$

Example 5.2. *The joint probability of item I_3 and I_4 is listed in Table 2.*

Table 2. The joint probability of item I_3 and I_4

$P(I_3 = x, I_4 = y)$	$I_4 = 2$	$I_4 = 1$	$I_4 = 0$	$P(I_3 = x)$
$I_3 = 1$	0.10	0.20	0.30	0.60
$I_3 = 0$	0.20	0.10	0.10	0.40
$P(I_4 = y)$	0.30	0.30	0.40	1

$$d_3 = 0.6 > d_4 = 0.45, \quad \frac{1}{2}E(I_3I_4) = 0.2 \Rightarrow \rho_{3,4}^{(L)} = -0.07 < 0 \tag{28}$$

$$\begin{aligned} \gamma_{34}^{(GIRS)} \ &=\ 1 - \frac{1 \times \frac{0.2}{0.4\times0.3} + \frac{1}{2} \times \frac{0.1}{0.4\times0.3}}{1+\frac{1}{2}} = -\frac{34}{36} \\ &<\ \gamma_{3}^{(GIRS)} = 1 - \frac{1 \times \frac{0.2}{0.6\times0.3} + \frac{1}{2} \times \frac{0.3}{0.6\times0.2}}{1+\frac{1}{2}} = -\frac{31}{54} \end{aligned} \tag{29}$$

$$\gamma_{34}^{(GLIRS)} = 1 - \frac{1 \times f\left(\frac{5}{3}\right) + \frac{1}{2} \times f\left(\frac{5}{6}\right)}{1 + \frac{1}{2}} = \frac{1}{18} > \gamma_{43}^{(GLIRS)} = 1 - \frac{\frac{1}{2} \times f\left(\frac{10}{9}\right) + 1 \times f\left(\frac{5}{2}\right)}{\frac{1}{2} + 1} = 0 \quad (30)$$

These examples show that GIRS does not satisfy completeness, normalization and strict consistency criteria, but GLIRS can satisfy three criteria of completeness, normalization and consistency.

6. **Conclusion.** Based on weighted mean, this study extends two item relational structure theories: Takeya's IRS and Liu et al.'s LIRS, from the dichotomous scoring to the polytomous scoring, denoted GIRS and GLIRS respectively. However, before this study, there is no valid criterion to know which one of the two different generalized item relational structure theories is better. The important definitions of finite correlation coefficient, item difficulty, and three criteria of completeness, normalization and consistency are also extended from dichotomous scoring to poytomous scoring. And then, we find that the new joint probability, $f\left(\frac{P(I_i=x_s,I_j=y_t)}{P(I_i=x_s)P(I_j=y_t)}\right)$, is not less than $P(I_i = x_s, I_j = y_t)$, and we can obtain GLIRS is not less than GIRS and zero, and then, the value of GLIRS is no more negative. Some other important properties and counter examples are provided. This paper points out that GIRS does not satisfy the generalized criteria of completeness, normalization and consistency for polytomous scoring just as IRS does not satisfy three criteria of completeness, normalization and consistency for dichotomous scoring, only GLIRS can satisfy all above-mentioned generalized criteria for polytomous scoring.

In the future, we will develop the integrated model of improved DEMATEL [6,7] and this new method.

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