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A NEW GENERALIZED ALTERNATING DIRECTION METHOD OF MULTIPLIERS FOR SEPARABLE CONVEX PROGRAMMING AND ITS APPLICATIONS TO IMAGE DEBLURRING WITH WAVELETS

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Abstract. *The classical alternating direction method of multipliers (ADMM), which is a special case of the famous proximal point algorithm, is well studied in the literature. In this paper, we propose a new generalized ADMM for separable convex programming, which is a modification of the classical ADMM by introducing an acceleration factor, and includes the latter as a special case. We prove the global convergence of the proposed method under some standard assumptions. Numerical experiments about image deblurring with wavelets are included to illustrate the efficiency of the new method.*

Keywords: Alternating direction method of multipliers, Convex programming, Variational inequalities, Image deblurring with wavelets

1. **Introduction.** Let l, n_1, n_2 be three positive integers, $\theta_i : \mathcal{R}^{n_i} \to \mathcal{R}$ $(i = 1, 2)$ be closed proper convex functions (not necessarily smooth), $A_i \in \mathcal{R}^{l\times n_i}$ $(i = 1, 2)$ be given matrices, $b \in \mathcal{R}^l$ be a given vector, and $\mathcal{X}_i \subseteq \mathcal{R}^{n_i}$ $(i = 1, 2)$ be nonempty closed convex sets. In this paper, we consider the linearly constrained separable convex programming problem of the form

$$\min \left\{\theta_1(x_1) + \theta_2(x_2) | A_1x_1 + A_2x_2 = b, x_1 \in \mathcal{X}_1, x_2 \in \mathcal{X}_2\right\}. \tag{1}$$

Throughout, the solution set of (1) is assumed to be nonempty, and A_1, A_2 are assumed to be full column rank matrices. Problem (1) has numerous applications, such as, statistical learning, sparse signal/image recovery, and transportation problems.

The augmented Lagrangian function of (1) can be written as

$$\mathcal{L}_{\mathcal{A}}(x, y, \lambda) = \theta_1(x_1) + \theta_2(x_2) - \langle\lambda, A_1x_1 + A_2x_2 - b\rangle + \frac{\beta}{2}\|A_1x_1 + A_2x_2 - b\|^2,$$

where λ is the Lagrange multiplier and $\beta > 0$ is a penalty parameter for the violation of the linear constraints. Given (x_2^k, λ^k), the famous alternating direction method of multipliers (ADMM) [1] for solving (1) updates the primal and the dual variables via

$$\begin{cases} x_1^{k+1} = \operatorname{argmin}_{x_1\in\mathcal{X}_1} \left\{\theta_1(x_1) - x_1^\top A_1^\top \lambda^k + \frac{\beta}{2}\|A_1x_1 + A_2x_2^k - b\|^2\right\}, \\ x_2^{k+1} = \operatorname{argmin}_{x_2\in\mathcal{X}_2} \left\{\theta_2(x_2) - x_2^\top A_2^\top \lambda^k + \frac{\beta}{2}\|A_1x_1^{k+1} + A_2x_2 - b\|^2\right\}, \\ \lambda^{k+1} = \lambda^k - \beta\left(A_1x_1^{k+1} + A_2x_2^{k+1} - b\right). \end{cases} \tag{2}$$

Obviously, the classical ADMM solves two subproblems in (2) in the Gauss-Seidel manner at each iteration, and thus the variables x_1 and x_2 can be solved separably in the alternating order. This treatment is meaningful, since in many concrete applications of (1), the subproblems in (2) are often simple enough to have closed-form solutions or can be easily solved up to a high precision. ADMM can be viewed as an application of the Douglas Rachford splitting method (DRSM) [2] to the dual of (1), and DRSM is in turn a special case of the proximal point algorithm [3, 4].

Inspired by the work [6], a generalized ADMM was proposed in [5] and it generates the new iterative $(x_1^{k+1}, x_2^{k+1}, \lambda^{k+1})$ via the following procedure:

$$\left\{\begin{array}{l} x_1^{k+1} = \text{argmin}_{x_1\in\mathcal{X}_1} \left\{\theta_1(x_1) - x_1^\top A_1^\top \lambda^k + \frac{\beta}{2}\|A_1x_1 + A_2x_2^k - b\|^2\right\}, \\ x_2^{k+1} = \text{argmin}_{x_2\in\mathcal{X}_2} \left\{\theta_2(x_2) - x_2^\top A_2^\top \lambda^k + \frac{\beta}{2}\|\alpha A_1x_1^{k+1} \right. \\ \qquad\qquad \left. -(1-\alpha)(A_2x_2^k - b) + A_2x_2 - b\|^2\right\}, \\ \lambda^{k+1} = \lambda^k - \beta\left[\alpha A_1x_1^{k+1} - (1-\alpha)(A_2x_2^k - b) + A_2x_2^{k+1} - b\right], \end{array}\right. \tag{3}$$

where $\alpha \in (0, 2)$ is an acceleration parameter [4].

Notice that the two subproblems related to x_1 in (2) and (3) are the same. That is to say the first subproblem of (3) is irrelevant to the acceleration parameter α. In this paper, we propose a new generalized ADMM, whose two subproblems both incorporate the important parameter α. More specifically, our new iterative scheme is

$$\left\{\begin{array}{l} x_1^{k+1} = \text{argmin}_{x_1\in\mathcal{X}_1} \left\{\theta_1(x_1) - x_1^\top A_1^\top \lambda^k + \frac{\alpha\beta}{2}\|A_1x_1 + A_2x_2^k - b\|^2\right\}, \\ x_2^{k+1} = \text{argmin}_{x_2\in\mathcal{X}_2} \left\{\theta_2(x_2) - x_2^\top A_2^\top \lambda^k + \frac{(2\alpha-1)\beta}{2}\|A_1x_1^{k+1} + A_2x_2 - b\|^2\right\}, \\ \lambda^{k+1} = \lambda^k - \beta\left[\alpha A_1x_1^{k+1} - (1-\alpha)(A_2x_2^k - b) + A_2x_2^{k+1} - b\right]. \end{array}\right. \tag{4}$$

Similar to (3), (4) also reduces to (2) when $\alpha = 1$.

2. **Preliminaries.** In this section, we provide some preliminaries which are useful for further discussions. In particular, we characterize problem (1) by a variational inequality problem.

We define some auxiliary variables which will help us alleviate the notation in the following analysis. First of all, we introduce a new vector $x = (x_1, x_2)$, which is a column vector by stacking vectors x_1, x_2. Similarly, we use the following notations $w = (x, \lambda)$, $v = (x_2, \lambda)$ and $\theta(x) = \theta_1(x_1) + \theta_2(x_2)$. Now, by invoking the first-order optimality condition for convex programming, we reformulate problem (1) as the following variational inequality problem (denoted by VI$(\mathcal{W}, F, \theta)$): finding a vector $w^* \in \mathcal{W}$ such that

$$\theta(x) - \theta(x^*) + (w - w^*)^\top F(w^*) \geq 0, \quad \forall w \in \mathcal{W}, \tag{5}$$

where $\mathcal{W} = \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{R}^l$, and

$$F(w) = \begin{pmatrix} -A_1^\top\lambda \\ -A_2^\top\lambda \\ A_1x_1 + A_2x_2 - b \end{pmatrix}. \tag{6}$$

Obviously, the just defined mapping $F(w)$ is affine with a skew-symmetric matrix; it is thus monotone. The solution set of (5) is denoted by $\mathcal{W}^*$. Then, $\mathcal{W}^*$ is nonempty under assumption of the solution set of problem (1).

Now, let us define some matrices in order to present our analysis in a compact way. Let

$$M = \begin{pmatrix} I_{n_2} & 0 \\ -\beta A_2 & I_l \end{pmatrix} \tag{7}$$

and for $\alpha \in [1, 2)$, set

$$Q = \begin{pmatrix} (2\alpha-1)\beta A_2^\top A_2 & \frac{1-\alpha}{\alpha}A_2^\top \\ -A_2 & \frac{1}{\alpha\beta}I_l \end{pmatrix}. \tag{8}$$

Last, we define a symmstric matrix

$$H = \begin{pmatrix} \frac{2\alpha^2-2\alpha+1}{\alpha}\beta A_2^\top A_2 & \frac{1-\alpha}{\alpha} A_2^\top \\ \frac{1-\alpha}{\alpha} A_2 & \frac{1}{\alpha\beta} I_l \end{pmatrix}. \tag{9}$$

The matrices M, Q, H just defined satisfy the following properties.

Lemma 2.1. *The matrix H defined in (9) is positive definite (if A_2 is a full column rank matrix) for $\alpha \in [1, 2)$.*

Proof: Set $t = 2\alpha^2 - 2\alpha + 1$. Obviously, for all $\alpha \in \mathcal{R}$, $t > 0$. By (9), we have

$$H = \frac{1}{\alpha} \begin{pmatrix} \sqrt{\beta} A_2^\top & 0 \\ 0 & \frac{1}{\sqrt{\beta}} I_l \end{pmatrix} \begin{pmatrix} tI_l & (1-\alpha)I_l \\ (1-\alpha)I_l & I_l \end{pmatrix} \begin{pmatrix} \sqrt{\beta} A_2 & 0 \\ 0 & \frac{1}{\sqrt{\beta}} I_l \end{pmatrix}.$$

Note that the matrix

$$\begin{pmatrix} t & 1-\alpha \\ 1-\alpha & 1 \end{pmatrix}$$

is positive definite when $\alpha \in [1, 2)$, therefore, H is also positive definite. The proof is then complete.

Lemma 2.2. *The matrices M, Q, H defined, respectively, in (7), (8), (9) have the following relationships:*

$$HM = Q \tag{10}$$

and

$$Q^\top + Q - M^\top HM \succeq \frac{\alpha - 1}{2\alpha} M^\top HM. \tag{11}$$

Proof: Elementary.

3. **Algorithm and Global Convergence.** In this section, we first describe our new generalized alternating direction method of multipliers (NGADMM) for VI$(\mathcal{W}, F, \theta)$ formally, followed by some remarks on its properties. Then, we establish its global convergence in a contraction perspective.

Algorithm 3.1. NGADMM

Step 0. Choose the parameters $\alpha \in [1, 2)$, $\beta > 0$, the tolerance $\varepsilon > 0$ and the initial iterate $(x_2^0, \lambda^0) \in \mathcal{X}_2 \times \mathcal{R}^l$. Set $k := 0$.

Step 1. Solve (4), and get the new iteration $w^{k+1} = \left(x_1^{k+1}, x_2^{k+1}, \lambda^{k+1}\right)$.

Step 2. Set $\hat{\lambda}^k = \lambda^k - \alpha\beta\left(A_1 x_1^{k+1} + A_2 x_2^k - b\right)$. If

$$\max\left\{\left\|A_2 x_2^k - A_2 x_2^{k+1}\right\|, \left\|\lambda^k - \hat{\lambda}^k\right\|\right\} < \epsilon, \tag{12}$$

then stop and return an approximate solution $\left(x_1^{k+1}, x_2^{k+1}, \hat{\lambda}^k\right)$ of VI$(\mathcal{W}, F, \theta)$; else set $k := k + 1$, and go to Step 1.

Remark 3.1. *Obviously, if both parameters $\alpha = 1$, then the classical ADMM (2) is recovered. Therefore, in the following, we only consider the case $\alpha \in (1, 2)$.*

Remark 3.2. *For NGADMM, x_1 is the intermediate variable, and $v = (x_2, \lambda)$ are the essential variables. For this reason, we further define the notation $v^k = \left(x_2^k, \lambda^k\right)$, $\mathcal{V} = \mathcal{X}_2 \times \mathcal{R}^l$,*

$$\mathcal{V}^* := \{v^* = (x_2^*, \lambda^*) | w^* = (x_1^*, x_2^*, \lambda^*) \in \mathcal{W}^*\}.$$

For further analysis, we also need to define an auxiliary sequence $\{\hat{w}^k\}$ as

$$\hat{w}^k = \begin{pmatrix} \hat{x}_1^k \\ \hat{x}_2^k \\ \hat{\lambda} \end{pmatrix} = \begin{pmatrix} x_1^{k+1} \\ x_2^{k+1} \\ \lambda^k - \alpha\beta(A_1 x_1^{k+1} + A_2 x_2^k - b) \end{pmatrix}. \tag{13}$$

Lemma 3.1. *For the two sequences* $\{\lambda^{k+1}\}$ *and* $\left\{\hat{\lambda}^k\right\}$ *generated by NGADMM, we have*

$$\lambda^{k+1} = \hat{\lambda}^k - \beta\left(A_2\hat{x}_2^k - A_2x_2^k\right), \tag{14}$$

and

$$\hat{\lambda}^k - \left(\frac{1}{\alpha} - 1\right)\left(\hat{\lambda}^k - \lambda^k\right) = \lambda^k - (2\alpha-1)\beta\left(A_1\hat{x}_1^k + A_2x_2^k - b\right). \tag{15}$$

Proof: Elementary.

Then, by (16) and (17), we have the relationship between $\{v^k\}$ and $\{\hat{v}^k\}$:

$$\begin{pmatrix} x_2^{k+1} \\ \lambda^{k+1} \end{pmatrix} = \begin{pmatrix} x_2^k \\ \lambda^k \end{pmatrix} - \begin{pmatrix} I_{n_2} & 0 \\ -\beta A_2 & I_l \end{pmatrix}\begin{pmatrix} x_2^k - \hat{x}_2^k \\ \lambda^k - \hat{\lambda}^k \end{pmatrix},$$

which can be rewritten in a compact form by using the notation of v^k and $\hat{v}^k$:

$$v^{k+1} = v^k - M\left(v^k - \hat{v}^k\right), \tag{16}$$

where M is as defined in (7).

Now, we are going to prove the global convergence of the NGADMM in the analytic framework of contraction type methods, and begin our proof by showing the stopping criterion (7) is reasonable.

Lemma 3.2. *If* $A_2x_2^k = A_2\hat{x}_2^k$ *and* $\lambda^k = \hat{\lambda}^k$, *then* $\hat{w}^k = \left(\hat{x}_1^k, \hat{x}_2^k, \hat{\lambda}^k\right)$ *produced by NGADMM is a solution of VI*$(\mathcal{W}, F, \theta)$.

Proof: The optimality condition of the three subproblems in (4) can be characterized by the following variational inequality problems: for any $w = (x_1, x_2, \lambda) \in \mathcal{W}$,

$$\begin{cases} \theta_1(x_1) - \theta_1\left(\hat{x}_1^k\right) - \left(x_1 - \hat{x}_1^k\right)^\top\left\{A_1^\top\left[\lambda^k - \alpha\beta(A_1\hat{x}_1^k + A_2x_2^k - b)\right]\right\} \geq 0, \\ \theta_2(x_2) - \theta_2\left(\hat{x}_2^k\right) - \left(x_2 - \hat{x}_2^k\right)^\top\left\{A_2^\top\left[\lambda^k - (2\alpha-1)\beta\left(A_1\hat{x}_1^k + A_2\hat{x}_2^k - b\right)\right]\right\} \geq 0, \\ \left(\lambda - \hat{\lambda}^k\right)^\top\left[\alpha A_1\hat{x}_1^k - (1-\alpha)\left(A_2x_2^k - b\right) + A_2\hat{x}_2^k - b - \left(\lambda^k - \lambda^{k+1}\right)/\beta\right] \geq 0. \end{cases}$$

Adding the above three inequalities and by (14), (15), we have

$$\begin{aligned}&\theta(x) - \theta\left(\hat{x}^k\right) \\ &+ \left(w - \hat{w}^k\right)^\top\left\{F(\hat{w}^k) + \begin{pmatrix} 0 \\ (2\alpha-1)\beta A_2^\top\left(A_2\tilde{x}_2^k - A_2x_2^k\right) + (1-\alpha)A_2^\top\left(\hat{\lambda}^k - \lambda^k\right)\big/\alpha \\ -\left(A_2\hat{x}_2^k - A_2x_2^k\right) + \left(\hat{\lambda}^k - \lambda^k\right)\big/(\alpha\beta) \end{pmatrix}\right\} \geq 0. \end{aligned} \tag{17}$$

In addition, by (8) (the definition of Q), the inequality (17) can be written as

$$\theta(x) - \theta\left(\hat{x}^k\right) + \left(w - \hat{w}^k\right)^\top F\left(\hat{w}^k\right) \geq \left(v - \hat{v}^k\right)^\top Q\left(v^k - \hat{v}^k\right), \tag{18}$$

for any $w \in \mathcal{W}$. Therefore, if $A_2x_2^k = A_2\hat{x}_2^k$ and $\lambda^k = \hat{\lambda}^k$, we have

$$Q(v^k - \hat{v}^k) = 0.$$

Then, (18) reduces to

$$\theta(x) - \theta\left(\hat{x}^k\right) + \left(w - \hat{w}^k\right)^\top F\left(\hat{w}^k\right) \geq 0, \ \forall w \in \mathcal{W},$$

which implies that $\hat{w}^k$ is a solution of VI$(\mathcal{W}, F, \theta)$. This completes the proof.

According to (18), the accuracy of $\hat{w}^k$ to a solution of VI$(\mathcal{W}, F, \theta)$ is measured by the quantity $\left(v - \hat{v}^k\right)^\top Q\left(v^k - \hat{v}^k\right)$.

Theorem 3.1. *Let* $\{w^k\}$ *be the sequence generated by NGADMM. Then, if* $\alpha > 1$, *the corresponding sequence* $\{v^k\}$ *converges to a some* v^∞, *which belongs to* $\mathcal{V}^*$.

Proof: Elementary.

4. **Numerical Experiments.** In this section, we conduct some numerical experiments about image deblurring with wavelets to verify the efficiency of the proposed NGADMM. All the codes were written by Matlab 7.0 and performed on a ThinkPad computer equipped with Windows XP, 997 MHz and 1.60 GB of memory.

Given an original image P and concatenate it into an n-vector $\bar{p} \in \mathcal{R}^n$. Let $D \in \mathcal{R}^{n\times n}$ be the matrix representation of a distortion operator acting on the image, such as a blurring, vignetting, inpainting or zooming operator. Let $\omega \in \mathcal{R}^n$ be the noise added onto the image. The observed image $p \in \mathcal{R}^n$ can be modeled by $p = D\bar{p} + \omega$. Let $x \in \mathcal{R}^d$ be the vector of wavelet coefficients of the original image $\bar{p}$ under a wavelet dictionary. Let $W \in \mathcal{R}^{n\times d}$ be the matrix of a wavelet dictionary. Then, we have $\bar{p} = Wx$. The mathematical model of image deblurring with wavelets is

$$\min_{x\in\mathcal{R}^d} \frac{1}{2}\|DWx - p\|^2 + \gamma\|x\|_1, \tag{19}$$

where $\|x\|_1 := \sum_{i=1}^d |x_i|$; $\gamma > 0$ is the regularization parameter.

Now, we show that the model (19) can be reformulated as a special case of (1). In fact, by setting $x_1 = x$, $x_2 = x$, we can reformulate (19) as

$$\begin{aligned} &\min \frac{1}{2}\|DWx - p\|^2 + \gamma\|y\|_1 \\ &\text{s.t. } x - y = 0, \quad x \in \mathcal{R}^d, \quad y \in \mathcal{R}^d, \end{aligned}$$

which is a special case of the abstract model (1) with the following specifications:

$$\begin{aligned} &x_1 = x, \quad x_2 = y, \quad \theta_1(x_1) = \frac{1}{2}\|DWx - p\|^2, \\ &\theta_2(x_2) = \gamma\|y\|_1, \quad A_1 = I_d, \quad A_2 = -I_d, \quad b = 0. \end{aligned}$$

Below, we elaborate on how to derive the closed-form solutions for the subproblems resulted by the NGADMM.

• With the given x_2^k and λ^k, the x_1-subproblem in (4) is

$$x_1^{k+1} = \operatorname{argmin}_{x_1\in\mathcal{R}^d} \left\{ \frac{1}{2}\|DWx_1 - p\|^2 + \frac{\alpha\beta}{2}\left\| x_1 - x_2^k - \frac{1}{\alpha\beta}\lambda^k \right\|^2 \right\},$$

which has the following closed-form solution:

$$x_1^{k+1} = \left((DW)^\top(DW) + \alpha\beta I_d\right)^{-1}\left((DW)^\top p + \alpha\beta x_2^k + \lambda^k\right).$$

Obviously, the closed-form solution of x_1 needs to compute $\left((DW)^\top(DW) + \alpha\beta I_d\right)^{-1}$, and under the periodic boundary conditions for x_1, the coefficient in the above equation can be diagonalized easily by fast Fourier transforms (FFT). Consequently, the solution of the above equation can be accomplished by two FFTs (including one inverse FFT).

• With the updated x_1^{k+1}, the x_2-subproblem in (4) is

$$x_2^{k+1} = \operatorname{argmin}_{x_2\in\mathcal{R}^d} \left\{ \gamma\|x_2\|_1 + \frac{(2\alpha-1)\beta}{2}\left\| x_2 - x_1^{k+1} + \frac{\lambda^k}{(2\alpha-1)\beta} \right\|^2 \right\},$$

and its closed-form solution is given by

$$x_2^{k+1} = \operatorname{shrink}_{\frac{\gamma}{(2\alpha-1)\beta}}\left(x_1^{k+1} - \frac{\lambda^k}{(2\alpha-1)\beta} \right),$$

where, for any $c > 0$, $\operatorname{shrink}_c(\cdot)$ is defined as

$$\operatorname{shrink}_c(g) := g - \min\{c, |g|\}\frac{g}{|g|}, \quad \forall g \in \mathcal{R}^n,$$

and $(g/|g|)_i$ should be taken as 0 if $|g|_i = 0$.

• Then, with the newly generated x_1^{k+1} and x_2^{k+1}, the Lagrange multiplier λ is updated via

$$\lambda^{k+1} = \lambda^k - \beta \left[\alpha x_1^{k+1} + (1-\alpha)x_2^k - x_2^{k+1}\right].$$

In the experiment, we consider the deblurring problems with 9×9 uniform blur kernel, and W is a redundant Harr wavelet frame with four levels. The test image is the well-known Cameraman image which with size 256-by-256 pixels. Accordingly, $d = 256^2$ in model (19). The Gaussian noise is generated from $\mathtt{N}(0, 0.5550)$. In the following, we set $\gamma = 0.0075$, $\alpha = 1.2$, $\beta = 0.0075$. The stop criterion for the method is

$$\frac{\left|\text{objective}^{k+1} - \text{objective}^k\right|}{\text{objective}^k} \leq 10^{-3},$$

where objective^k represents the objective function value at the kth iterate for the model (19). To assess the restoration performance qualitatively, we use the error of the reconstructed image is measured by the mean squared error (MSE) and the improved signal-to-noise ratio (ISNR) which are defined as follows:

$$\text{MSE} = \frac{\|\bar{p} - \hat{p}\|^2}{d}, \quad \text{ISNR} = 10\log_{10}\frac{\|p - \bar{p}\|^2}{\|\hat{p} - \bar{p}\|^2},$$

where $\bar{p}$ and $\hat{p}$ are the original image and the reconstructed image, respectively. The initial points of all the variables are set as zero vectors with corresponding dimension.

In the experiment, the number of calls to the operators D, $D^\top$ is 68, the number of iterations is 33, computation times in seconds is 10.3, the MSE is 92.6 and the ISNR is 7.69 dB. Figures 1 and 2 indicate that NGADMM can recover the blurred image efficiently.

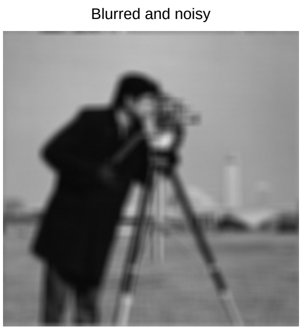

FIGURE 1. Blurred and noisy image

FIGURE 2. Estimated image

5. **Conclusions.** In this paper, we have proposed a new generalized ADMM for separable convex programming, and proved its global convergence under mild conditions. Furthermore, its numerical efficiency is verified by the image deblurring. In the future, we shall further study the generalized ADMM, and apply it to other domains, such as compressive sensing, signal recovery, and machine learning.

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