ICIC Express Letters ICIC International ©2016 ISSN 1881-803X
Volume **10**, Number **3**, March **2016** pp. **621–628**

FUZZY SHAPLEY VALUE BASED ON LINEAR SOLVABLE FORMULATION

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Received August 2015; accepted October 2015

Abstract. *Consensus-building is an important matter in real world and social science. For consensus-decision making, Shapley value can assign total profit or risk if their data of characteristic function can be observed accurately. However, we often face ambiguous case such that we can grasp data only without accuracy in real world. In this paper we proposed fuzzy Shapley value derived by solving linear programming and discussed its possibility of equivalent and necessity of inclusion.*
Keywords: Decision making, Shapley value, Fuzziness, Possibility of equivalent, Necessity of inclusion, Linear programming

1. **Introduction.** For cooperative decision making, it is important to search for equilibrium solution that can explain the final imputation. Decision makers in an effective consensus process should strive to reach the best possible decision for the group and of all its members, rather than competing for personal preferences, i.e., agreement seeking. In transferable utility game, assignment of total profit or risk that occurs after cooperation is a difficult problem. It is known that the Shapley value gives one rational solution for the problem [1,2].

This paper deals with strategic decision making problems. We first introduce a previous related study that is a linear solvable model of Shapley value for decision-making game [3]. As a method for checking rational weights, the model has advantages such that computation becomes easy by using solver even if target case is a large scale problem. The innovation of this research is that we show a natural extension to consider ambiguities by deriving fuzzy Shapley value.

For the main goal of this research, the paper is organized as follows. Section 2 briefly describes the outline of Shapley value based on the transferable utility game. Then we describe the Shapley value obtained by a linear programming problem based on least-squares error minimization. This leads to us expansion considering fuzziness to Shapley value in Section 3. Also the possibility of equivalent for fuzzy Shapley value and the necessity of inclusion of fuzzy Shapley value will be discussed. The practical application to prove the main result is shown in Section 4. Finally, the paper concludes in Section 5.

2. **Least Square Value in Transferable Utility Game.**

2.1. **Shapley value.** An n-person game in characteristic function form or transferable utility game is a pair (K, v) where $K = \{1, 2, 3, \cdots, k\}$ where $k = |K|$ means the number of members of set K and v is a function such that $v(\emptyset) = 0$. In case of $S \cap T = \emptyset$, then the condition like $v(S \cup T) \geq v(S) + v(T)$ is called superadditivity.

In transferable utility game (K, v), the evaluation of benefit for player d can be represented by $z_d(K, v)$ and the set $\boldsymbol{z}(K, v)$ is called a payoff vector. The condition of linearity is as follows

$$\boldsymbol{z}\left(K, v + v'\right) = \boldsymbol{z}(K, v) + \boldsymbol{z}(K, v') \tag{1}$$

In case of the game satisfying

$$v(S \cup d) = v\left(S \cup d'\right), \quad (\forall S \in K - \{d,\ d'\}) \tag{2}$$

and $\boldsymbol{z}(K, v)$ keeps the following equation

$$z_d(K, v) = z_{d'}(K, v), \quad (d, d' = 1, 2, 3, \cdots, k \text{ and } d \neq d') \tag{3}$$

then it is assumed to have the equal treatment property.

An imputation of the game (K, v) is a vector $\boldsymbol{z}(K, v) = (z_1(K, v), z_2(K, v), z_3(K, v), \cdots, z_k(K, v))$ which satisfies the following individual and grand coalition rationalities:

- Individual rationality: $z_d(K, v) \geq v(d)$, $(d = 1, 2, 3, \cdots, k)$.
- Grand coalition rationality: $\sum_{d \in K} z_d(K, v) = v(K)$.

Such coalition $z_d(K, v)$ can be obtained by the Shapley value of game (K, v).

$$\phi_d(K, v) = \sum_{S \backslash d \in S \subset K} \frac{(s-1)!(k-s)!}{k!} \{v(S) - v(S \backslash d)\} \tag{4}$$

where $s = |S|$ is the number of members of coalition set S. $\phi_d(K, v)$ is the mathematical expectation of the marginal contribution of player d when all orders of formation of the grand coalition are equi-probable [1].

2.2. **Shapley value by least-squares error minimization.** Now we consider the set of weight $M_{r,s}$, $(r = 2, 3, 4, \cdots, k;\ s = 1, 2, 3, \cdots, r-1)$ which takes non negative value and has at least one positive value for each k, and consider the following problem,

$$\begin{aligned} &\text{Minimize} \quad \sum_{S \subset K,\, S \neq K} M_{r,s} \left(v(S) - \sum_{d \in S} z_d(K, v) \right)^2 \\ &\text{Subject to} \quad \sum_{d=1}^{k} z_d(K, v) = v(K) \end{aligned} \tag{5}$$

The solution can be given by Ruiz et al. [4] as follows,

$$z_d^*(K, v) = \frac{1}{k} \left\{ v(K) + \sum_{r \in K} (c_{dr} - c_{rd'}) \right\} \tag{6}$$

where $c_{dd'}$ is derived by Namekata [5] and

$$c_{dd'} = \sum_{S \in \Gamma(d^+,\, d'^-)} M_{r,s} v(S) \tag{7}$$

and $\Gamma(d^+, d'^-) = \{S \subset K |\ d \in S,\ d' \notin S\}$. In order that the set of weight satisfies

$$\sum_{S \in \Gamma(d^+,\, d'^-)} M_{r,s} = \sum_{s=1}^{r-1} M_{r,s} \sum_{S \in \Gamma(d^+,\, d'^-),\, s = |S|} 1 = \sum_{s=1}^{r-1} M_{r,s} \{{}_{r-2}C_{s-1}\} = 1 \tag{8}$$

We can determine the following weight as one of them

$$M_{r,s} = \frac{1}{r-1} \left\{ {}_{r-2}C_{s-1} \right\}^{-1} \tag{9}$$

In this case the solution of the objective function can be shown that it is equivalent to Shapley value

$$z_d^*(K, v) = \phi_d(K, v) \tag{10}$$

2.3. **Shapley value defined by linear problem.** We propose the linear solvable formation to obtain allocation based on the least square value in transferable utility game. The solution of weighted least squared error, if it satisfies the constraint condition, can be given by the following inner product form,

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{A}\boldsymbol{z} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{v} \tag{11}$$

where, for example of case $K = \{\varphi_1, \varphi_2, \varphi_3, \cdots, \varphi_k\}$ and $q = \sum_{j=1}^{k-1} {}_kC_j$ then

$$\boldsymbol{z} = \begin{bmatrix} z_{\varphi_1}(K,v) \\ z_{\varphi_2}(K,v) \\ z_{\varphi_3}(K,v) \\ \vdots \\ z_{\varphi_k}(K,v) \end{bmatrix} \in \Re^k \times \Re^1, \quad \boldsymbol{v} = \begin{bmatrix} v(\varphi_1) \\ v(\varphi_2) \\ \vdots \\ v(\varphi_k) \\ v(\varphi_1, \varphi_2) \\ v(\varphi_1, \varphi_3) \\ \vdots \\ v(\varphi_1, \varphi_k) \\ \vdots \\ v(\varphi_{k-1}, \varphi_k) \\ \vdots \\ v(\varphi_1, \varphi_2, \cdots, \varphi_{k-1}) \\ \vdots \\ v(\varphi_2, \varphi_3, \cdots, \varphi_k) \end{bmatrix} \in \Re^q \times \Re^1 \tag{12}$$

and

$$\boldsymbol{M} = \begin{bmatrix} M_{k,1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & M_{k,1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{k,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M_{k,2} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & M_{k,k-1} \end{bmatrix} \in \Re^q \times \Re^q, \tag{13}$$

$$\boldsymbol{A}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \ddots & 0 & \vdots & \cdots & \ddots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & \cdots & 1 \end{bmatrix} \in \Re^k \times \Re^q \tag{14}$$

Taking a sample set d from the data, $\boldsymbol{v}$ and $\boldsymbol{z}$ as the value of referred sample the error obtained is the expression in

$$e_d = f_d(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{v}) = \left(\boldsymbol{B}^{\mathrm{T}}\boldsymbol{v} - \boldsymbol{C}^{\mathrm{T}}\boldsymbol{z}\right)_d \tag{15}$$

where $\boldsymbol{B}^{\mathrm{T}} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{M} \in \Re^k \times \Re^q$, $\boldsymbol{C}^{\mathrm{T}} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{B} \in \Re^k \times \Re^k$ and $(\ )_d$ denotes selection of the d-th row value, thus the sum of all error functions $\epsilon = \sum_d |e_d|$ using the multiple linear regression model that minimize the sum of the absolute values of the residuals.

Linear solvable formulation to obtain Shapley value [3] is defined by

$$
\begin{aligned}
&\text{Minimize} && \epsilon \\
&\text{Subject to} && \boldsymbol{c}_d^{\mathrm{T}}\boldsymbol{z} + s_d^+ - s_d^- = \boldsymbol{b}_d^{\mathrm{T}}\boldsymbol{v}, \quad (d = 1, 2, 3, \cdots, k) \\
& && \sum_{d=1}^{k} z_d(K, v) = v(K) \\
& && z_d \geq 0,\ s_d^+ \leq \epsilon,\ s_d^- \leq \epsilon, \quad (d = 1, 2, 3, \cdots, k)
\end{aligned}
\tag{16}
$$

3. Derivation of Fuzzy Shapley Value.

3.1. Definition of fuzzy Shapley value.

We introduce fuzzy Shapley value $Z_d = (z_d, \zeta_d)_L$ where its center is z_d and width is ζ_d and $(\ )_L$ denotes the triangle type, i.e., L-L type, fuzzy membership function $\mu_{Z_d}(y_d)$ represented by $L(x)$ which satisfies

$$
\begin{aligned}
&L(x) = L(-x), \\
&L(0) = 1, \\
&L(x) \text{ is non-increasing function in } x \in [0, \infty).
\end{aligned}
$$

For example, in case of $L(x) = \max(0, 1 - |x|)$ then the fuzzy membership function is defined by $\mu_{Z_d}(y_d) = L\left(\frac{z_d - y_d}{\zeta_d}\right)$.

Moreover, the characteristic function is also assumed to be generated by fuzzy value $V(S) = (v(S), \pi_S)_L$.

Then we can define fuzzy vector $\boldsymbol{Y} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{v}$ and $\boldsymbol{X} = \boldsymbol{C}^{\mathrm{T}}\boldsymbol{z}$ whose elements are given by the fuzzy variables $\boldsymbol{Y} = [Y_1, Y_2, Y_3, \cdots, Y_k]$ and

$$
Y_d = \left(\boldsymbol{b}_d^{\mathrm{T}}\boldsymbol{v}, |\boldsymbol{b}_d^{\mathrm{T}}|\boldsymbol{\pi}\right)_L = \left(\sum_{S \subset K, S \neq K} b_{dS} v(S), \sum_{S \subset K, S \neq K} |b_{dS}| \pi_S\right)_L, \quad (d = 1, 2, 3, \cdots, k) \tag{17}
$$

where $\boldsymbol{b}_d^{\mathrm{T}} \in \Re^1 \times \Re^q$ is a d-th row vector of matrix $\boldsymbol{B}^{\mathrm{T}}$ and $\left|\boldsymbol{b}_d^{\mathrm{T}}\right| = [|b_{d,1}|, |b_{d,2}|, |b_{d,3}|, \cdots, |b_{d,q}|]$. The vector $\boldsymbol{\pi} \in \Re^q \times \Re^1$ is the width representing ambiguity of fuzzy variables for $V(S)$.

Also $\boldsymbol{X} = [X_1, X_2, X_3, \cdots, X_k]$ is

$$
X_d = \left(\boldsymbol{c}_d^{\mathrm{T}}\boldsymbol{z}, |\boldsymbol{c}_d^{\mathrm{T}}|\boldsymbol{\zeta}\right)_L = \left(\sum_{S \subset K, S \neq K} c_{dS} z_d, \sum_{S \subset K, S \neq K} |c_{dS}| \zeta_d\right)_L, \quad (d = 1, 2, 3, \cdots, k) \tag{18}
$$

where $\boldsymbol{c}_d^{\mathrm{T}} \in \Re^1 \times \Re^d$ is a d-th row vector of matrix $\boldsymbol{C}^{\mathrm{T}}$ and $\left|\boldsymbol{c}_d^{\mathrm{T}}\right| = [|c_{d,1}|, |c_{d,2}|, |c_{d,3}|, \cdots, |b_{d,k}|]$. The vector $\boldsymbol{\zeta} \in \Re^k \times \Re^1$ is the width representing ambiguity of fuzzy variables for z_d.

3.2. Possibility of equivalent for fuzzy Shapley value.

Here we consider the following possibility of equivalent $Pos(Y_d = X_d)$ for fuzzy variables, Y_d and X_d, as follows,

$$
Pos(Y_d = X_d) = \sup_{\theta \in \Re} \min\left(\mu_{Y_d}(\theta), \mu_{X_d}(\theta)\right), \tag{19}
$$

and the condition for α level set

$$
Pos\left(Y_d = X_d\right) \geq \alpha. \tag{20}
$$

Such definition implies the sets of vectors $\boldsymbol{z}$ and $\boldsymbol{\zeta}$ consisting of the fuzzy vector $\boldsymbol{X}$ under the condition as the possibility of equivalent between Y_d and X_d is larger than α.

Now that we can derive an LP problem to obtain fuzzy Shapley value by

$$\begin{aligned}
\text{Minimize} \quad & \sum_{d=1}^{k} \left|\boldsymbol{c}_d^{\mathrm{T}}\right| \boldsymbol{\zeta} + \beta\epsilon \qquad (21)\\
\text{Subject to} \quad & \boldsymbol{c}_d^{\mathrm{T}} \boldsymbol{z} - L^{-1}(\alpha) \left|\boldsymbol{c}_d^{\mathrm{T}}\right| \boldsymbol{\zeta} + s_d^+ - s_d^- \le \boldsymbol{b}_d^{\mathrm{T}} \boldsymbol{v} + L^{-1}(\alpha) \left|\boldsymbol{b}_d^{\mathrm{T}}\right| \boldsymbol{\pi}, \\
& \qquad (d = 1, 2, 3, \cdots, k) \\
& \boldsymbol{c}_d^{\mathrm{T}} \boldsymbol{z} - L^{-1}(\alpha) \left|\boldsymbol{c}_d^{\mathrm{T}}\right| \boldsymbol{\zeta} + s_d^+ - s_d^- \ge \boldsymbol{b}_d^{\mathrm{T}} \boldsymbol{v} - L^{-1}(\alpha) \left|\boldsymbol{b}_d^{\mathrm{T}}\right| \boldsymbol{\pi}, \\
& \qquad (d = 1, 2, 3, \cdots, k) \\
& \sum_{d=1}^{k} z_d(K, v) = v(K) \\
& z_d \ge 0,\ \zeta_d \ge 0,\ s_d^+ \le \epsilon,\ s_d^- \le \epsilon \qquad (d = 1, 2, 3, \cdots, k)
\end{aligned}$$

In above LP model, the coefficient β is assumed to be given by an enough large value and ϵ works significantly in case of $\alpha = 1$.

3.3. **Necessity of inclusion for fuzzy Shapley value.** Next we consider the following necessity of inclusion $Nec(Y_d \supset X_d)$ for fuzzy variables, Y_d and X_d,

$$Nec(Y_d \supset X_d) = \inf_{\theta \in \Re} \max\left(\mu_{Y_d}(\theta), 1 - \mu_{X_d}(\theta)\right), \qquad (22)$$

and the condition for α level set

$$Nec\left(Y_d \supset X_d\right) \ge \alpha. \qquad (23)$$

Such definition implies the sets of vectors $\boldsymbol{z}$ and $\boldsymbol{\zeta}$ consisting of the fuzzy vector $\boldsymbol{X}$ under the condition as the degree of necessity that X_d is included by Y_d is larger than α.

In that case, we can derive an LP problem to obtain fuzzy Shapley value by

$$\begin{aligned}
\text{Minimize} \quad & -\sum_{d=1}^{k} \left|\boldsymbol{c}_d^{\mathrm{T}}\right| \boldsymbol{\zeta} + \beta\epsilon \qquad (24)\\
\text{Subject to} \quad & \boldsymbol{c}_d^{\mathrm{T}} \boldsymbol{z} + L^{-1}(1-\alpha) \left|\boldsymbol{c}_d^{\mathrm{T}}\right| \boldsymbol{\zeta} + s_d^+ - s_d^- \le \boldsymbol{b}_d^{\mathrm{T}} \boldsymbol{v} + L^{-1}(\alpha) \left|\boldsymbol{b}_d^{\mathrm{T}}\right| \boldsymbol{\pi}, \\
& \qquad (d = 1, 2, 3, \cdots, k) \\
& \boldsymbol{c}_d^{\mathrm{T}} \boldsymbol{z} - L^{-1}(1-\alpha) \left|\boldsymbol{c}_d^{\mathrm{T}}\right| \boldsymbol{\zeta} + s_d^+ - s_d^- \ge \boldsymbol{b}_d^{\mathrm{T}} \boldsymbol{v} - L^{-1}(\alpha) \left|\boldsymbol{b}_d^{\mathrm{T}}\right| \boldsymbol{\pi}, \\
& \qquad (d = 1, 2, 3, \cdots, k) \\
& \sum_{d=1}^{k} z_d(K, v) = v(K) \\
& z_d \ge 0,\ \zeta_d \ge 0,\ s_d^+ \le \epsilon,\ s_d^- \le \epsilon \qquad (d = 1, 2, 3, \cdots, k)
\end{aligned}$$

4. **Numerical Example and Discussion.** We will show the difference between proposed fuzzy Shapley value and normal Shapley value for three players, φ_1, φ_2, φ_3. The values of characteristic function are assumed fuzzy values, then their center $\boldsymbol{v}$ and width $\boldsymbol{\pi}$ are observed as follows,

$$\boldsymbol{v} = \begin{bmatrix} v(\varphi_1) \\ v(\varphi_2) \\ v(\varphi_3) \\ v(\varphi_1, \varphi_2) \\ v(\varphi_1, \varphi_3) \\ v(\varphi_2, \varphi_3) \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.1667 \\ 0.3182 \\ 0.5 \\ 0.6444 \\ 0.5455 \end{bmatrix}, \quad \boldsymbol{\pi} = \begin{bmatrix} \pi(\varphi_1) \\ \pi(\varphi_2) \\ \pi(\varphi_3) \\ \pi(\varphi_1, \varphi_2) \\ \pi(\varphi_1, \varphi_3) \\ \pi(\varphi_2, \varphi_3) \end{bmatrix} = \begin{bmatrix} 0.02 \\ 0.01 \\ 0.03 \\ 0.05 \\ 0.06 \\ 0.05 \end{bmatrix} \qquad (25)$$

Matrices $\boldsymbol{M}$ and $\boldsymbol{A}^{\mathrm{T}}$ are defined by

$$\boldsymbol{M} = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 \end{bmatrix} \in \Re^6 \times \Re^6, \tag{26}$$

$$\boldsymbol{A}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \in \Re^3 \times \Re^6 \tag{27}$$

And matrices $\boldsymbol{B}^{\mathrm{T}}$ and $\boldsymbol{C}^{\mathrm{T}}$ are calculated by

$$\boldsymbol{B}^{\mathrm{T}} = \begin{bmatrix} 0.5 & 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0.5 & 0.5 \end{bmatrix} \in \Re^3 \times \Re^6, \tag{28}$$

$$\boldsymbol{C}^{\mathrm{T}} = \begin{bmatrix} 1.5 & 0.5 & 0.5 \\ 0.5 & 1.5 & 0.5 \\ 0.5 & 0.5 & 1.5 \end{bmatrix} \in \Re^3 \times \Re^3 \tag{29}$$

Then the fuzzy variable $\boldsymbol{Y}$ is obtained as

$$\boldsymbol{Y} = \left(\begin{bmatrix} 0.67220 \\ 0.60610 \\ 0.75405 \end{bmatrix}, \begin{bmatrix} 0.065 \\ 0.055 \\ 0.070 \end{bmatrix} \right)_L \tag{30}$$

POS	z1	z2	z3	zeta1	zeta2	zeta3	s1+	s2+	s3+	s1−	s2−	s3−	ε	α	L^−1(α)	L^−1(1−α)
Fuzzy Shapley value	0.328083	0.261983	0.409933	0	0	0	0	0	0	0.155883	0.155883	0.155883	0.155883	1	0	
min	0	0	0	2.5	2.5	2.5	0	0	0	0	0	0	100	15.58833		
s.t.	1.5	0.5	0.5	0	0	0	1	0	0	−1	0	0	0	0.6722	<=	0.6722
	0.5	1.5	0.5	0	0	0	0	1	0	0	−1	0	0	0.6061	<=	0.6061
	0.5	0.5	1.5	0	0	0	0	0	1	0	0	−1	0	0.75405	<=	0.75405
	1.5	0.5	0.5	0	0	0	1	0	0	−1	0	0	0	0.6722	>=	0.6722
	0.5	1.5	0.5	0	0	0	0	1	0	0	−1	0	0	0.6061	>=	0.6061
	0.5	0.5	1.5	0	0	0	0	0	1	0	0	−1	0	0.75405	>=	0.75405
	1	1	1	0	0	0	0	0	0	0	0	0	0	1	=	1
	0	0	0	0	0	0	1	0	0	0	0	0	−1	−0.15588	<=	0
	0	0	0	0	0	0	0	1	0	0	0	0	−1	−0.15588	<=	0
	0	0	0	0	0	0	0	0	1	0	0	0	−1	−0.15588	<=	0
	0	0	0	0	0	0	0	0	0	1	0	0	−1	0	<=	0
	0	0	0	0	0	0	0	0	0	0	1	0	−1	0	<=	0
	0	0	0	0	0	0	0	0	0	0	0	1	−1	0	<=	0

FIGURE 1. Fuzzy Shapley value based on possibility of equivalent ($\alpha = 1$)

NEC	z1	z2	z3	zeta1	zeta2	zeta3	s1+	s2+	s3+	s1−	s2−	s3−	ε	α	L^−1(α)	L^−1(1−α)
Fuzzy Shapley value	0.328083	0.261983	0.409933	0	0	0	0	0	0	0.155883	0.155883	0.155883	0.155883	1	0	1
min	0	0	0	−2.5	−2.5	−2.5	0	0	0	0	0	0	100	15.58833		
s.t.	1.5	0.5	0.5	1.5	0.5	0.5	1	0	0	−1	0	0	0	0.6722	<=	0.6722
	0.5	1.5	0.5	0.5	1.5	0.5	0	1	0	0	−1	0	0	0.6061	<=	0.6061
	0.5	0.5	1.5	0.5	0.5	1.5	0	0	1	0	0	−1	0	0.75405	<=	0.75405
	1.5	0.5	0.5	−1.5	−0.5	−0.5	1	0	0	−1	0	0	0	0.6722	>=	0.6722
	0.5	1.5	0.5	−0.5	−1.5	−0.5	0	1	0	0	−1	0	0	0.6061	>=	0.6061
	0.5	0.5	1.5	−0.5	−0.5	−1.5	0	0	1	0	0	−1	0	0.75405	>=	0.75405
	1	1	1	0	0	0	0	0	0	0	0	0	0	1	=	1
	0	0	0	0	0	0	1	0	0	0	0	0	−1	−0.15588	<=	0
	0	0	0	0	0	0	0	1	0	0	0	0	−1	−0.15588	<=	0
	0	0	0	0	0	0	0	0	1	0	0	0	−1	−0.15588	<=	0
	0	0	0	0	0	0	0	0	0	1	0	0	−1	0	<=	0
	0	0	0	0	0	0	0	0	0	0	1	0	−1	0	<=	0
	0	0	0	0	0	0	0	0	0	0	0	1	−1	0	<=	0

FIGURE 2. Fuzzy Shapley value based on necessity of inclusion ($\alpha = 1$)

In case of normal Shapley value with no ambiguity such as $\alpha = 1$, the results for both possibility of equality and necessity of inclusion are the same as follows,

$$\boldsymbol{z} = \begin{bmatrix} 0.3281 \\ 0.2620 \\ 0.4099 \end{bmatrix} \tag{31}$$

Naturally the result is also the same as usual Shapley value without ambiguity (see Figures 1 and 2).

In case of fuzzy Shapley value with some ambiguity such as $\alpha = 0.7$, the results for possibility of equality and necessity of inclusion are different. The fuzzy Shapley value based on possibility of equality is as follows (see Figure 3),

$$\boldsymbol{z} = \begin{bmatrix} 0.4381 \\ 0.2047 \\ 0.3572 \end{bmatrix}, \quad \boldsymbol{\zeta} = \begin{bmatrix} 0.5475 \\ 0 \\ 0 \end{bmatrix} \tag{32}$$

This result about $\boldsymbol{\zeta}$ can be controlled to other width pattern if we add adequate conditions about them, because $\boldsymbol{C}^{\mathrm{T}}$ has no inverse matrix.

Also the fuzzy Shapley value based on necessity of inclusion is as follows (see Figure 4),

$$\boldsymbol{z} = \begin{bmatrix} 0.3286 \\ 0.2595 \\ 0.4119 \end{bmatrix}, \quad \boldsymbol{\zeta} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{33}$$

From this result, we can find that even if there is ambiguity in observed characteristic function; however, necessity of inclusion for Shapley value has not always ambiguity.

POS	z1	z2	z3	zeta1	zeta2	zeta3	s1+	s2+	s3+	s1−	s2−	s3−	ε	α	L^−1(α)	L^−1(1−α)
Fuzzy Shapley value	0.43809	0.20473	0.35718	0.547533	0	0	0	0	0	0	0	0	0	0.7	0.3	
min	0	0	0	2.5	2.5	2.5	0	0	0	0	0	0	100	1.368833		
s.t.	1.5	0.5	0.5	−0.45	−0.15	−0.15	1	0	0	−1	0	0	0	0.6917	<=	0.6917
	0.5	1.5	0.5	−0.15	−0.45	−0.15	0	1	0	0	−1	0	0	0.6226	<=	0.6226
	0.5	0.5	1.5	−0.15	−0.15	−0.45	0	0	1	0	0	−1	0	0.77505	<=	0.77505
	1.5	0.5	0.5	−0.45	−0.15	−0.15	1	0	0	−1	0	0	0	0.6917	>=	0.6527
	0.5	1.5	0.5	−0.15	−0.45	−0.15	0	1	0	0	−1	0	0	0.6226	>=	0.5896
	0.5	0.5	1.5	−0.15	−0.15	−0.45	0	0	1	0	0	−1	0	0.77505	>=	0.73305
	1	1	1	0	0	0	0	0	0	0	0	0	0	1	=	1
	0	0	0	0	0	0	1	0	0	0	0	0	−1	0	<=	0
	0	0	0	0	0	0	0	1	0	0	0	0	−1	0	<=	0
	0	0	0	0	0	0	0	0	1	0	0	0	−1	0	<=	0
	0	0	0	0	0	0	0	0	0	1	0	0	−1	0	<=	0
	0	0	0	0	0	0	0	0	0	0	1	0	−1	0	<=	0
	0	0	0	0	0	0	0	0	0	0	0	1	−1	0	<=	0

FIGURE 3. Fuzzy Shapley value based on possibility of equivalent ($\alpha = 0.7$)

NEC	z1	z2	z3	zeta1	zeta2	zeta3	s1+	s2+	s3+	s1−	s2−	s3−	ε	α	L^−1(α)	L^−1(1−α)
Fuzzy Shapley value	0.328583	0.259483	0.411933	0	0	0	0	0	0	0.136883	0.136883	0.136883	0.136883	0.7	0.3	0.7
min	0	0	0	−2.5	−2.5	−2.5	0	0	0	0	0	0	100	13.68833		
s.t.	1.5	0.5	0.5	1.05	0.35	0.35	1	0	0	−1	0	0	0	0.6917	<=	0.6917
	0.5	1.5	0.5	0.35	1.05	0.35	0	1	0	0	−1	0	0	0.6226	<=	0.6226
	0.5	0.5	1.5	0.35	0.35	1.05	0	0	1	0	0	−1	0	0.77505	<=	0.77505
	1.5	0.5	0.5	−1.05	−0.35	−0.35	1	0	0	−1	0	0	0	0.6917	>=	0.6527
	0.5	1.5	0.5	−0.35	−1.05	−0.35	0	1	0	0	−1	0	0	0.6226	>=	0.5896
	0.5	0.5	1.5	−0.35	−0.35	−1.05	0	0	1	0	0	−1	0	0.77505	>=	0.73305
	1	1	1	0	0	0	0	0	0	0	0	0	0	1	=	1
	0	0	0	0	0	0	1	0	0	0	0	0	−1	−0.13688	<=	0
	0	0	0	0	0	0	0	1	0	0	0	0	−1	−0.13688	<=	0
	0	0	0	0	0	0	0	0	1	0	0	0	−1	−0.13688	<=	0
	0	0	0	0	0	0	0	0	0	1	0	0	−1	0	<=	0
	0	0	0	0	0	0	0	0	0	0	1	0	−1	0	<=	0
	0	0	0	0	0	0	0	0	0	0	0	1	−1	0	<=	0

FIGURE 4. Fuzzy Shapley value based on necessity of inclusion ($\alpha = 0.7$)

5. **Conclusion.** In this paper we first introduce the Shapley value derived by the linear solvable formulation. From the linear model, the expansion formulation considering fuzziness to Shapley value can be obtained. The difference between proposed fuzzy Shapley value and normal Shapley value for three players are shown in the numerical example. Our model can derive fuzzy Shapley value for data with ambiguity in the possibility of equivalent case and the necessity of inclusion case.

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