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STABILITY OF COMPETITIVE SYSTEM IN A PATCHY ENVIRONMENT BASED UPON MATRIX THEORY AND GRAPH THEORY

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Abstract. *Competition possesses an important role in ecology. Population diffusion is a common phenomenon. In this paper, we systematically study the dynamics of a competitive system with linear diffusion. The explorations involve the existence and global asymptotic stability. Based upon M-matrix theory and graph theory, we obtain conditions for equilibriums existence and global-stability. These results extend some recent known ones for diffusion system.*

Keywords: Linear diffusion, Positive equilibrium, Irreducibility, Lyapunov functions, Global stability

1. **Introduction.** Competition possesses an important role in the study of ecological problems, which have been widely explored mathematically and biologically, and lots of excellent results are obtained [1-5]. Indeed, a famous model for population dynamics is the Lotka-Volterra competition system.

In real life, there are all kinds of diffusion effects. It is reasonable and practical to study Lotka-Volterra competition systems with linear diffusion. However, most are just for competitive systems without diffusion. Allen [6], by demanding a comparison theorem, obtained a partial answer to the persistent and extinction problem for single-species discrete diffusion systems.

Based upon M-matrix theory and graph-theory, Li and Fan [7,8] obtain conditions for the system with linear diffusion of equilibriums existence and global-stability. These results extend some recent known ones for prey-predator diffusion system.

In this paper, we utilize the graph-theory approach to investigate the global stability of the competition system with one population travel among n patches. We consider the following system:

$$\begin{cases} \dot{x}_i = x_i(f_i(x_i) - e_i y_i) + \sum\limits_{j\neq i}^{n} \alpha_{ij}(x_j - x_i), \\ \dot{y}_i = y_i(g_i(y_i) - \varepsilon_i x_i) \end{cases} \text{for } i = 1, \cdots, n \tag{1}$$

where x_i and y_i stand for the population densities of competitors in the i-th patch and $f_i(x_i)$ and $g_i(y_i)$ are the specific growth rates of competitors in the i-th patch. α_{ij} is a nonnegative diffusion coefficient for one competitor from j-th patch to i-th patch ($i \neq j$). The parameters in the system are nonnegative, and e_i, ε_i are positive. Much of the previous work related to (1) can be found in [1-11] and references cited therein.

We want to find all equilibriums by utilizing M-matrixs different from [7,8] and investigate stability of some equilibriums by utilizing assumptions different from [7,8].

The structure of this paper is as follows. In Section 2, we study equilibriums of system (1). In Section 3, asymptotic stability of some equilibriums is demonstrated. The example and conclusions are given in Section 4 and Section 5, respectively. For convenience and simplicity in the following discussion, we always use the notations:

$$L=\begin{pmatrix} \sum_{i=2}^{n}\alpha_{1i} & -\alpha_{12} & \cdots & -\alpha_{1n}\\ -\alpha_{21} & \sum_{i\neq 2}^{n}\alpha_{2i} & \cdots & -\alpha_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ -\alpha_{n1} & -\alpha_{n2} & \cdots & \sum_{i=1}^{n-1}\alpha_{ni} \end{pmatrix}$$

$$M=\begin{pmatrix} f_1(0)-\sum_{i=2}^{n}\alpha_{1i} & \alpha_{12} & \cdots & \alpha_{1n}\\ \alpha_{21} & f_2(0)-\sum_{i\neq 2}^{n}\alpha_{2i} & \cdots & \alpha_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ \alpha_{n1} & \alpha_{n2} & \cdots & f_n(0)-\sum_{i=1}^{n-1}\alpha_{ni} \end{pmatrix}$$

$$M_1=\begin{pmatrix} f_1(0)-\sum_{i=2}^{n}\alpha_{1i} & \cdots & \alpha_{1j} & \alpha_{1,j+1} & \cdots & \alpha_{1n}\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \alpha_{k1} & \cdots & f_k(0)-\sum_{i\neq j}^{n}\alpha_{ki} & \alpha_{k,j+1} & \cdots & \alpha_{kn}\\ \alpha_{k+1,1} & \cdots & \alpha_{k+1,j} & f_{k+1}(0)-e_{k+1}g_{k+1}^{-1}(0)-\sum_{i\neq j+1}^{n}\alpha_{k+1,i} & \cdots & \alpha_{k+1,n}\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \alpha_{n1} & \cdots & \alpha_{nj} & \alpha_{n,j+1} & \cdots & f_n(0)-e_ng_n^{-1}(0)-\sum_{k=1}^{n-1}\alpha_{ni} \end{pmatrix}$$

$$M_2=\begin{pmatrix} f_1(0)-e_1g^{-1}(0)-\sum_{i=2}^{n}\alpha_{1i} & \cdots & \alpha_{1j} & \cdots & \alpha_{1n}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ \alpha_{k1} & \cdots & f_k(0)-e_kg^{-1}(0)-\sum_{i\neq j}^{n}\alpha_{ki} & \cdots & \alpha_{kn}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ \alpha_{n1} & \cdots & \alpha_{nj} & \cdots & f_n(0)-e_ng_n^{-1}(0)-\sum_{k=1}^{n-1}\alpha_{ni} \end{pmatrix}$$

and $s(M)$ denotes the maximum real part of all eigenvalues of matrix M. In system (1), we assume that

$(H1)$ $f_i(0)>0$, $g_i(0)>0$, $\dot{f}(x_i)<0$, $\dot{g}(y_i)<0$, for $i=1,\cdots,n$;
$(H2)$ $f_i(0)-e_ig_i^{-1}(0)>0$, $\dot{f}(x_i)-e_i\dot{g}_i^{-1}(\varepsilon_ix_i)<0$ for $i=k+1,\cdots,n$;
$(H3)$ $f_i(0)-e_ig_i^{-1}(0)>0$, $\dot{f}(x_i)-e_i\dot{g}_i^{-1}(\varepsilon_ix_i)<0$ for $i=1,\cdots,n$;
$(H4)$ $\dot{f}(x_i)\dot{g}_i(y_i)-e_i\varepsilon_i<0$ for any x_i, y_i;
$(H5)$ $f_i(0)-\sum_{j\neq i}^{n}\alpha_{1i}<0$.

In the first section, we introduce the background of the problem (1) and some notations.

2. **Existence of Equilibriums.** We study equilibriums of system (1). In order to find the equilibriums of (1), we give the following system

$$\begin{cases} x_i(f_i(x_i)-e_iy_i)+\sum_{j\neq i}^{n}\alpha_{ij}(x_j-x_i)=0, \\ y_i(g_i(y_i)-\varepsilon_ix_i)=0. \end{cases} \text{for } i=1,\cdots,n \tag{2}$$

Theorem 2.1. *The system (1) always has a trial equilibrium* $E_0 = (0, 0, \cdots, 0, 0)$.

Clearly, $E_0 = (0, 0, \cdots, 0, 0)$ is zero-solution of system (2), which means $E_0 = (0, 0, \cdots, 0, 0)$ is trial equilibrium of system (1).

Theorem 2.2. *The system (1) has an equilibrium* $E_1 = (x_{10}, 0, \cdots, x_{n0}, 0)$ *if the following conditions are satisfied*
(1) L is irreducible,
(2) $(H1)$ *holds,*
(3) $s(M) > 0$.

In fact, $(x_{10}, \cdots, x_{n0})$ is a positive equilibrium of

$$\dot{x}_i = x_i(f_i(x_i)) + \sum_{j\neq i}^{n} \alpha_{ij}(x_j - x_i) \text{ for } i = 1, \cdots, n. \tag{3}$$

Proof: Readers can read reference [7-9] for proof of the theorem.

Theorem 2.3. *The system (1) has an equilibrium* $E_1^* = (0, y_{10}, \cdots, 0, y_{n0})$ *if the following conditions are satisfied*
(1) L is irreducible,
(2) $(H1)$ *holds.*

In fact, $(y_{10}, \cdots, y_{n0})$ is a positive equilibrium of

$$\dot{y}_i = y_i g_i(y_i) \text{ for } i = 1, \cdots, n. \tag{4}$$

We know (4) always has a trial equilibrium, also (4) has a positive equilibrium $(y_{10}, \cdots, y_{n0})$ if $g_i(0) > 0$ and $\dot{g}_i(x_i) < 0$ by $H(1)$.

Theorem 2.4. *The system (1) has an equilibrium* $E_2 = (x_{10}, 0, \cdots, x_{k0}, 0, x_{k+1}^*, y_{k+1}^* \cdots, x_n^*, y_n^*)$, *if the following conditions are satisfied*
(1) L is irreducible,
(2) $(H1)$, $(H2)$ *hold,*
(3) $s(M_1) > 0$.

Here $x_i^* > 0$, $y_i^* > 0$, for $i = k+1, \cdots, n$. We change the order of the equations, and there is always an equilibrium $E_2 = (x_{10}, 0, \cdots, x_{k0}, 0, x_{k+1}^*, y_{k+1}^* \cdots, x_n^*, y_n^*)$.

Proof: In this section, we prove that E_2 exists if $s(M_1) > 0$ and $y_i = 0$ for $i = 1, \cdots, k$. Therefore, we need to consider the following system.

$$\begin{cases} \dot{x}_i = x_i(f_i(x_i)) + \sum\limits_{j\neq i}^{n} \alpha_{ij}(x_j - x_i) \\ \dot{y}_i = 0 \text{ for } i = 1, \cdots, k \\ \dot{x}_i = x_i(f_i(x_i) - e_i y_i) + \sum\limits_{j\neq i}^{n} \alpha_{ij}(x_j - x_i) \\ \dot{y}_i = y_i(g_i(y_i) - \varepsilon_i x_i), \text{ for } i = k+1, \cdots, n. \end{cases} \tag{5}$$

We know g_i^{-1} exists and $\dot{g}_i^{-1} < 0$ by $H(1)$.

$$\begin{cases} \dot{x}_i = x_i(f_i(x_i)) + \sum\limits_{j\neq i}^{n} \alpha_{ij}(x_j - x_i) \text{ for } i = 1, \cdots, k \\ \dot{x}_i = x_i(f_i(x_i) - e_i g_i^{-1}(\varepsilon_i x_i)) + \sum\limits_{j\neq i}^{n} \alpha_{ij}(x_j - x_i), \text{ for } i = k+1, \cdots, n. \end{cases} \tag{6}$$

System (6) has a positive equilibrium $(x_{10}, \cdots, x_{k0}, x_{k+1}^*, \cdots, x_n^*)$ if $s(M_1) > 0$ and $(H1)$, $(H2)$ hold [7-9]; therefore, system (1) has an equilibrium

$$E_2 = (x_{10}, 0, \cdots, x_{k0}, 0, x_{k+1}^*, y_{k+1}^* \cdots, x_n^*, y_n^*),$$

where $y_i^* = g_i^{-1}(\varepsilon_i x_i^*)$ for $i = k+1, \cdots, n$. Readers may prove $y_i^* > 0$ by themselves [8].

Theorem 2.5. *The system (1) does not have equilibrium*

$$E_2^* = (0, y_{10}, \cdots, 0, y_{k0}, x_{k+1}^*, y_{k+1}^*, \cdots, x_n^*, y_n^*).$$

Readers may prove this theorem by themselves [7].

Theorem 2.6. *The system (1) has a positive equilibrium $E^* = (x_1^*, y_1^*, \cdots, x_n^*, y_n^*)$, if the following conditions are satisfied*
(1) L is irreducible,
(2) $(H1)$, $(H3)$ hold,
(3) $s(M_2) > 0$.

Proof: Applying Theorem 2.2, we give following form:

$$\dot{x}_i = x_i(f_i(x_i) - e_i g_i^{-1}(\varepsilon_i x_i)) + \sum_{j\neq i}^{n} \alpha_{ij}(x_j - x_i), \text{ for } i = 1, \cdots, n. \quad (7)$$

System (7) has a positive equilibrium $(x_1^*, \cdots, x_n^*)$ by $s(M_2) > 0$ and $(H1)$, $(H3)$ [9]; therefore, system (1) has an equilibrium $E_2 = (x_n^*, y_n^* \cdots, x_n^*, y_n^*)$, where $y_i^* = g_i^{-1}(\varepsilon_i x_i^*)$ for $i = 1, \cdots, n$.

Theorem 2.7. *Suppose $x_i(0), y_i(0) > 0$ for $i = 1, \cdots, n$, $\Gamma : \{((x_1, y_1, \cdots, x_n, y_n) \in R^{2n}/x_i, y_i > 0, x_i < x_{i0}, y_i < y_{i0})\}$ is positive invariance of system (1).*

Proof: First, we consider for all i, for all t, $x_i(t), y_i(t) > 0$ in condition $x_i(0), y_i(0) > 0$, suppose there exists $T > 0$ and $x_i(T) = 0, x_j(T) > 0$, then $\dot{x}_i(T) = \sum_{j=1}^{n} \alpha_{ij} x_j(T) > 0$. Similar steps $y_i > 0$, then

$$\dot{x}_i = x_i(f_i(x_i) - e_i y_i) + \sum_{j\neq i}^{n} \alpha_{ij}(x_j - x_i) < x_i f_i(x_i) + \sum_{j\neq i}^{n} \alpha_{ij}(x_j - x_i).$$

Suppose $x_i = x_{i0}$, $x_j < x_{j0}$,

$$\dot{x}_i \mid_{x_i = x_{i0}} < x_{i0} f_i(x_{i0}) + \sum_{j=1}^{n} \alpha_{ij}(x_{j0} - x_{i0}) = 0.$$

So, for all i, there exists x_{i0}, $x_i < x_{i0}$.
Similar steps for all i, there exists y_{i0}, $y_i < y_{i0}$.
Therefore, $\Gamma : \{((x_1, y_1, \cdots, x_n, y_n) \in R^{2n}/x_i, y_i > 0, x_i < x_{i0}, y_i < y_{i0})\}$ is positive invariance of system (1), which means uniform boundedness of solution in $\Gamma/\{E_0, E_1, E_1^*, E_2\}$.

3. Global Asymptotic Stability of Equilibriums.

3.1. Boundary equilibriums.

In this section, we will prove the global asymptotic stability of boundary equilibriums E_0, E_1, E_1^*.

Theorem 3.1. *If $s(M) < 0$ and $(H1)$ holds, E_0 is global asymptotic stability, and if $s(M) > 0$, E_0 is unstable.*

Readers can read reference [7,8] for proof of the theorem.

Theorem 3.2. *Suppose assumption $(H1)$, $(H4)$ hold, E_1 is global asymptotic stability.*

Proof: Denote the boundary equilibrium $E_1 = (x_{10}, 0, \cdots, x_{n0}, 0)$ about system (1). Here

$$x_{i0}(f_i(x_{i0})) + \sum_{j\neq i}^{n} \alpha_{ij}(x_{j0} - x_{i0}) = 0 \text{ for } i = 1, \cdots, n.$$

Consider a Lyapunov function for a single patch predator-prey model

$$V_i(x_i, y_i) = \varepsilon_i \left(x_i - x_{i0} + x_{i0} \ln \frac{x_i}{x_{i0}} \right) + e_i y_i. \tag{8}$$

We show that V_i satisfies the assumption of Lemma 1 [9],

$$\begin{aligned}
\dot{V}_i &= \varepsilon_i (x_i - x_{i0}) \frac{\dot{x}_i}{x_i} + e_i \dot{y}_i \\
&= \varepsilon_i \dot{f}(\xi)(x_i - x_{i0})^2 - 2e_i\varepsilon_i(x_i - x_{i0})y_i + e_i\dot{g}(\eta)y_i^2 \\
&\quad + \sum_{j=1}^{n} \alpha_{ij}\varepsilon_i x_{j0} \left(\frac{x_j}{x_{j0}} - \frac{x_i}{x_{i0}} + 1 - \frac{x_j x_{i0}}{x_{j0} x_i} \right).
\end{aligned}$$

By $(H1)$, $(H4)$ assumptions, then

$$\begin{aligned}
\dot{V}_i &< \sum_{j=1}^{n} \alpha_{ij}\varepsilon_i x_{j0} \left(1 - \frac{x_j x_{i0}}{x_{j0} x_i} + \ln \frac{x_j x_{i0}}{x_{j0} x_i} \right) + \sum_{j=1}^{n} \alpha_{ij}\varepsilon_i x_{j0} \left[\left(\frac{x_j}{x_{j0}} + \ln \frac{x_j}{x_{j0}} \right) - \left(\frac{x_i}{x_{i0}} + \ln \frac{x_i}{x_{i0}} \right) \right] \\
&< \sum_{j=1}^{n} \alpha_{ij}\varepsilon_i x_j^* [(H_j(x_j)) - (H_i(x_i))]
\end{aligned}$$

and $H_i(x_i)$ and α_{ij} satisfy the assumption of Lemma 1 and Lemma 2 [9], then $\dot{V}_i < 0$.

Therefore, the function

$$V(x_1, y_1, \cdots, x_n, y_n) = \sum_{i=1}^{n} c_i V_i(x_i, y_i), \quad i = 1, \cdots, n$$

as defined in Lemma 2 [9] is a Lyapunov function for (1), and $\dot{V} < 0$ for all $(x_1, y_1, \cdots, x_n, y_n) \in R_+^{2n}$.

This also implies that the only compact invariant set on which $\dot{V} = 0$ is $x_i = x_{i0}, y_i = 0, i = 1, \cdots, n$ is the singleton E_1. The LaSalle Invariance principle [12] implies that E_1 is global asymptotically stable. This also implies that E_1 is unique, completing the proof of Theorem 3.2.

Theorem 3.3. *Suppose $s(M) > 0$ and $(H5)$ holds, if $s(N) < 0$, E_1^* is global asymptotic stability, and if $s(N) > 0$, E_1^* is unstable.*

Here

$$N = \begin{pmatrix}
f_1(0) + e_1 y_{10} - \sum_{i=2}^{n} \alpha_{1i} & \alpha_{12} & \cdots & \alpha_{1n} \\
\alpha_{21} & f_2(0) + e_2 y_{20} - \sum_{i \neq 2}^{n} \alpha_{2i} & \cdots & \alpha_{2n} \\
\cdots & \cdots & \cdots & \cdots \\
\alpha_{n1} & \alpha_{n2} & \cdots & f_n(0) + e_n y_{n0} - \sum_{i=1}^{n-1} \alpha_{ni}
\end{pmatrix}.$$

Proof: First let $N = diag\{e_i y_{i0}\} - F$, where

$$diag\{e_i y_{i0}\} = \begin{pmatrix}
e_1 y_{10} & 0 & \cdots & 0 \\
0 & e_2 y_{20} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & e_n y_{n0}
\end{pmatrix}$$

and

$$F=\begin{pmatrix} -f_1(0)+\sum\limits_{i=2}^{n}\alpha_{1i} & -\alpha_{12} & \cdots & -\alpha_{1n} \\ -\alpha_{21} & -f_2(0)+\sum\limits_{i\neq 2}^{n}\alpha_{2i} & \cdots & -\alpha_{2n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ -\alpha_{n1} & -\alpha_{n2} & \cdots & -f_n(0)+\sum\limits_{i=1}^{n-1}\alpha_{ni} \end{pmatrix}$$

Since $diag\{e_i y_{i0}\}$ is a non-negative and F is non-singular M-matrix. $N = diag\{e_i y_{i0}\} - F$ has Z sign pattern [7,8,13].

$$s(N) < 0 \Longleftrightarrow \rho(diag^{-1}\{e_i y_{i0}\}F) > 1.$$

Let $(\omega_1, \cdots, \omega_n)$ be left eigenvalue of $diag^{-1}\{e_i y_{i0}\}F$ corresponding $\rho(diag^{-1}\{e_i y_{i0}\}F) > 1$. Since $diag^{-1}\{e_i y_{i0}\}F$ is irreducible, we know $\omega_i > 0, i = 1, \cdots, n$ [13].

Set

$$V = \sum_{i=1}^{n} \frac{\omega_i}{e_i y_{i0}} x_i$$

we obtain

$$\begin{aligned} \dot{V} &= \sum_{i=1}^{n} \frac{\omega_i}{e_i y_{i0}} \left(x_i(f_i(x_i) - e_i y_i) + \sum_{j\neq i}^{n} \alpha_{ij}(x_j - x_i) \right) \\ &< \sum_{i=1}^{n} \frac{\omega_i}{e_i y_{i0}} \left(x_i(f_i(0) + e_i y_{i0}) + \sum_{j\neq i}^{n} \alpha_{ij}(x_j - x_i) \right) \\ &= \left(\frac{\omega_1}{e_1 y_{10}}, \cdots, \frac{\omega_n}{\varepsilon_n y_{n0}} \right) (diag\{e_i y_{i0}\} - F)(x_1, \cdots, x_n)^T \\ &= (\omega_1, \cdots, \omega_n)(1 - \rho(diag^{-1}\{e_i y_{i0}\}F))(x_1, \cdots, x_n)^T \\ &< 0 \end{aligned} \tag{9}$$

and the equal sign holds if and only if $y_i = 0, i = 1, \cdots, n$. Therefore, by LaSalle Invariantce principle [12], E_1 is global asymptotic stability.

We will discuss the stability of E_2 in future.

3.2. **Positive equilibrium.**

In this section, we will prove the positive equilibrium of the system (1) is global asymptotic stability if it exists.

Theorem 3.4. *Suppose assumption* $(H1)$, $(H4)$ *hold, then positive equilibrium* $E^* = (x_1^*, y_1^*, \cdots, x_n^*, y_n^*)$ *of system (1) is global asymptotically stable.*

Proof: First, we suppose the positive equilibrium exists.

Denote the positive equilibrium $E^* = (x_1^*, y_1^*, \cdots, x_n^*, y_n^*), x_i^*, y_i^* > 0$, for $i = 1, \cdots, n$ about system (1). Here

$$\begin{cases} x_i^*(f_i(x_i^*) - e_i y_i^*) + \sum\limits_{j=1}^{n} \alpha_{ij}(x_j^* - x_i^*) = 0, \\ y_i^*(g_i(y_i^*) - \varepsilon_i x_i^*) = 0. \end{cases} \quad \text{for } i = 1, \cdots, n \tag{10}$$

Consider a Lyapunov function in [7-9] for a single patch predator-prey model

$$V_i(x_i, y_i) = \varepsilon_i \left(x_i - x_i^* + x_i^* \ln \frac{x_i}{x_i^*} \right) + e_i \left(y_i - y_i^* + y_i^* \ln \frac{y_i}{y_i^*} \right). \tag{11}$$

We show that V_i satisfies the assumption of Lemma 1 [9],

$$\begin{aligned}\dot{V}_i &= \varepsilon_i(x_i - x_i^*)\frac{\dot{x}_i}{x_i} + e_i(y_i - y_i^*)\frac{\dot{y}_i}{y_i} \\ &= \varepsilon_i \dot{f}(\xi)(x_i - x_i^*)^2 - 2e_i\varepsilon_i(x_i - x_i^*)(y_i - y_i^*) + e_i\dot{g}(\eta)(y_i - y_i^*)^2 \\ &\quad + \sum_{j=1}^{n} \alpha_{ij}\varepsilon_i x_j^* \left(\frac{x_j}{x_j^*} - \frac{x_i}{x_i^*} + 1 - \frac{x_j x_i^*}{x_j^* x_i}\right).\end{aligned}$$

By $(H1)$, $(H4)$ assumptions, then

$$\begin{aligned}\dot{V}_i &< \sum_{j=1}^{n} \alpha_{ij}\varepsilon_i x_j^* \left(1 - \frac{x_j x_i^*}{x_j^* x_i} + \ln\frac{x_j x_i^*}{x_j^* x_i}\right) + \sum_{j=1}^{n} \alpha_{ij}\varepsilon_i x_j^* \left[\left(\frac{x_j}{x_j^*} + \ln\frac{x_j}{x_j^*}\right) - \left(\frac{x_i}{x_i^*} + \ln\frac{x_i}{x_i^*}\right)\right] \\ &< \sum_{j=1}^{n} \alpha_{ij}\varepsilon_i x_j^* [(G_j(x_j)) - (G_i(x_i))]\end{aligned}$$

and $G_i(x_i)$ and α_{ij} satisfy the assumption of Lemma 1 and Lemma 2 [9], then $\dot{V}_i < 0$.

Therefore, the function

$$V(x_1, y_1, \cdots, x_n, y_n) = \sum_{i=1}^{n} c_i V_i(x_i, y_i), \quad i = 1, \cdots, n$$

as defined in Lemma 2 [9] is a Lyapunov function for (1), and $\dot{V} < 0$ for all $(x_1, y_1, \cdots, x_n, y_n) \in R_+^{2n}$.

This also implies that the only compact invariant set on which $\dot{V} = 0$ is $x_i = x_i^*, y_i = y_i^*, i = 1, \cdots, n$ is the singleton E^*. The LaSalle Invariance principle [12] implies that E^* is global asymptotically stable. This also implies that E^* is unique, completing the proof of Theorem 3.4.

Remark 3.1. *When β_{ij} is a nonnegative diffusion coefficient for another competitor from j-th patch to i-th patch $(i \neq j)$, system (1) becomes the following n-patch competitive system*

$$\begin{cases} \dot{x}_i = x_i(f_i(x_i) - e_i y_i) + \sum_{j=1}^{n} \alpha_{ij}(x_j - x_i), \\ \dot{y}_i = y_i(g_i(y_i) - \varepsilon_i x_i) + \sum_{j=1}^{n} \beta_{ij}(y_j - y_i). \end{cases} \quad \text{for } i = 1, \cdots, n \tag{12}$$

We can prove the positive equilibrium not only exists but is globally asymptotically stable under simple assumptions.

4. **Example.** Consider the following coupled system

$$\begin{cases} \dot{x}_1 = x_1(1 - x_1 - y_1) + 2(x_2 - x_1), \\ \dot{y}_1 = y_1(1 - y_1 - 2x_1), \\ \dot{x}_2 = x_2(1 - x_2 - 3y_2) + 2(x_1 - x_2), \\ \dot{y}_2 = y_2(1 - 2y_2 - 3x_2). \end{cases} \tag{13}$$

where $M = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$, $s(M) > 0$, $N = \begin{pmatrix} 0 & 2 \\ 2 & 0.5 \end{pmatrix}$, $s(N) > 0$, $M_1 = \begin{pmatrix} -1 & 2 \\ 2 & -2.5 \end{pmatrix}$, $s(M_1) > 0$, $M_2 = \begin{pmatrix} -2 & 2 \\ 2 & -2.5 \end{pmatrix}$, $s(M_2) > 0$. E_0, E_1^* are unstable, E_1, E^* are stable.

5. **Conclusions.** In this paper, we consider a competitive dynamical system in a patchy environment where population individuals in each compartment can travel among n patches. Under some minimal assumptions, we obtain conditions for boundary equilibriums existence. We also obtain sufficient conditions under which the positive equilibrium is asymptotically stable as long as it exists. Many factors influence the local persistence or extinction of a particular species. One of these factors is the species dispersal pattern.

The movement of some species can be described as linear diffusion or nonlinear diffusion. We will study nonlinear diffusion system in the future.

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