ICIC Express Letters
Volume **10**, Number **4**, April **2016**

ICIC International ©2016 ISSN 1881-803X
pp. **889–896**

STOCHASTIC SERVICE SYSTEMS OF DISCRETE-TIME: QUEUE LENGTH AND BUSY PERIOD

Yabo Wang, Yao Lin, Sufang Han and Zhenting Hou

Department of Mathematics and Statistics
Central South University
Shiji Building 1401, Shaoshan Road, Changsha 410000, P. R. China
yaboowang@csu.edu.cn

Received August 2015; accepted November 2015

Abstract. *In this paper, we aim to study further theoretical properties of discrete-time queuing systems. Taking into consideration the restriction of existing methods that the arrival time interval of the customer or the customer service time is negative exponential distributed, we use Markov skeleton processes to remove such restriction and obtain (1) the instantaneous distribution of the queue length; (2) the stationary distribution of the queue length; and (3) the busy period. In addition, the train-ticketing system is taken as an example to illustrate our results, which is a reference for transportation and management systems.*

Keywords: Markov skeleton process, Stochastic service systems, Queue length

1. **Introduction.** Through the development for a century, “queuing theory”, as an important branch of the randomized operational research and application of probability theory, has become a mature theoretical system. The achievements are widely used in all kinds of fields, such as transportation system, supply chain in logistics system, medical services system, computer network, communications, and other public services. Based on the previous results on discrete-time queuing systems [1-3], Tian et al. studied systematically the discrete-time stochastic service systems [4-8], while Liu et al. [9] conducted research on the performance of admission control for multi-traffic in wireless communication network based on discrete time queue. And Ma et al. [10,11] studied discrete-time gated service system and discrete-time Geom/G/1 Queuing System. However, there is a limitation in Tian’s work that the distribution of arrival time interval of customer and the customer service time is assumed to be geometric distribution.

We conduct a thorough research on the discrete-time stochastic service systems, and present the stability and the instantaneous form of the queue length by using Markov skeleton process [12] which is a widely used method in queuing theory and other areas [13]. Recently, the behavior of a discrete-time single-server queuing system with general service times was analyzed by Feyaerts et al. [14], and the discrete-time queuing system where each customer has a maximum allowed sojourn time was given by Bruneel and Maertens [15]. Wu et al. [16] considered a discrete-time Geo/G/1 retrial queue with impatient customers.

In this work, the distributions of arrival time interval of customer and customer service time are given without too many restrictions (different from Tian’s work, which limits one of the two distributions to geometric distribution). Furthermore, the existence conditions of the instantaneous distribution (Section 3) and stationary distribution (Section 5) of the queue length of discrete-time stochastic service system are presented, as well as the busy period (Section 4). Eventually, the applications of our results and conclusions are given in Section 6 and Section 7.

2. Preliminaries of Markov Skeleton Process.

Definition 2.1. $\{X(t),\, t < T\}$ *is a left-limit and right-continuous stochastic process with value in* E, *if there exists a sequence of stopping times* (τ_n), *satisfying:*
(i) $\tau_n \equiv 0$ *and* $\tau_n \uparrow T$;
(ii) $X(t)$ *has Markov properties in* τ_n $(n \geq 1)$;
(iii) $\tau_{n+1} = \tau_1 + \theta_{\tau_1}\tau_n$, *where* θ_t *denotes the shift operator:* $(\theta_t\omega)_s = \omega_{t+s}$.
$\{X(t),\, t < T\}$ *is called a Markov skeleton process, and* (τ_n) *is called the skeleton time sequence of* $X(t)$.

Lemma 2.1. *If* $X(t)$ *is a Markov skeleton process with* (τ_n) *as its skeleton time sequence, then* $\tau_{n+1} - \tau_1 = \theta_{\tau_1}\tau_n$ *is* $\sigma(X(\tau_1 + t),\, t \geq 0)$*-measurable.*

Proof: τ_n is an F_∞-measurable stopping time, and thereby there is a measurable function $f(x_1, x_2, \ldots)$ and a sequence $\{t_1, t_2, \ldots\}$, such that:

$$\tau_n(\omega) = f(x(t_1, \omega), x(t_2, \omega), \ldots)$$

$$\tau_{n+1} - \tau_1 = \theta_{\tau_1}\tau_n = f(x(\tau_1 + t_1, \omega), x(\tau_1 + t_2, \omega), \ldots)$$

Thus, we get our lemma proved.

A random process is (only) determined by its finite-dimensional joint distribution, that is, two random processes having an identical finite dimensional distribution can be seen as an identical process. Hence, the determination of the finite dimensional joint distribution, especially its one dimensional distribution, is a key step in the research of a stochastic process. The tool we used is the backward equation.

Let $X(t)$ be a Markov skeleton process with skeleton time sequence (τ_n), and let

$$h(x, t, A) = P(X(t) \in A,\, t < \tau_1 | X(0) = x)$$

$$q(x, t, A) = P(X(\tau_1) \in A,\, \tau_1 \leq t | X(0) = x)$$

$$EN = \exp\left\{\sum_{k=1}^{\infty} \frac{1 - a_k}{k}\right\} = \exp\left\{\sum_{k=1}^{\infty} \frac{x^k}{k}\right\}$$

Namely,

$$\begin{aligned} q(x, ds, A) &= P(X(\tau_1) \in A,\, \tau_n \in ds | X(0) = x) \\ P(x, t, A) &= P(X(t) \in A | X(0) = x) \\ P^{(n)}(x, t, A) &= P(X(t) \in A,\, t < \tau_n | X(0) = x) \end{aligned}$$

Theorem 2.1. *Let* $X = \{X(t),\, t < T\}$ *be a Markov skeleton process with skeleton time sequence* $\{\tau_n\}_{n=0}^{\infty}$, *then for any* $x \in E$, $t \geq 0$, $A \in \mathcal{E} = \mathcal{B}(E)$, *we have*

$$\begin{cases} P^{(1)}(x, t, A) = h(x, t, A) \\ P^{(n+1)}(x, t, A) = h(x, t, A) + \int_E \int_0^t (q(x, ds, dy)) P^{(n)}(y, t - s, A) \end{cases} \tag{1}$$

Thereby, $\{P(x, t, A)\}$ *is a minimal non-negative solution of the following non-negative equation system:*

$$P(x, t, A) = h(x, t, A) + \int_E \int_0^t q(x, ds, dy) P(y, t - s, A) \quad x \in E,\ t \geq 0,\ A \in \mathcal{E} \tag{2}$$

Proof: Apparently, the first equation in (1) is correct.

For any $x \in E$, $t \geq 0$, $A \in \mathcal{E}$, $n \in N$,

$$\begin{aligned} &P^{(n+1)}(x, t, A) \\ &= P(X(t) \in A,\, t < \tau_1) + P(X(t) \in A,\, \tau_1 \leq t < \tau_{n+1} | X(0) = x) \\ &= h(x, t, A) + \int_E \int_0^t P(X(t) \in A,\, t - s < \tau_{n+1} - \tau_1 | X(\tau_1) = y,\, \tau_1 = s,\, X(0) = x) \end{aligned}$$

$$\begin{aligned} &\times P(X(\tau_1) \in dy,\, \tau_1 \in ds | X(0) = x) \\ = \; & h(x,t,A) + \int_E \int_0^t P(X(t-s+\tau_1) \in A,\, t-s < \theta_{\tau_1}\tau_n | X(\tau_1) = y,\, \tau_1 = s,\, X(0) = x) \\ &\times q(x, ds, dy) \end{aligned}$$

By the homogeneity property and the Markov property at τ_1 of $X = \{X(t),\, t < T\}$ in Lemma 2.1, we have

$$\begin{aligned} & P(X(t-s+\tau_1) \in A,\, t-s < \theta_{\tau_1}\tau_n | X(\tau_1) = y,\, \tau_1 = s,\, X(0) = x) \\ = \; & P\left(X(t-s) \in A,\, t-s < \tau_n | X(0) = y,\, P^{(n)}(y, t-s, A)\right) \end{aligned}$$

Hence,

$$P(X(t) \in A,\, t < \tau_{n+1} | X(0) = x) = h(x,t,A) + \int_E \int_0^t q(x, ds, dy) P^{(n)}(y, t-s, A)$$

Then the second equation in (1) is proved. By the theory of smallest non-negative solution, $\{P(x,t,A)\}$ is the smallest non-negative solution of Equation (2). Equation (2) is called the backward equation of the Markov skeleton process $\{X(t),\, t < T\}$.

Definition 2.2. *Suppose $X(t)$ is a stochastic process on $(\Omega, \mathcal{F}, P)$ taking values in $(E, \mathcal{E})$. If for any $x \in E$, $A \in \mathcal{E}$, $\lim_{t\to\infty} P(x,t,A)$ exists and does not depend on x, thus $P(A) \equiv \lim_{t\to\infty} P(x,t,A)$, $(A \in \mathcal{E})$ is a probability distribution in $(E, \mathcal{E})$, then the limit (probability) distribution of $X(t)$ exists, and $P(\cdot)$ is called the limit (probability) distribution of $X(t)$:*

$$P(x,t,A) = P(X(t) \in A | X(0) = x)$$

Definition 2.3. *Suppose $X(t)$ is a Markov skeleton process with skeleton time sequence $\{\tau_n\}_{n=0}^{\infty}$. If there exists a probability measure $\pi(\cdot)$ on $(E, \mathcal{E})$ such that for any $A \in \mathcal{E}$,*

$$P(X(\tau_1) \in A | X(0) = x,\, \tau_1 = s) = P(X(\tau_1) \in A) = \pi(A) \tag{3}$$

then, $X(t)$ is called a Doob skeleton process, $\pi(\cdot)$ is the characteristic measure of $X(t)$, and $\{\tau_n\}_{n=0}^{\infty}$ are the regeneration points of $X(t)$.

Let

$$\begin{aligned} F(x,t) &= P(\tau_1 \le t | X(0) = x) \\ F(t) &= \int_0^\infty \pi(dx) F(x,t) \\ m &= \int_0^\infty t dF(t) \end{aligned}$$

Thus, we have the definition as follows.

Definition 2.4. *Suppose $X(t)$ is a Doob skeleton process, if $m < \infty$ and for any $x \in E$, $F(x,0) = 0$, $F(x,\infty) \equiv 1$, then we call $X(t)$ a positive recurrent Doob skeleton process.*

Theorem 2.2. *Suppose $X(t)$ is a positive recurrent Doob skeleton process. If $F(t)$ is non-grid distribution, then for any $A \in \mathcal{E}$, $\lim_{t\to\infty} P(x,t,A)$ exists:*

$$P(A) =: \lim_{t\to\infty} P(x,t,A) = \frac{\int_0^\infty \int_E h(y,t,A)\pi(dy)dt}{m}, \quad \forall A \in \mathcal{E} \tag{4}$$

and $P(\cdot)$ is the characteristic measure on $(E, \mathcal{E})$.

3. **Instantaneous Distribution of the Queue Length.** The stochastic service system is also called the crowded system and queuing system. A queuing system is called the GI/G/1 queuing system if it satisfies:

(i): The customers arrive at the reception time $\ldots \tau_{-2}, \tau_{-1}, \tau_0, \tau_1, \tau_2, \ldots$ (where τ_i ($i = \ldots, -2, -1, 0, 1, 2, \ldots$) is integer), the arrival time intervals

$$t_n = \tau_n - \tau_{n-1} \ (n \in Z = (\ldots, -2, -1, 0, 1, 2, \ldots))$$

are independently and identically distributed random variables, and their common distribution is

$$\alpha_k = P(t_n = k) \quad (k = 1, 2, \ldots)$$

(ii): The customer's service times, $\ldots, v_{-3}, v_{-2}, v_{-1}, v_0, v_1, v_2, \ldots$, are independently and identically distributed random variables taking integer values, and mutually independent with $\{t_n, n \in Z\}$. Their common distribution is

$$\beta_k = P(v_n = k) \quad (k = 1, 2, \ldots)$$

(iii): There is a waiter with first-come-first-served rules.

Let $L(n)$ be the number of customers in the queue at time n (the sum of the number of customers waiting and at the front of the service at time n), i.e., the queue length of a GI/G/1 queuing system, and $D_1(n)$ be the time from the arrival time of the last customer before time n to time n, thus $D_2(n)$ is defined as follows: if the reception is idle at time n, then $D_2(n) = 0$; else $D_2(n)$ equals the service hours completed by time n. As known, $L(n)$ may not be a Markov chain, but must be a Markov skeleton process with $0, \ldots, r_1, r_2, \ldots$ as its skeleton time sequence, where $r_1 = \min$ (the time of first customer's arrival after time 0, service is finished of the first customer after time 0).

Let

$$D_1(0) = D_1, \quad D_2(0) = D_2$$

Thus, obviously, $D_1(n)$, $D_2(n)$ are random variables taking integer values.

Let

$$\begin{aligned} h(i, D_1, D_2, j, n) &= P(L(n) = j,\, n < r_1 | L(0) = i,\, D_1(0) = D_1,\, D_2(0) = D_2) \\ q(i, D_1, D_2, s, m) &= P(r_1 = s,\, L(s) = m | L(0) = i,\, D_1(0) = D_1,\, D_2(0) = D_2) \\ P(i, D_1, D_2, j, n) &= P(L(n) = j | L(0) = i,\, D_1(0) = D_1,\, D_2(0) = D_2) \end{aligned}$$

Apparently

$i = 0$,

$$h(0, D_1, D_2, j, n) = \begin{cases} 1 - \dfrac{\sum\limits_{k=1}^{n} \alpha_{D_1+k}}{\sum\limits_{k=1}^{\infty} \alpha_{D_1+k}}, & j = 0,\ D_2 = 0 \\ 0, & j \neq 0,\ \text{or } D_2 \neq 0,\ \text{or } D_1 = 0 \end{cases}$$

$$q(0, D_1, D_2, s, m) = \begin{cases} \dfrac{\alpha_{D_1+s}}{\sum\limits_{k=1}^{\infty} \alpha_{D_1+k}}, & m = 1 \\ 0, & m > 1 \end{cases}$$

$i > 0$,

$$h(i, D_1, D_2, j, n) = \begin{cases} \left(1 - \dfrac{\sum\limits_{k=1}^{n} \alpha_{D_1+k}}{\sum\limits_{k=1}^{\infty} \alpha_{D_1+k}}\right)\left(1 - \dfrac{\sum\limits_{k=1}^{n} \beta_{D_2+k}}{\sum\limits_{k=1}^{\infty} \beta_{D_2+k}}\right), & j = i \\ 0, & j \neq i \end{cases}$$

$$q(i,D_1,D_2,s,m)I_{(1,2,\dots)}(i) = \begin{cases} \left(\dfrac{\alpha_{D_1+s}}{\sum_{k=1}^{\infty}\alpha_{D_1+k}}\right)\left(1-\dfrac{\beta_{D_2+s}}{\sum\limits_{k=1}^{\infty}\beta_{D_2+k}}\right), & m=i+1 \\ \left(1-\dfrac{\alpha_{D_1+s}}{\sum\limits_{k=1}^{\infty}\alpha_{D_1+k}}\right)\left(\dfrac{\beta_{D_2+s}}{\sum\limits_{k=1}^{\infty}\beta_{D_2+k}}\right), & m=i-1 \\ \left(\dfrac{\alpha_{D_1+s}}{\sum\limits_{k=1}^{\infty}\alpha_{D_1+k}}\right)\left(\dfrac{\beta_{D_2+s}}{\sum\limits_{k=1}^{\infty}\beta_{D_2+k}}\right), & m=i \\ 0, & \text{others} \end{cases}$$

From Theorem 2.1, we obtain:

Theorem 3.1. *$\{P(i,D_1,D_2,j,n)\}$ is the smallest non-negative solution of the following linear equation. It is also the unique bounded solution and the unique finite solution.*

$$\begin{aligned} &P(i,D_1,D_2,j,n) \\ &= h(i,D_1,D_2,j,n) + I_{(0)}(i)\sum_{s=1}^{n-1}\frac{\alpha_{D_1+s}}{\sum\limits_{k=1}^{\infty}\alpha_{D_1+k}}P(i,0,0,j,n-s) \\ &\quad + I_{(1,2,\dots)}(i)\left(\sum_{s=1}^{n-1}\left(\frac{\alpha_{D_1+s}}{\sum\limits_{k=1}^{\infty}\alpha_{D_1+k}}\left(1-\frac{\beta_{D_2+s}}{\sum\limits_{k=1}^{\infty}\beta_{D_2+k}}\right)P(i+1,0,D_2+s,j,n-s)\right.\right. \\ &\quad + \frac{\beta_{D_2+s}}{\sum\limits_{k=1}^{\infty}\beta_{D_2+k}}\left(1-\frac{\alpha_{D_1+s}}{\sum\limits_{k=1}^{\infty}\alpha_{D_1+k}}\right)P(i-1,D_1+s,0,j,n-s) \\ &\quad \left.\left. + \frac{\alpha_{D_1+s}}{\sum\limits_{k=1}^{\infty}\alpha_{D_1+k}}\frac{\beta_{D_2+s}}{\sum\limits_{k=1}^{\infty}\beta_{D_2+k}}P(i,0,0,j,n-s)\right)\right) \end{aligned} \tag{5}$$

4. **Busy Period.** The busy period means: when a customer arrives at the idle desk, the busy period begins, and lasts until the service desk becomes idle once again which means the desk busy period has been finished.

Let $A(x)$ and $B(x)$ represent the distribution of the arrival time interval and the service time for each customer, respectively, and thus we have

$$\left.\begin{aligned} A(x) &= \sum_{k=1}^{[x]}\alpha_k \\ B(x) &= \sum_{k=1}^{[x]}\beta_k, \end{aligned}\right. \qquad [x] \text{ is an integer not greater than } x$$

Let

$$\lambda = \frac{1}{\int_{0-}^{\infty} x dA(x)} = \frac{1}{E(t_n)}$$

$$\begin{aligned}
\mu &= \frac{1}{\int_{0-}^{\infty} x dB(x)} = \frac{1}{E(\nu_n)} \\
\rho &= \frac{\lambda}{\mu} \\
a_k &= \int_{0-}^{\infty} \left[1 - A^{(k)}(x)\right] dB^{(k)}(x)
\end{aligned}$$

where $A^{(k)}(x)$ and $B^{(k)}(x)$ represent k fold convolution of $A(x)$ and $B(x)$, respectively.

Let N denote the number of customers served in a busy period, and $\bar{D}$ represent the average length of the busy period; thus through [17] (there is no other assumptions but $\frac{1}{\lambda} < +\infty$, $\frac{1}{\mu} < +\infty$ in the results presented in the paper), we get:

Theorem 4.1. *If* $\rho = \frac{\lambda}{\mu} < 1$,

$$EN = \exp\left\{\sum_{k=1}^{\infty} \frac{1-a_k}{k}\right\} < \infty \tag{6}$$

$$\bar{D} = \frac{1}{\mu} \exp\left\{\sum_{k=1}^{\infty} \frac{1-a_k}{k}\right\} < \infty \tag{7}$$

5. **Stationary Distribution of Queue Length.** Let $\xi_0 = 0$, ξ_i $(i = 1, 2, \ldots)$ denote the time of the i-th occurrence of the state $(i, 0, 0)$. Apparently, $(L(n), D_1(n), D_2(n))$ is a Markov skeleton process with (ξ_n) as the skeleton time sequence, and it is a Doob skeleton process as well. There exists a characteristic measure:

$$\begin{aligned}
\pi(i, D_1, D_2) &= P(L(\xi_n) = i,\ D_1(\xi_n) = D_1,\ D_2(\xi_n) = D_2) \\
&= \begin{cases} 1, & i = 1,\ D_1 = 0,\ D_2 = 0 \\ 0, & \text{otherwise} \end{cases} \\
F(n) &= P(\xi_2 - \xi_1 \le n)
\end{aligned}$$

Let

$$\lambda = \frac{1}{E(t_n)}, \quad \mu = \frac{1}{E(\nu_n)}, \quad \rho = \frac{\lambda}{\mu}$$

Thus, we have the following results directly from the results presented in [18-20].

Lemma 5.1. $(\xi_2 - \xi_1)$ *is non-grid distribution if and only if* (α_k) *is non-grid distribution.*

By Theorems 2.2 and 4.1, we obtain

Theorem 5.1. *If* $\rho = \frac{\lambda}{\mu} < 1$, (α_k) *and* (β_k) *are not one-point distributions, and* (α_k) *is a non-grid distribution, then* $\lim_{n\to\infty} P(i, D_1, D_2, j, n) = P_j$, *and* $(P_j)_{0 \le j < +\infty}$ *is a probability distribution.*

6. **Applications in Transportation Management.** Suppose in a ticket window of a train station, the distribution of the customers' arrival time is assumed to be

$$\alpha_k = \begin{cases} \dfrac{\delta}{100}, & k = 1 \\ \dfrac{100-\delta}{100}, & k = 2, \quad 0 < \delta < 100 \\ 0, & k \ge 3 \end{cases}$$

and the distribution of the customer's service time is $\beta_k = 1$ (time unit for three minutes). Now the busy period of this system can be calculated using the results in Section 4:

$$E\alpha_k = \frac{\delta}{100} \cdot 1 + \frac{100-\delta}{100} \cdot 2 + 0 = \frac{200-\delta}{100}$$

$$
\begin{aligned}
\lambda &= \frac{100}{200-\delta},\ \mu = 1 \\
\rho &= \frac{100}{200-\delta} < 1,\ 1 < \delta < 100 \\
a_k &= \int_{0-}^{\infty} \left[1 - A^{(k)}(x)\right] dB^{(k)}(x) \\
&= \sum_{x=0}^{\infty} P(t_1 + t_2 + \cdots + t_k > x) P(v_1 + v_2 + \cdots + v_k = x) \\
&= P(t_1 + t_2 + \cdots + t_k > k) \cdot 1 \\
&= 1 - P(t_1 + t_2 + \cdots + t_k \le k) \\
&= 1 - P(t_1 + t_2 + \cdots + t_k = k) \\
&= 1 - \left(\frac{\delta}{100}\right)^k
\end{aligned}
$$

Let

$$a_k = 1 - x^k, \quad 0 < x < 1$$

Thus,

$$\int_0^x \left(\sum_{k=1}^{\infty} \frac{x^k}{k}\right)' dx = \int_0^x \sum_{k=1}^{\infty} x^{k-1} dx = \int_0^x \frac{1}{1-x} dx = \int_0^x \frac{1}{1-t} dt = -\ln(1-x)$$

$$EN = \exp\left\{\sum_{k=1}^{\infty} \frac{1-a_k}{k}\right\} = \exp\{-\ln(1-x)\} = \frac{1}{1-x}, \quad EN = \frac{100}{100-\delta}$$

Similarly,

$$\bar{D} = \frac{1}{\mu} \exp\left\{\sum_{k=1}^{\infty} \frac{1-a_k}{k}\right\} = \frac{1}{1-x}, \quad \bar{D} = \frac{100}{100-\delta}$$

If $\delta = 99$,

$$EN = 100, \quad \bar{D} = 100$$

The busy period is 300 minutes ($EN \times 3$).

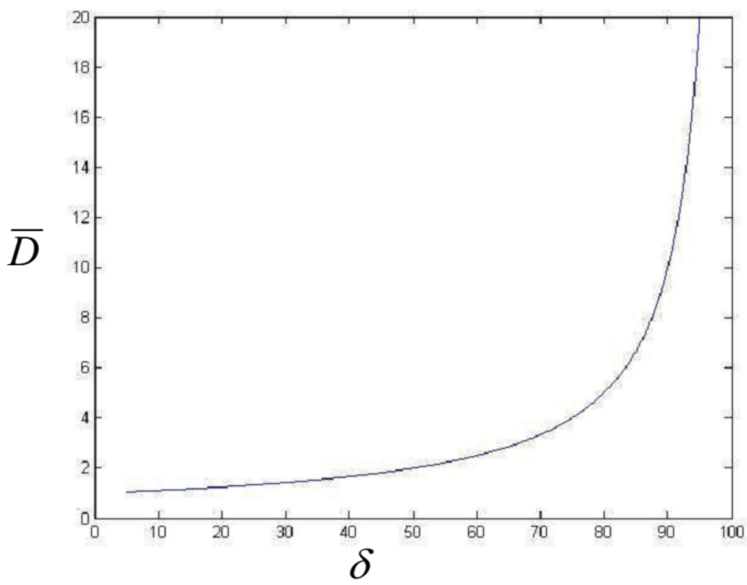

FIGURE 1. $\bar{D}$ against δ

7. Conclusions.

(1) Using Markov skeleton processes, we analyze the discrete-time queuing systems without putting any restriction, which is necessary in previous researches by using other methods, on the distribution of the arrival time interval of customer or the customer service time. In our work, we obtain (a) the instantaneous distribution of the queue length; (b) the stationary distribution of the queue length; and (c) the busy period.

(2) The train-ticketing system is taken as an example to illustrate our results, which is a reference for transportation and management systems.

REFERENCES

[1] X. Chang and D. W. Choi, Performance analysis of a finite-buffer discrete-time queue with bulk arrival, bulk service and vacations, *Computers & Oper. Res.*, pp.365-407, 2005.

[2] M. L. Chaudhry and U. Gupta, Performance analysis of the discrete-time GI/Geom/1/N queue, *J. Appl. Prob.*, vol.33, no.1, pp.239-255, 1996.

[3] A. S. Alfa, Vacation models in discrete-time, *Queueing Systems*, vol.44, pp.5-30, 2003.

[4] N. Tian, *Discrete-Time Queuing Theory*, Science Press, Beijing, 2008 (in Chinese).

[5] J. Li and N. Tian, The discrete-time GI/Geom/1 queue with working vacations and vacation interruption, *Applied Mathematics and Computer*, vol.185, no.1, pp.1-10, 2007.

[6] J. Li and N. Tian, Analysis of the discrete time Geo/Geo/1 queue with single working vacation, *Quality Technology and Quantitative Management*, vol.5, no.1, pp.77-89, 2008.

[7] J. Li, N. Tian and W. Liu, The discrete-time GI/Geo/1 queue with multiple working vacations, *Queueing Systems*, vol.56, no.1, pp.53-63, 2007.

[8] N. Tian, M. Liu, Z. Ma and X. Xu, Performance analysis of discrete-time Erlang loss system, *Systems Engineering and Electronics*, vol.28, no.6, pp.623-626, 2006 (in Chinese).

[9] M. Liu, Z. Ma and N. Tian, Performance of admission control for multi-traffic in wireless communication network based on discrete time queue, *Journal on Communications*, vol.27, no.7, pp.100-105, 2006 (in Chinese).

[10] Z. Ma, S. Jin and M. Liu, Modeling of discrete-time gated service system, *Proc. of 2006 Chinese Control and Decision Conference*, Tianjin, China, pp.265-268, 2006 (in Chinese).

[11] Z. Ma and N. Tian, Discrete-time Geom/G/1 queueing system with single vacation and gated service discipline, *Or Transactions*, vol.11, no.4, pp.70-76, 2007 (in Chinese).

[12] Z. Hou and G. Liu, *Markov Skeleton Processes and Their Application*, Science Press, Beijing, 2005.

[13] Z. Hou, X. Zhang and X. Kong, A new analytical algorithm for computing probability distribution of project completion time, *J. Cent. South Univ. Technol.*, vol.17, pp.1006-1010, 2010.

[14] B. Feyaerts, S. Vuyst, H. Brunneel and S. Wittevrongel, The impact of the NT-policy on the behaviour of a discrete-time queue with general service times, *Journal of Industrial & Management Optimization*, vol.10, no.1, pp.131-149, 2014.

[15] H. Bruneel and T. Maertens, A discrete-time queue with customers with geometric deadlines, *Performance Evaluation*, vols.85-86, pp.52-70, 2015.

[16] J. Wu, J. Wang and Z. Liu, A discrete-time Geo/G/1 retrial queue with preferred and impatient customers, *Applied Mathematical Modelling*, vol.37, pp.2552-2561, 2013.

[17] P. D. Finch, On the busy period in the queuing system GI/G/1, *J. Aust. Math. Soc.*, vol.2, pp.217-228, 1961.

[18] S. Asmussen, *Applied Probability and Queues*, John Wiley, New York, 1987.

[19] H. Dong, Z. Hou and G. Jiang, Limit distribution of Markov skeleton processes, *Acta Mathematicae Applicatae Sinica*, vol.33, no.2, pp.290-296, 2010 (in Chinese).

[20] H. Dong, *Markov Skeleton Process Limit Theory*, Ph.D. Thesis, Central South University, 2008 (in Chinese).