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NEIMARK-SACKER BIFURCATION IN A NEW THREE-DIMENSIONAL DISCRETE CHAOTIC SYSTEM

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Abstract. *A new three-dimensional discrete chaotic system is proposed by using forward-Euler scheme. The dynamics of this system is considered, and the existence and stability of fixed points are also discussed. Based on explicit Neimark-Sacker bifurcation criterion, center manifold theory and normal form theory, the existence, stability and direction of Neimark-Sacker bifurcation are studied. Finally, a numerical example is provided for justifying the validity of the theoretical analysis.*
Keywords: Stability, Center manifold theorem, Forward-Euler scheme, Neimark-Sacker bifurcation

1. **Introduction.** The dynamical system refers to the dynamic system of change over time, which includes continuous dynamical systems and discrete dynamical systems. There are many scholars focusing on the continuous system's bifurcation, but the research about bifurcation of discrete systems is relatively few. Compared with the continuous system, the discrete systems possess their unique dynamic characteristics. In the real life, many practical problems can be depicted by the discrete systems, and we can also discretize the continuous systems. Therefore, the study of discrete system is very important and achieved great development in the field of mathematics, physics and engineering. Hu et al. [1] obtained an epidemic model by using Euler difference method, and discussed the Neimark-Sacker bifurcation of the system based on the center manifold theorem and the bifurcation theory. Wang and Feng [2] used the Euler difference method to get a discrete BVP oscillator, and studied the existence and stability of the fixed points. Xin et al. [3] proposed a financial system by using forward-Euler difference method, and investigated the Neimark-Sacker bifurcation of the system based on normal form method, center manifold theory and Neimark-Sacker bifurcation theory. Elabbasy et al. [4] focus on a two-dimensional discrete Lorenz system, and mainly studied the Pitchfork bifurcation, Flip bifurcation and Neimark-Sacker bifurcation of the system. At present, the researches on the bifurcation of discrete systems are mostly concentrated in the two-dimensional systems [5-10], and there are few studies focusing on the three dimensional discrete system.

Lei and Wang [11] proposed a new three-dimensional continuous system as follows:

$$\begin{cases} \dot{x} = a(y-x), \\ \dot{y} = cx - dy - xz, \\ \dot{z} = x^2 - bz, \end{cases} \tag{1}$$

where $(x, y, z) \in R^3$ are the state variables and $(a, b, c, d) \in R^4$ are real parameters. We use the forward-Euler difference method to the system (1), and obtain a three-dimensional discrete system as follows:

$$\begin{cases} x_{n+1} = x_n + ha\,(y_n - x_n) \\ y_{n+1} = y_n + h\,(cx_n - dy_n - x_n z_n) \\ z_{n+1} = z_n + h\,(x_n^2 - bz_n) \end{cases} \tag{2}$$

where $h\,(0 < h < 1)$ is the step size, and $(a, b, c, d) \in R^4$ are real parameters. In this paper, we will study the existence and stability of the fixed points of system (2), and the existence, stability and direction of Neimark-Sacker bifurcation.

The paper organizes as follows. We discuss the existence and local stability of equilibria in system (2) in Section 2. In Section 3, we study Neimark-Sacker bifurcation for system (2) by choosing h as a bifurcation parameter. We present the numerical simulations illustrating our results with the theoretical analysis in Section 4. In Section 5, we conclude the paper.

2. **Existence and Stability of the Fixed Points.**

The fixed points of system (2) satisfy:

$$\begin{cases} x_n = x_n + ha\,(y_n - x_n) \\ y_n = y_n + h\,(-x_n z_n + cx_n - dy_n) \\ z_n = z_n + h\,(x_n^2 - bz_n) \end{cases} \tag{3}$$

Lemma 2.1. *(1) For any parameters, the system (2) has only one fixed point:* $E_0 = (0, 0, 0)$; *(2) If* $b\,(c - d) > 0$, *the three-dimensional discrete system (2) has three fixed points:* $E_0 = (0, 0, 0)$, $E_\pm = \left(\pm\sqrt{b\,(c - d)}, \pm\sqrt{b\,(c - d)}, c - d\right)$.

Next, we study the stability of E_0 and $E_\pm$. The Jacobian matrix at $E = (x^*, y^*, z^*)$ is

$$J\,(E) = \begin{bmatrix} 1 - ha & ha & 0 \\ -hz^* + hc & 1 - hd & -hx^* \\ 2hx^* & 0 & 1 - hb \end{bmatrix} \tag{4}$$

From the local stability theory of fixed point, it is easy to obtain the following lemmas.

Lemma 2.2. *(1) If one of the following conditions is satisfied, the fixed point E_0 is local asymptotically stable:*

(a) $(a - d)^2 + 4ac \geq 0$, $0 < h < \min\left\{\frac{2}{b}, \frac{a+d+\sqrt{(a-d)^2+4ac}}{ad-ac}, \frac{a+d-\sqrt{(a-d)^2+4ac}}{ad-ac}\right\}$;

(b) $(a - d)^2 + 4ac < 0$, $0 < h < \frac{2}{b}$, $(ad - ac)\,h < a + d$.

(2) If one of the following conditions is satisfied, the fixed point E_0 is not stable:

(a) $(a - d)^2 + 4ac \geq 0$, $h > \frac{2}{b}$, $\left(a + d + \sqrt{(a - d)^2 + 4ac}\right) h < 0$ *or* $\left(a + d + \sqrt{(a - d)^2 + 4ac}\right) h > 4$, $\left(a + d - \sqrt{(a - d)^2 + 4ac}\right) h < 0$ *or* $\left(a + d - \sqrt{(a - d)^2 + 4ac}\right) h > 4$;

(b) $(a - d)^2 + 4ac < 0$, $(ad - ac)\,h > a + d$, $hb < 0$ *or* $hb > 2$.

(3) If one of the following conditions is satisfied, the fixed point E_0 is non-hyperbolic point:

(a) $(a - d)^2 + 4ac \geq 0$, $h = \frac{2}{b}$, *or* $h = \frac{a+d+\sqrt{(a-d)^2+4ac}}{ad-ac}$, *or* $h = \frac{a+d-\sqrt{(a-d)^2+4ac}}{ad-ac}$;

(b) $(a - d)^2 + 4ac < 0$, $h = 2/b$, $h = (a + d)/(ad - ac)$.

Lemma 2.3. *(1) If* $0 < h\,(a + b + d) < 6$, $0 < 2ab\,(c - d)\,h^2 - (ab + bd)\,h + (a + b + d) < 2$, *the fixed point E_+ is local asymptotically stable;*

(2) If $h\,(a + b + d) > 6$ *or* $h\,(a + b + d) < 0$, $2ab\,(c - d)\,h^2 - (ab + bd)\,h + (a + b + d) > 2$ *or* $2ab\,(c - d)\,h^2 - (ab + bd)\,h + (a + b + d) < 0$, *the fixed point E_+ is unstable.*

3. **Neimark-Sacker Bifurcation Analysis.** When $(a-d)^2+4ac<0$, the eigenvalues of system (2) can be written as

$$\lambda_{1,2}=(2-ha-hd)/2\pm ih\sqrt{-(a-d)^2-4ac}\Big/2,\quad \lambda_3=1-hb. \tag{5}$$

Assume that $h_0=(a+d)/(ad-ac)$, and $b(a+d)/a(d-c)\neq 0,2,a(d-c)\neq 0$, and we can get:

$$|\lambda_{1,2}(h_0)|=\sqrt{1-(a+d)h_0+(ad-ac)h_0^2}=1,\quad |\lambda_3(h_0)|=\left|1-\frac{b(a+d)}{a(d-c)}\right|. \tag{6}$$

$$d(|\lambda_{1,2}|)/dh|_{h_0=a+d/a(d-c)}=\frac{a+d}{2}\neq 0, \tag{7}$$

$$\lambda_{1,2}(h_0)=1-\frac{(a+d)^2}{2a(d-c)}\pm\frac{(a+d)\sqrt{-(a-d)^2-4ac}}{2a(d-c)}i,\quad \lambda_3(h_0)=1-\frac{b(a+d)}{a(d-c)}. \tag{8}$$

And by calculation we can get $\lambda_{1,2}^m(h_0)\neq 1,2,3,4$. According to Neimark-Sacker bifurcation theory [12], the Neimark-Sacker bifurcation occurs at the fixed point E_0.

Next, we will analyze the stability and direction of the Neimark-Sacker bifurcation. First, the system (2) can be written as

$$X_{n+1}=JX_n+\frac{1}{2}B(X_n,X_n)+\frac{1}{6}C(X_n,X_n,X_n)+O(X_n^4), \tag{9}$$

where J is the Jacobin matrix at the fixed point E_0, $O(X_n^4)$ is the 4 order indefinite small of X_n. And for $i=1,2,3$, we can get:

$$B_i(x,y)=\sum_{j,k=1}^{3}\left.\frac{\partial^2X_i(\xi,0)}{\partial\xi_j\partial\xi_k}\right|_{\xi=0}x_jy_k,\quad C_i(x,y,z)=\sum_{j,k,l=1}^{3}\left.\frac{\partial^3X_i(\xi,0)}{\partial\xi_j\partial\xi_k\partial\xi_l}\right|_{\xi=0}x_jy_kz_l, \tag{10}$$

Let $p,q\in C^3$ be vectors such that:

$$Jq=\lambda_1q,\quad J^{\mathrm{T}}p=\lambda_2p,\quad \langle p,q\rangle=\sum_{i=1}^{3}\bar{p}_iq_i=1, \tag{11}$$

where J^{T} is the transpose of the J, and λ_1, λ_2 is a pair of complex conjugate eigenvalues at the fixed point E_0. For the system (2), we can get

$$B(x,y)=\left(0,-\frac{a+d}{a(d-c)}(x_1y_3+x_3y_1),\frac{2(a+d)}{a(d-c)}x_1y_1\right)^{\mathrm{T}},\quad C(x,y,z)=(0,0,0)^{\mathrm{T}}, \tag{12}$$

$$q=\left(\frac{d-a}{2c}+\frac{\sqrt{-(a-d)^2-4ac}}{2c}i,1,0\right)^{\mathrm{T}},\quad p=(\xi_3+\xi_4i,\xi_1+\xi_2i,0)^{\mathrm{T}}, \tag{13}$$

where

$$\xi_1=\frac{(a-d)^2+3ac}{2(a-d)^2+9ac},\quad \xi_2=\frac{(a-d)\sqrt{-(a-d)^2-4ac}}{2(a-d)^2+9ac},$$

$$\xi_3=\frac{d-a}{2a}\xi_1+\frac{\sqrt{-(a-d)^2-4ac}}{2a}\xi_2,\quad \xi_4=\frac{d-a}{2a}\xi_2-\frac{\sqrt{-(a-d)^2-4ac}}{2a}\xi_1.$$

Thus, we can obtain

$$g_{20}=\langle p,B(q,q)\rangle=0,\quad g_{11}=\langle p,B(q,\bar{q})\rangle=0,\quad g_{02}=\langle p,B(\bar{q},\bar{q})\rangle=0,$$

$$g_{21} = \langle p, C(q,q,\bar{q})\rangle + 2\left\langle p, B\left(q,(I_n-J)^{-1}B(q,\bar{q})\right)\right\rangle + \left\langle p, B\left(\bar{q},\left(\lambda_1^2 I_n - J\right)^{-1}B(q,q)\right)\right\rangle$$
$$+\frac{\lambda_2(1-2\lambda_1)}{1-\lambda_1}g_{20}g_{11} + \frac{2}{1-\lambda_1}|g_{11}|^2 + \frac{\lambda_1}{\lambda_1^3-1}|g_{02}|^2$$
$$= \frac{(a+d)^2(d-a)}{abc^2h(d-c)^2}\xi_1 + \frac{(a+d)^2\sqrt{-(a-d)^2-4ac}}{abc^2h(d-c)^2}\xi_2 + \xi_1\varphi_1 - \xi_2\varphi_2$$
$$+\left[\frac{(a+d)^2\sqrt{-(a-d)^2-4ac}\xi_1 - (a+d)^2(d-a)\xi_2}{abc^2h(d-c)^2} + \xi_2\varphi_1 + \xi_1\varphi_2\right]i,$$

where

$$\omega_1 = \frac{A_1(B_1-1+hb)+A_2B_2}{(B_1-1+hb)^2-B_2^2},\ \omega_2 = \frac{A_2(B_1-1+hb)-A_2B_2}{(B_1-1+hb)^2-B_2^2},$$
$$A_1 = \frac{(a-d)^2(a+d)+2ac(a+d)}{a^2c^2(d-c)},\ A_2 = \frac{(d-a)^2\sqrt{-(a-d)^2-4ac}}{2ac^2(d-c)},$$
$$B_1 = \frac{2+h^2(a^2+d^2+2ac)-2h(a+d)}{2},\ B_2 = \frac{h(2-ha-hd)\sqrt{-(a-d)^2-4ac}}{2},$$
$$\varphi_1 = \frac{(a^2-d^2)\omega_1 - \omega_2(a+d)\sqrt{-(a-d)^2-4ac}}{2ac(d-c)},$$
$$\varphi_2 = \frac{(a^2-d^2)\omega_2 + \omega_1(a+d)\sqrt{-(a-d)^2-4ac}}{2ac(d-c)}.$$

Then, by calculation one has

$$l_1(h_0) = \mathrm{Re}\left(\frac{\lambda_2 g_{21}}{2}\right) - \mathrm{Re}\left(\frac{\lambda_2^2(1-2\lambda_1)}{2(1-\lambda_1)}g_{20}g_{11}\right) - \frac{1}{2}|g_{11}|^2 - \frac{1}{4}|g_{02}|^2 = \frac{\delta}{4abc^2h_0(d-c)^2}, \tag{14}$$

where

$$\delta = (2-h_0a-h_0d)\left[4abc^2h_0(d-c)^2(\xi_1\varphi_1-\xi_2\varphi_2) + (a+d)^2\sqrt{-(a-d)^2-4ac}\xi_2 + (a+d)^2(d-a)\xi_1\right] + h_0\sqrt{-(a-d)^2-4ac}\left[(a+d)^2\sqrt{-(a-d)^2-4ac}\xi_1 - (a+d)^2(d-a)\xi_2 + 4bc^2h_0(d-c)^2(\xi_2\varphi_1-\xi_1\varphi_2)\right].$$

Theorem 3.1. *The direction and stability of Neimark-Sacker bifurcation at the fixed point E_0 are determined by $l_1(h_0)$. If $l_1(h_0) < 0\ (> 0)$, the Neimark-Sacker bifurcation of system (2) at the fixed point $h_0 = (a+d)/(ad-ac)$, is supercritical (subcritical), and the unique closed invariant curve bifurcating from E_0 is asymptotically stable (unstable).*

4. **Numerical Simulation.** In order to justify the theoretical analysis, we choose one group parameters: $a = -0.1$, $b = 2$, $c = 4.7$, $d = 0.3$ and we can get the critical value of Neimark-Sacker bifurcation $h_0 = 0.4545$. We fixed the parameter $a = -0.1$, $b = 2$, $c = 4.7$, $d = 0.3$, and draw the global bifurcation diagram when $h \in [0.2, 1]$, as shown in Figure 1. From Figure 1, we see that the equilibrium E_0 is stable when $h < 0.4545$, and loses its stability when $h = 0.4545$. And there is a Neimark-Sacker bifurcation when $h > 0.4545$. Figure 2 and Figure 3 show the time history and phase portraits of system (2) for different values of h corresponding to Figure 1. The fixed point is stable when

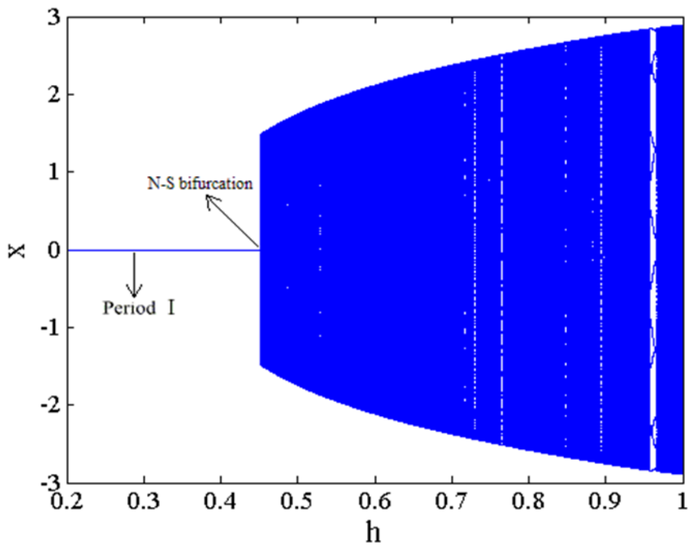

FIGURE 1. Bifurcation diagram in the (x, h) plane for $a = -0.1$, $b = 2$, $c = 4.7$, $d = 0.3$

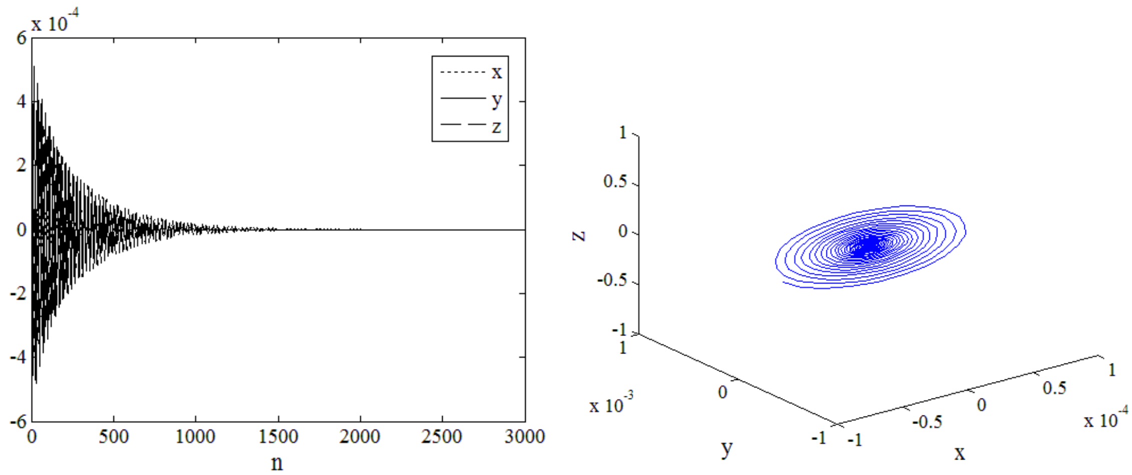

FIGURE 2. Time history and phase diagram of system (2) with $h = 0.3$

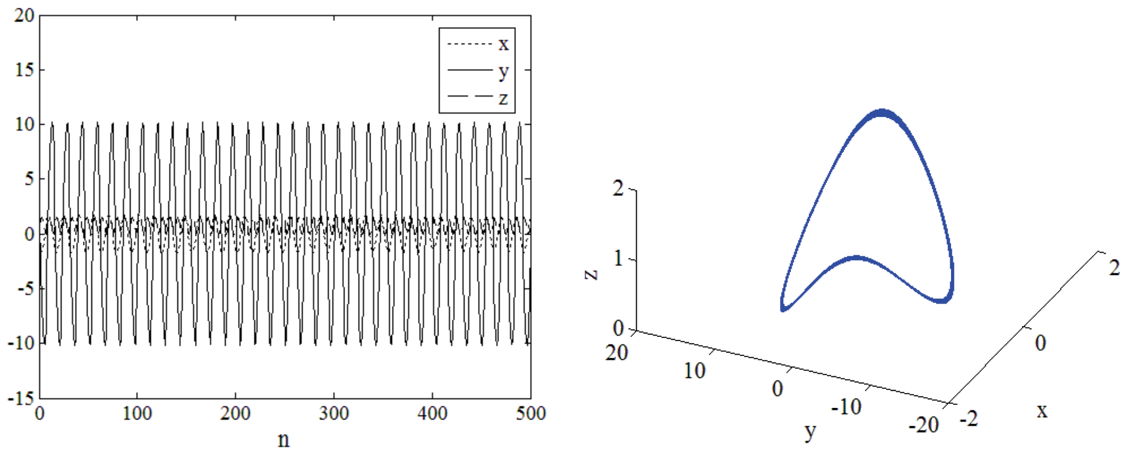

FIGURE 3. Time history and phase diagram of system (2) with $h = 0.6$

$h = 0.3 < h_0$, and unstable when $h = 0.6 > h_0$, as shown in Figure 2 and Figure 3, respectively. Based on the previous conclusions, and through complex calculations, we can get $l_1(h_0) = 0.0528 > 0$, and thus the Neimark-Sacker bifurcation of system (2) at the fixed point E_0 is subcritical, and the unique invariant curve which is resulting from the bifurcation at fixed point is unstable.

5. **Conclusions.** In this paper, we use the forward-Euler difference method to construct a new three-dimensional discrete chaotic system, and study the complex dynamical properties of the system. The existence and stability of the fixed points are considered. And the existence, stability and direction of Neimark-Sacker bifurcation are investigated in detail by using the center manifold theory and normal form theory. Finally, numerical results are given to illustrate the correctness of theoretical analysis. The study shows that the discrete systems have rich dynamic characteristics, and possess their unique properties compared with continuous systems. Therefore, the study of dynamics characteristic of discrete system has a great significance both in theory and in engineering application. Apparently there are more interesting problems about this discrete system in terms of complexity, control and synchronization, codimension-two bifurcation, which deserve further investigation.

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