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A SYSTEM WITH THE WHOLE SPACE BEING WEAKLY MIXING BUT NOT DISTRIBUTIONALLY CHAOTIC

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Abstract. *As we all know, the whole space could not be a distributionally scrambled set in a compact dynamical system. However, in a noncompact dynamical system, there were examples which are weakly mixing and the whole space is a distributionally scrambled set. Meanwhile, in this paper, we construct a noncompact metric space Y with the whole space being weakly mixing but not distributionally chaotic. Moreover, the orbit of each point $x \in Y$ is dense in Y.*
Keywords: Weakly mixing, Distributionally chaotic, Dense

1. **Introduction.** The complexity of dynamical systems can be well described by chaos. Maps with different properties (such as weakly mixing, mixing) may generate different chaotic behaviors. For example, in a compact system (X, f), if f is weakly mixing, then f is distributional chaos in a sequence. We are interested in the relationship between weakly mixing property and distributional chaos. In this paper, we construct a noncompact metric space Y with the whole space being weakly mixing but not distributionally chaotic. And, the orbit of each point $x \in Y$ is dense in Y.

This paper is organized as follows. In Section 1, we briefly introduce our main purpose and result. In Section 2, we summarize the recent progress in the study of distributional chaos, and introduce some basic definitions which will be used in this paper. In Section 3, we construct a noncompact system, and prove that the system is weakly mixing, the orbit of each point is dense in the system, but no distributionally chaotic pairs in the whole space. In Section 4, we draw a conclusion: there exists a weakly mixing system which does not exhibit distributional chaos no matter whether the system is compact or noncompact.

2. **Problem Statement and Preliminaries.** Li and Yorke first put forward mathematical definition of chaos in 1975 [1]. Since then, chaos has had a more and more important status in the research of the dynamical systems. The experts gave many definitions of chaos from different angles, such as distributional chaos [2], distributional chaos in a sequence [3], and Kato chaos [4]. Among them, distributional chaos is a very important concept which is introduced by Schweizer and Smital. Distributional chaos is defined by scrambled sets, and the scrambled sets with special properties imply additional characteristics of chaos. For example, Oprocha investigated some kinds of special distributionally scrambled sets, such as extremal distributionally scrambled sets, transitive distributionally scrambled sets in [5, 6]. So the size and property of scrambled set are crucial parts in the research of distributional chaos.

It is known that the whole space cannot be distributionally chaotic in a compact dynamical system [7]. However, we can find many examples such that the whole space is a distributionally scrambled set in a noncompact dynamical system. Wang proved that the whole space can be an extremal distributionally scrambled set in a weakly mixing symbolic dynamical system [8]. Martinez-Gimenze et al. gave an example of linear operator on Banach space which is weakly mixing and the whole space is a transitive distributionally scrambled set [9]. However, in [10], Oprocha inductively constructed a family of sequences $\{w_n\}_{n=0}^{\infty}$, the limit of the sequences is x, and he proved that the compact system (X, σ) (X is the closure of the orbit of x, σ is the shift map) is weakly mixing, and does not exhibit distributional chaos. In this paper, we use a different method to construct a noncompact system. And we prove that the noncompact system is weakly mixing but not distributionally chaotic.

2.1. **Several definitions.** Throughout this paper, we always suppose that X is a metric space with metric d, and $f : X \to X$ is a continuous map.

Definition 2.1. *Suppose (X, f) is a dynamical system. (X, f) is called a compact system, if X is compact.*

Definition 2.2. *Suppose (X, f) is a dynamical system. A nonempty subset $Y \subset X$ is called an invariant set, if Y satisfies $f(Y) \subset Y$.*

Definition 2.3. *Suppose $x \in X$. A point $y \in X$ is called a ω-limit point of x, if there exists a sequence of positive integers $\{n_i\}$ such that $f^{n_i}(x) \to y (n_i \to \infty)$. A set consisting of all the ω-limit points of x is called a ω-limit set of x, denoted by $\omega_f(x)$.*

Definition 2.4. *A set $S \subset X$ (containing at least two points) is called a distributionally scrambled set, if for all $x, y \in S$, $x \neq y$, such that*

(1) $F^*_{xy}(t) = \limsup_{n\to\infty} \frac{1}{n} \sharp\{i \mid d(f^i(x), f^i(y)) < t, 0 \le i \le n-1\} = 1$ *for all* $t > 0$,

(2) $F_{xy}(\epsilon) = \liminf_{n\to\infty} \frac{1}{n} \sharp\{i \mid d(f^i(x), f^i(y)) < \epsilon, 0 \le i \le n-1\} = 0$ *for some* $\epsilon > 0$.

where $\sharp C$ is the cardinal number of C. And the pair of points (x, y) is called distributionally chaotic pair. f is called distributional chaotic, if it has an uncountable distributionally scrambled set.

Definition 2.5. *A map f is called weakly mixing, if for any four nonempty open subsets I_1, I_2, K_1, K_2 of X, there exists a positive integer n such that $f^n(I_1) \cap K_1 \neq \emptyset$ and $f^n(I_2) \cap K_2 \neq \emptyset$.*

Definition 2.6. *Suppose $S = \{0, 1\}$, $\Sigma_2 = \{x = x_0x_1x_2 \ldots \mid x_i \in S, i = 0, 1 \ldots\}$. The shift map $\sigma : \Sigma_2 \to \Sigma_2$ is defined by: $\sigma(x) = x_1x_2 \ldots$ for any $x = x_0x_1x_2 \ldots \in \Sigma_2$.*

3. Main Result.

Theorem 3.1. *There exists a noncompact dynamical system with the whole space being weakly mixing but not distributionally chaotic. And, the orbit of each point is dense in the system.*

Proof: The first step is that we construct a weakly mixing dynamical system.

In this paper, the metric d of (Σ_2, d) is defined as follows: for any $x = x_0x_1x_2 \ldots, y = y_0y_1y_2 \ldots \in \Sigma_2$,

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{m+1} & \text{if } x \neq y \text{ and } m = \min\{i | x_i \neq y_i\} \end{cases}$$

As we all know, σ is continuous and (Σ_2, σ) is a compact dynamical system.

For any $x = x_0x_1x_2 \ldots \in \Sigma_2$, the finite sequence from the $(i+1)$th symbol to the $(j+1)$th symbol of x is denoted by $x[i,j]$, i.e., $x[i,j] = x_ix_{i+1}\ldots x_j$. If H is a finite sequence, then $|H|$ is its length.

We suppose that A is a finite sequence with length $m+1$, denoted by $A = x_0x_1\ldots x_m$. For $0 \le i \le j \le m$, we similarly define $A[i,j]$. If B is another finite sequence, AB is defined the new sequence that directly combines A with B. If B appears in A, i.e., there exist i and j such that $B = A[i,j]$, we denote it by $B \prec A$. We define $[A] = \{y \in \Sigma_2 \mid y_i = x_i, 0 \le i \le m\}$. If there exists some $i \ge 0$ such that $B = A[i,m]$ (i.e., B appears at the tail of A), we denote it by $B \rhd A$. For convenience, we use $\emptyset$ to denote the empty sequence of length 0, and assume that $\emptyset \rhd A$ for any finite sequence A.

At first, we suppose that $\{(a_n^1, a_n^2, a_n^3, a_n^4)\}_{n=1}^{\infty}$ is a sequence of arrangements of four nonnegative integers. It is constructed as follows:

(1) $(a_1^1, a_1^2, a_1^3, a_1^4) = (0,0,0,0)$;
(2) $(a_2^1, a_2^2, a_2^3, a_2^4), \ldots, (a_{17}^1, a_{17}^2, a_{17}^3, a_{17}^4)$ are exactly all the 16 arrangements with $a_n^i \in \{0,1\}$, $i = 1,2,3,4$, $n = 2,3,\ldots,17$;
(3) $(a_{18}^1, a_{18}^2, a_{18}^3, a_{18}^4), \ldots, (a_{98}^1, a_{98}^2, a_{98}^3, a_{98}^4)$ are exactly all the 81 arrangements with $a_n^i \in \{0,1,2\}$, $i = 1,2,3,4$, $n = 18,19,\ldots,98$;
(4) $\ldots$

From the construction above, we can see that for any arrangements of four nonnegative integers (i,j,l,k), there exist infinitely many n such that $(a_n^1, a_n^2, a_n^3, a_n^4) = (i,j,l,k)$.

We use u to denote the one-sided infinite sequence that contains only one symbol 0, that is $u = 000\ldots \in \Sigma_2$. Similarly, $v = 111\ldots \in \Sigma_2$.

Now, we construct a family of finite sequences $\{P_n\}_{n=1}^{\infty}$. Let P_1 be the sequence which only has one symbol 0, that is $P_1 = 0$. And we define $p_1 = |P_1|$. Then for any $n \ge 1$, we inductively define P_{n+1} and p_{n+1} as follows:

(1) At first, let $E_n = P_nP_n$ and $F_n^i = E_n[a_n^i, a_n^i + p_n - 1]$, $i = 1,2,3,4$;
(2) Next, let $G_n = F_n^1F_n^2F_n^3F_n^4J_n$, where $J_n = v[0, 6p_n - 1]$;
(3) At last, define $P_{n+1} = P_nP_nG_n$ and $p_{n+1} = |P_{n+1}|$.

Thus, for any positive integer n, the finite sequence P_n has been defined. Obviously, for any positive integer $n \ge 1$, the sequence P_{n+1} begins with P_n. Therefore, when $n \to \infty$, the finite sequence P_n extends to one-sided infinite sequence $w \in \Sigma_2$.

Let $Y = \{x \in \Sigma_2 \mid x = DT_sG_sT_{s+1}G_{s+1}\ldots T_nG_n\ldots,$ where $s \ge 1, D \rhd P_s, T_n = P_n$ or $\phi, \forall n \ge s\}$. Obviously, Y is an uncountable set.

For any $x = DT_sG_sT_{s+1}G_{s+1}\ldots T_nG_n\ldots \in Y$, obviously, $v = 111\ldots \in \omega_\sigma(x)$, but $v \notin Y$. Therefore, (Y,σ) is a noncompact dynamical system.

For any $x = DT_sG_sT_{s+1}G_{s+1}\ldots T_nG_n\ldots \in Y$, define $x^{(s)} = D$. And for any $n \ge s+1$, we denote $x^{(n)} = DT_sG_sT_{s+1}G_{s+1}\ldots T_{n-1}G_{n-1}$. Because $P_n = P_{n-1}P_{n-1}G_{n-1}$ for any $n \ge 2$, it is easy to prove that $x^{(n)} \rhd P_n$ for any $n \ge s$ by the previous induction. Moreover, $P_n \prec x$ for all positive integer n large enough. Therefore, for any $x \in Y$, we can get three conclusions as follows:

(1) $\sigma(x) \in Y$, thus $\sigma(Y) \subseteq Y$,
(2) $x \in \omega_\sigma(w)$, thus $Y \subseteq \omega_\sigma(w)$,
(3) $w \in \omega_\sigma(x)$, thus the orbit of x is dense in Y.

Next, we prove that (Y,σ) is weakly mixing.

For any four nonempty open sets U_1, U_2, V_1, V_2 of Y, as $Y \subseteq \omega_\sigma(w)$, there must exist finite sequences $C_1, C_2, D_1, D_2 \prec w$ such that $[C_1] \subseteq U_1$, $[C_2] \subseteq U_2$, $[D_1] \subseteq V_1$, $[D_2] \subseteq V_2$. There exists large enough k such that $C_1, C_2, D_1, D_2 \prec P_k$. We suppose

$$C_1 = P_k[m_1, m_1 + |C_1| - 1] \qquad C_2 = P_k[m_2, m_2 + |C_2| - 1]$$

$$D_1 = P_k[l_1, l_1 + |D_1| - 1] \qquad C_2 = P_k[l_2, l_2 + |D_2| - 1]$$

As $P_k = P_n[0, |P_k| - 1]$ for any $n \geq k$, we have that for any $n \geq k$,

$$C_1 = P_n[m_1, m_1 + |C_1| - 1] \qquad C_2 = P_n[m_2, m_2 + |C_2| - 1]$$
$$D_1 = P_n[l_1, l_1 + |D_1| - 1] \qquad C_2 = P_n[l_2, l_2 + |D_2| - 1]$$

By the structure of the sequences $\{(a_n^1, a_n^2, a_n^3, a_n^4)\}_{n=1}^{\infty}$, there must be some $n \geq k$ such that $(a_n^1, a_n^2, a_n^3, a_n^4) = (m_1, l_1, m_2, l_2)$. As $w = P_n P_n F_n^1 F_n^2 F_n^3 F_n^4 \dots$, we have $\sigma^{2p_n}(w) \in [C_1] \subseteq U_1$ and $\sigma^{3p_n}(w) \in [D_1] \subseteq V_1$. Then $\sigma^{3p_n}(w) \in \sigma^{p_n}(U_1) \cap V_1$, $\sigma^{p_n}(U_1) \cap V_1 \neq \phi$. Similarly, we can prove $\sigma^{p_n}(U_2) \cap V_2 \neq \phi$. Hence, (Y, σ) is weakly mixing.

The second step is that we prove the previous system is not distributionally chaotic.

Suppose (Y, σ) is the system which is constructed in the previous section. Next we prove that for any $x, y \in Y$, $x \neq y$, the pair of points (x, y) is not distributionally chaotic. That is, for any $x, y \in Y$, $x \neq y$, $F_{xy}(\epsilon) > 0$ for all $\epsilon > 0$.

For any $x, y \in Y$, $x \neq y$, we suppose $x = DT_s G_s T_{s+1} G_{s+1} \dots T_n G_n \dots$, where $s \geq 1, D \rhd P_s$, $T_n = P_n$ or ϕ, for all $n \geq s$. $y = D' T'_{s'} G_{s'} T'_{s'+1} G_{s'+1} \dots T'_n G_n \dots$, where $s' \geq 1, D' \rhd P_{s'}$, $T'_n = P_n$ or ϕ, for all $n' \geq s$. For any $n \geq \max\{s+1, s'+1\}$, we define

$$x^{(n-1)} = DT_s G_s T_{s+1} G_{s+1} \dots T_{n-2} G_{n-2}$$
$$y^{(n-1)} = D' T'_{s'} G_{s'} T'_{s'+1} G_{s'+1} \dots T'_{n-2} G_{n-2}$$

Then $x^{(n-1)} \rhd P_{n-1}$, $y^{(n-1)} \rhd P_{n-1}$.

Because $x \neq y$, there must exist $n_0 \geq \max\{s+1, s'+1\}$ such that $x^{(n)} \neq y^{(n)}$ for any $n \geq n_0$ (or else we must have $x = y$). Then for $n \geq n_0$,

$$x = x^{(n-1)} T_{n-1} G_{n-1} T_n G_n \dots = x^{(n-1)} T_{n-1} F_{n-1}^1 F_{n-1}^2 F_{n-1}^3 F_{n-1}^4 J_{n-1} T_n G_n \dots$$
$$y = y^{(n-1)} T'_{n-1} G_{n-1} T'_n G_n \dots = y^{(n-1)} T'_{n-1} F_{n-1}^1 F_{n-1}^2 F_{n-1}^3 F_{n-1}^4 J_{n-1} T'_n G_n \dots$$

Let $t_{n-1} = ||x^{(n-1)} T_{n-1}| - |y^{(n-1)} T'_{n-1}||$. It is obvious that $0 \leq |x^{(n-1)} T_{n-1}| \leq 2|P_{n-1}| = 2p_{n-1}$, and $0 \leq |y^{(n-1)} T'_{n-1}| \leq 2|P_{n-1}| = 2p_{n-1}$. Then $0 \leq t_{n-1} \leq 2p_{n-1}$. On the other hand, $|x^{(n-1)} T_{n-1}| \neq |y^{(n-1)} T'_{n-1}|$ (or else $|x^{(n)}| = |y^{(n)}|$, and hence $x^{(n)} = y^{(n)} \rhd P_n$). Thus $0 < t_{n-1} \leq 2p_{n-1}$.

Without loss of generality, we assume $|x^{(n-1)} T_{n-1}| > |y^{(n-1)} T'_{n-1}|$.

For any $\epsilon > 0$, there must exist a positive integer N satisfying $\frac{1}{N} < \epsilon$.

For $n \geq n_0$, let

$$r_{n-1} = |x^{(n-1)} T_{n-1} F_{n-1}^1 F_{n-1}^2 F_{n-1}^3 F_{n-1}^4|$$
$$q_{n-1} = |y^{(n-1)} T'_{n-1} F_{n-1}^1 F_{n-1}^2 F_{n-1}^3 F_{n-1}^4 J_{n-1}|$$

Then $r_{n-1} \leq 6|P_{n-1}| = 6p_{n-1}$, $q_{n-1} \geq 10|P_{n-1}| = 10p_{n-1}$, and

$$x[r_{n-1}, q_{n-1} - 1] = J_{n-1}[0, q_{n-1} - r_{n-1} - 1] = 11 \dots 1$$
$$y[r_{n-1}, q_{n-1} - 1] = J_{n-1}[t_{n-1}, q_{n-1} + t_{n-1} - r_{n-1} - 1] = 11 \dots 1$$

When n is large enough, for any $r_{n-1} \leq i \leq q_{n-1} - N$, $d(\sigma^i(x), \sigma^i(y)) < \frac{1}{N} < \epsilon$. Thus

$$\begin{aligned} F_{xy}(\epsilon) &= \liminf_{n \to \infty} \frac{1}{q_{n-1}} \sharp\{i \mid d(\sigma^i(x), \sigma^i(y)) < \epsilon, 0 \leq i \leq q_{n-1} - 1\} \\ &\geq \liminf_{n \to \infty} \frac{1}{q_{n-1}} \sharp\{i \mid d(\sigma^i(x), \sigma^i(y)) < \epsilon, r_{n-1} \leq i \leq q_{n-1} - N\} \\ &= \liminf_{n \to \infty} \frac{q_{n-1} - N - r_{n-1} + 1}{q_{n-1}} \\ &\geq \liminf_{n \to \infty} \frac{10p_{n-1} - N - 6p_{n-1} + 1}{q_{n-1}} \\ &\geq \liminf_{n \to \infty} \frac{4p_{n-1} - N + 1}{12p_{n-1}} = \frac{1}{3} \end{aligned}$$

Then for any $x, y \in Y$, $x \neq y$, $F_{xy}(\epsilon) > 0$ for all $\epsilon > 0$.

From the above, we have proved that the noncompact system (Y, σ) is weakly mixing and the orbit of each point $x \in Y$ is dense in Y. However, there exists no distributionally chaotic pairs.

4. **Conclusions.** In a weakly mixing system, someone proves that the whole space can be a distributionally scrambled set. Oprocha proves that there exists a compact system which is weakly mixing but not distributionally chaotic. We prove that there exists a noncompact system with the whole space being weakly mixing but not distributionally chaotic. Moreover, we draw a conclusion: there exists a weakly mixing system which does not exhibit distributional chaos no matter whether the system is compact or noncompact.

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