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FINITE-TIME CONTROL OF NETWORKED CONTROL SYSTEMS WITH STOCHASTIC PACKET DROPOUTS

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Abstract. *In this paper, the finite-time stability and stabilization problems of a class of stochastic networked control systems (NCSs) with partly unknown transition probabilities are investigated. An iterative approach is proposed to model NCSs with bounded packet dropout as Markovian jump linear systems (MJLSs). The transition probabilities of MJLSs are partly unknown due to the complexity of network. The system under consideration is more general, which covers the systems with completely known and completely unknown transition probabilities as two special cases. Based on MJLSs theory and finite-time stability theory, the sufficient conditions for finite-time stability and stabilization of the underlying systems are derived via linear matrix inequalities (LMIs) formulation. Lastly, an illustrative example is given to demonstrate the effectiveness of the proposed results.*

Keywords: Networked control systems, Packet dropout process, Finite-time stabilization

1. **Introduction.** During the past two decades, NCSs have been widely studied, in which the feedback loops are implemented over some communication links [1]. The presence of such links enables the components (for instance, controllers, sensors and actuators) to be distributed in the different places. These components can share communication channels, which in general include both controller-to-actuator and sensor-to-controller channels. The sharing of the channels gives rise to many advantages including low cost, easy maintenance, flexible system structure, etc. [2]. Because of these attractive benefits, many industrial companies and institutes have shown interest in applying networks for remote industrial control purposes and factory automation [3]. However, as these components utilize common shared resources, i.e., the packet-based channels which are inherently imperfect, many new challenges are introduced including intermittent packet dropouts, the induced time delay, and the quantization errors [4]. Among all the challenging issues that emerged, packet dropouts are recognized to be one of the main causes for performance deterioration or even instability of NCS, and thus attract considerable research interests [5]. Various approaches have been proposed to model packet dropouts in NCS in order to better examine the role of packet dropouts with regard to system stability and performance, and develop control strategies to decrease their effects [6]. One of the most important approach is to model NCSs with packet dropout as MJLSs [7-10]. As a dominant factor, the transition probabilities in the jumping process determine the system behavior to a large extent. The analysis and synthesis results in [7-10] are based on the assumption that the complete knowledge of the transition probabilities is known. However, in almost all types of communication networks, either the variation of delays or the packet dropouts

are vague and random in different running periods of networks. Thus, all or part of the elements in the transition probabilities matrix are hard or costly to obtain.

On the other hand, the concept of Lyapunov asymptotic stability is largely known to the control community. However, Lyapunov asymptotic stability is not enough for practical applications, because there are some cases where large values of state variables are not acceptable. For this purpose, the concept of finite-time stability (FTS) can be used. Some early results on FTS can be found in [11], more recently the concept of FTS has been revisited in the light of recent results coming from LMIs theory, which has made it possible to find less conservative conditions for guaranteeing FTS and finite time stabilization [12,13]. In [14], the finite-time control problem of a class of networked control systems with time delay is investigated. In [15-17], sufficient conditions for finite time stability of networked control systems with packet dropout are provided. However, controller design methods are not given. In [18,19], the finite-time stabilization problems of a class of networked control systems with bounded Markovian packet dropout are investigated. In [20], the finite-time boundedness and stabilization problems of a class of networked control systems with bounded packet dropout are investigated. In [21], the stochastic finite-time stabilization and H_∞ control problem for one family of linear discrete-time systems over networks with packet loss are investigated. In [22], state feedback control for networked control systems with consideration of both network-induced delay and packet dropout is presented.

To date and to the best of our knowledge, the problems of stochastic networked control systems with partly unknown transition probabilities have not fully investigated and still remain challenging, although results related to systems over networks with packet loss are reported in the existing literature, which motivates the study of this paper. The main contributions are given as follows: (1) Definitions of finite-time stabilization are extended to stochastic networked control systems with packet dropout. (2) Sufficient conditions for finite-time stabilization in terms of linear matrix inequalities formulation are given.

This paper is organized as follows. An iterative method to model NCSs with bounded Markovian packet dropout as MJLSs is proposed in Section 2. The stochastic stability and stabilization conditions for NCSs are derived via LMIs in Section 3. In Section 4, a numerical example is provided to illustrate the effectiveness of our results. Concluding remarks are given in Section 5.

2. **Problem Formulation and Preliminaries.**

The framework of NCSs considered in this paper is depicted in Figure 1, and the plant to be controlled is described by the following linear discrete-time systems:

$$x(k+1) = Ax(k) + Bu(k) \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state and $u(k) \in \mathbb{R}^m$ is the control input. A and B are known real constant matrices with appropriate dimensions. We assume that (A, B) is controllable.

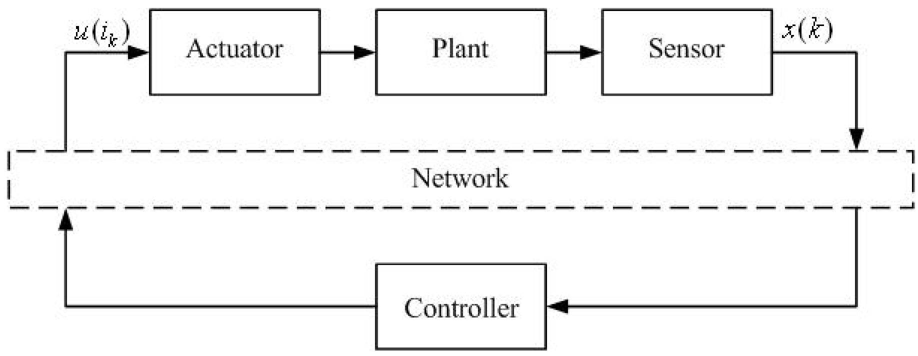

FIGURE 1. Illustration of NCSs over communication network

Let $\mathcal{I} = \{i_1, i_2, \cdots\}$, which is a subsequence of $\mathbb{N} = \{1, 2, \cdots\}$, denote the sequence of time points of successful date transmission from sensor to actuator. The state feedback controller law is

$$u(k) = Kx(k) \tag{2}$$

where $K \in \mathbb{R}^{m\times n}$ is to be designed. The control input is held at the previous value by the zero-order hold during the two successively successful transmitted instants, that is

$$u(l) = u(i_k) = Kx(i_k), \quad i_k \le l \le i_{k+1} - 1.$$

Thus the closed-loop system is

$$x(l+1) = Ax(l) + BKx(i_k), \quad i_k \le l \le i_{k+1} - 1. \tag{3}$$

Applying iteratively (3), we can obtain

$$x(i_{k+1}) = \left(A^{i_{k+1}-i_k} + \sum_{r=0}^{i_{k+1}-i_k-1} A^r BK \right) x(i_k), \quad i_k \in \mathcal{I}. \tag{4}$$

Define the packet dropout process as follows:

$$r(i_k) = i_{k+1} - i_k \tag{5}$$

which takes values in the finite state space $\mathcal{S} = \{1, 2, \cdots, s\}$, where s is defined as

$$s = \max_{i_k \in \mathcal{I}} (i_{k+1} - i_k).$$

Then the closed-loop system (4) can be rewritten as a jump linear system:

$$x(i_{k+1}) = \left(A^{r(i_k)} + \sum_{r=0}^{r(i_k)-1} A^r BK \right) x(i_k), \quad i_k \in \mathcal{I}. \tag{6}$$

The packet dropout process $\{r(i_k)\}$ is described by a discrete-time homogeneous Markov chain with mode transition probabilities

$$\pi_{ij} = \mathbb{P}(r(i_{k+1}) = j | r(i_k) = i) \ge 0, \quad \forall i, j \in \mathcal{S}.$$

The corresponding transition probabilities are defined as

$$\Pi = \begin{bmatrix} \pi_{11} & \pi_{12} & \cdots & \pi_{1s} \\ \pi_{21} & \pi_{22} & \cdots & \pi_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{s1} & \pi_{s2} & \dots & \pi_{ss} \end{bmatrix}.$$

Remark 2.1. *It is worth pointing out that the packet dropout process* $\{r(i_k), i_k \ge 0\}$ *includes both sensor-to-controller and controller-to-actuator packet dropouts.*

It is noticed that the ideal knowledge on the transition probabilities of packet dropout process is definitely expected to simplify the system analysis and design. However, the likelihood of obtaining such available knowledge is actually questionable, and the cost is probably expensive due to the complexity of networks [23]. Hence, it is necessary to discuss packet dropout process with partly unknown transition probabilities, i.e., some elements of matrix Π are unknown. For instance, the system (6) with $s = 4$, the transition probabilities matrix Π may be as

$$\Pi = \begin{bmatrix} \pi_{11} & \circ & \pi_{13} & \circ \\ \circ & \circ & \circ & \pi_{24} \\ \pi_{31} & \circ & \pi_{33} & \circ \\ \circ & \circ & \pi_{43} & \pi_{44} \end{bmatrix}$$

where "$\circ$" represents the inaccessible elements.

Denote

$$\mathcal{S} = \mathcal{S}^i_{\mathcal{K}} + \mathcal{S}^i_{\mathcal{UK}}, \quad \forall i \in \mathcal{S}$$

with

$$\begin{aligned} \mathcal{S}^i_{\mathcal{K}} &= \{j : \ \pi_{ij} \text{ is known}\} \\ \mathcal{S}^i_{\mathcal{UK}} &= \{j : \ \pi_{ij} \text{ is unknown}\} \end{aligned} \tag{7}$$

If $\mathcal{S}^i_{\mathcal{K}} \neq \emptyset$, it is further described as

$$\mathcal{S}^i_{\mathcal{K}} = \left(\mathcal{K}^i_1, \cdots, \mathcal{K}^i_m\right), \quad \forall 1 \leq m \leq N \tag{8}$$

where $\mathcal{K}^i_m \in \mathbb{N}$ represents the mth known element with the index $\mathcal{K}^i_m$ in the ith row of matrix Π.

The main aim of this paper is to find some sufficient conditions which guarantee that stochastic networked control system (6) with partly unknown transition probabilities is stable over a finite-time interval. This concept can be formalized through the following definition.

Definition 2.1. *Stochastic networked control system (6) is said to be finite-time stable with respect to* (α, β, R, N)*, where* R *is a positive-definite matrix,* $0 < \alpha < \beta$*, if for any* $k \in \{1, \cdots, N\}$

$$x^T(i_0)Rx(i_0) \leq \alpha^2 \Longrightarrow E\left\{x^T(i_k)Rx(i_k)\right\} \leq \beta^2.$$

To this end, the following lemmas will be essential for the proofs in the next section.

Lemma 2.1. *For a constant* $\gamma > 0$*, stochastic networked control system (6) with known transition probabilities is finite-time stable with respect to* (α, β, R, N) *if there exist matrix* $P_i > 0$*,* $i \in \mathcal{S}$*, and two positive scalars* λ_1*,* λ_2*, such that the following conditions hold*

$$\lambda_1 I \leq P_i \leq \lambda_2 I \tag{9}$$

$$(\gamma+1)^N \alpha^2 \lambda_2 - \beta^2 \lambda_1 < 0 \tag{10}$$

$$\sum_{j=1}^{s} \pi_{ij}(A^j + B_j K)^T \widetilde{P}_j (A^j + B_j K) - (\gamma+1)\widetilde{P}_i < 0 \tag{11}$$

where

$$\widetilde{P}_i = R^{\frac{1}{2}} P_i R^{\frac{1}{2}} \tag{12}$$

$$B_j = \sum_{r=0}^{j-1} A^r B \tag{13}$$

Lemma 2.2. (see [23]). *For any matrices* $U \in R^{n\times n}$ *and* $V \in R^{n\times n}$*, if the matrix* $V > 0$*, then we have*

$$U^T + U - V \leq U^T V^{-1} U.$$

Lemma 2.3. (Schur complement lemma, see [23]). *For a given symmetric matrix* $W = \begin{bmatrix} W_{11} & W_{12} \\ W_{12}^T & W_{22} \end{bmatrix}$*, where* $W_{11} \in \mathbb{R}^{p\times p}, W_{22} \in \mathbb{R}^{q\times q}$*, and* $W_{12} \in \mathbb{R}^{p\times q}$*, then the following three conditions are mutually equivalent*

1. $W < 0$,
2. $W_{11} < 0$, $W_{22} - W_{12}^T W_{11}^{-1} W_{12} < 0$,
3. $W_{22} < 0$, $W_{11} - W_{12} W_{22}^{-1} W_{12}^T < 0$.

3. **Main Results.** In this section, we will find a state feedback control matrix K, such that stochastic networked control system (6) with partly unknown transition probabilities is finite-time stable with respect to (α, β, R, N). In order to solve the problem, the following theorem will be essential.

Theorem 3.1. *For a constant $\gamma > 0$, stochastic networked control system (6) with partly unknown transition probabilities is finite-time stable with respect to (α, β, R, N) if there exist matrix $P_i > 0$, $i \in \mathcal{S}$, and two positive scalars λ_1, λ_2, such that the following conditions hold*

$$\lambda_1 I \leq P_i \leq \lambda_2 I \tag{14}$$

$$(\gamma+1)^N \alpha^2 \lambda_2 - \beta^2 \lambda_1 < 0 \tag{15}$$

$$\left(A^j + B_j K\right)^T \widetilde{P}_j \left(A^j + B_j K\right) - (\gamma+1)\widetilde{P}_i < 0, \quad \forall j \in \mathcal{S}^i_{\mathcal{UK}}, \tag{16}$$

$$\sum_{j \in \mathcal{S}^i_{\mathcal{K}}} \pi_{ij} \left(A^j + B_j K\right)^T \widetilde{P}_j \left(A^j + B_j K\right) - (\gamma+1) \left(\sum_{j \in \mathcal{S}^i_{\mathcal{K}}} \pi_{ij}\right) \widetilde{P}_i < 0, \quad \forall j \in \mathcal{S}^i_{\mathcal{K}} \tag{17}$$

where $\widetilde{P}_i$ and B_j are defined in (12) and (13) respectively.

Remark 3.1. *It is noticed that if $\mathcal{S}^i_{\mathcal{UK}} = \emptyset$, $\forall i \in \mathcal{S}$, the underlying system is the one with completely known transition probabilities, which is Markovian packet dropout process with known transition probabilities [17-19]. On the other hand, if $\mathcal{S}^i_{\mathcal{K}} = \emptyset$, $\forall i \in \mathcal{S}$, the underlying system is the one with completely unknown transition probabilities, which is arbitrary packet dropout process [17,19].*

Now we turn back to our original problem, that is to find sufficient conditions which guarantee that the system (6) with the controller (2) is finite-time stable with respect to (α, β, R, N). The solution of this problem is given by the following theorem.

Theorem 3.2. *For a constant $\gamma > 0$, stochastic networked control system (6) with partly unknown transition probabilities is finite-time stable with respect to (α, β, R, N) if there exist matrix $\widetilde{X}_i$, $i \in \mathcal{S}$, $G \in \mathbb{R}^{n \times n}$, $Y \in \mathbb{R}^{m \times n}$, and two positive scalars λ_1, λ_2, such that the following conditions hold*

$$\lambda_2^{-1} R^{-1} \leq \widetilde{X}_i \leq \lambda_1^{-1} R^{-1} \tag{18}$$

$$\lambda_1^{-1} (\gamma+1)^N \alpha^2 - \lambda_2^{-1} \beta^2 < 0 \tag{19}$$

$$\begin{bmatrix} -G - G^T + (\gamma+1)^{-1} \widetilde{X}_i & \left(A^j G + B_j Y\right)^T \\ * & -X_j \end{bmatrix} < 0, \quad \forall j \in \mathcal{S}^i_{\mathcal{UK}} \tag{20}$$

$$\begin{bmatrix} -G - G^T + (\gamma+1)^{-1} \left(\sum\limits_{j \in \mathcal{S}^i_{\mathcal{K}}} \pi_{ij}\right)^{-1} \widetilde{X}_i & \mathcal{L}^i_{\mathcal{K}} \\ * & -\mathcal{X}^i_{\mathcal{K}} \end{bmatrix} < 0, \quad \forall j \in \mathcal{S}^i_{\mathcal{K}} \tag{21}$$

where

$$\mathcal{L}^i_{\mathcal{K}} = \left[\sqrt{\pi_{i\mathcal{K}^i_1}} \left(A^{\mathcal{K}^i_1} G + B_{\mathcal{K}^i_1} Y\right)^T \ \cdots \ \sqrt{\pi_{i\mathcal{K}^i_m}} \left(A^{\mathcal{K}^i_m} G + B_{\mathcal{K}^i_m} Y\right)^T\right] \tag{22}$$

$$\mathcal{X}^i_{\mathcal{K}} = diag\left(\widetilde{X}_{\mathcal{K}^i_1} \ \cdots \ \widetilde{X}_{\mathcal{K}^i_m}\right), \quad \forall j \in \mathcal{S}^i_{\mathcal{K}} \tag{23}$$

Then the desired stabilizable control matrix is given by

$$K = Y G^{-1}. \tag{24}$$

Remark 3.2. *From the development in the above theorems, one can clearly see that our obtained stability and stabilization conditions actually cover the results for completely Markovian packet dropout process and arbitrary packet dropout process.*

Remark 3.3. *For the case of $i_k < l \le i_{k+1} - 1$, the system transient performance can also be accommodated. In fact, denote $h(i_k) = l - i_k \in \mathcal{S}$. Then we have*

$$x(l) = \left(A^{h(i_k)} + \sum_{r=0}^{h(i_k)-1} A^r BK \right) x(i_k), \quad i_k \in \mathcal{I}. \tag{25}$$

By Theorem 3.2, the system transient performance can be accommodated.

4. **Numerical Example.** In this section, to illustrate the effectiveness of the proposed method, we apply the results in Section 3 to a cart and inverted pendulum system. The state variables are x, $\dot{x}$, θ, and $\dot{\theta}$. Assume that $m_1 = 1$kg, $m_2 = 0.5$kg, $L = 1$m, and there are no friction surfaces. The sampling time is $T = 0.1$. The controllers are designed using the discretized linearized model, which has the state-space model as

$$x(k+1) = Ax(k) + Bu(k) \tag{26}$$

where

$$A = \begin{bmatrix} 1.0000 & 0.1000 & -0.0166 & -0.0005 \\ 0 & 1.0000 & -0.3374 & -0.0166 \\ 0 & 0 & 1.0996 & 0.1033 \\ 0 & 0 & 2.0247 & 1.0996 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.0045 \\ 0.0896 \\ -0.0068 \\ -0.1377 \end{bmatrix}.$$

The discretized system is unstable because the eigenvalues of A are $1, 1, 1.5569, 0.6423$. Furthermore, we assume that the packet dropout upper bound is $s = 4$ and the transition probabilities matrix is as follows

$$\Pi = \begin{bmatrix} 0.2 & \circ & 0.3 & \circ \\ \circ & \circ & 0.4 & 0.2 \\ \circ & 0.5 & \circ & 0.1 \\ 0.6 & \circ & \circ & \circ \end{bmatrix}.$$

For given $\gamma = 1$, $\alpha = 1$, $\beta = 5$, $R = I$, $N = 10$, according to Theorem 3.2, the control matrix is given by

$$K = YG^{-1} = \begin{bmatrix} 3.1623 & 3.6878 & -30.8732 & -5.6879 \end{bmatrix}.$$

The states θ and $\dot{\theta}$ of the closed-loop system caused by the discretized model and the obtained controller are shown in Figure 2.

5. **Conclusions.** In this paper, we have considered the finite-time stabilization problems of a class of stochastic NCSs with partly unknown transition probabilities. Based on the iterative approach, the NCSs with bounded packet dropout are modeled as jump linear systems. The sufficient conditions for finite-time stabilization of the underlying systems are derived via LMIs formulation. Lastly, an illustrative example is given to demonstrate the effectiveness of the proposed results. The finite-time stabilization problem for NCSs with both stochastic packet dropout and time delay is a future work.

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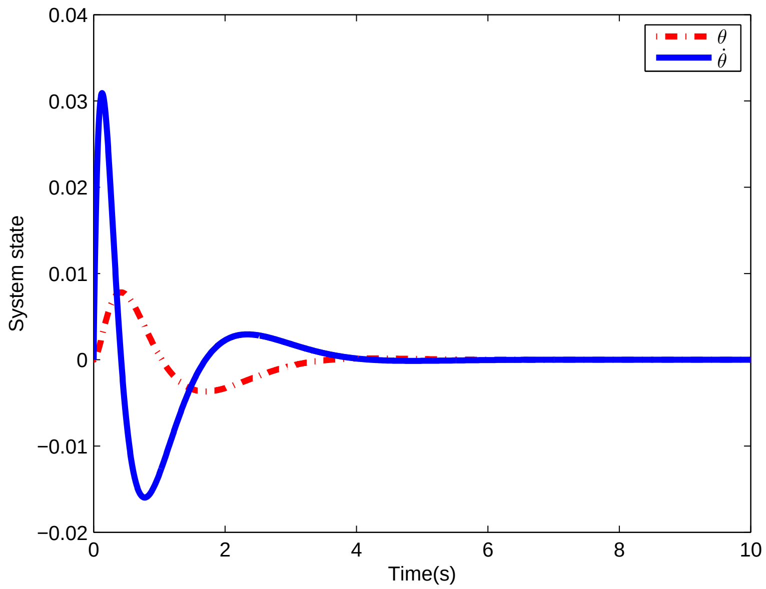

FIGURE 2. The states θ and $\dot{\theta}$ of the closed-loop system

REFERENCES

[1] W. Zhang, M. S. Branicky and S. M. Phillips, Stability of networked control systems, *IEEE Control Syst. Mag.*, vol.21, no.1, pp.84-99, 2001.

[2] N. W. Bauer, P. J. H. Maas and W. P. M. H. Heemels, Stability analysis of networked control systems: A sum of squares approach, *Automatica*, vol.48, no.8, pp.1514-1524, 2012.

[3] K. You and L. Xie, Survey of recent progress in networked control systems, *Acta Automatica Sinica*, vol.39, no.3, pp.101-117, 2013.

[4] Y. Sun and S. Qin, Stability analysis and controller design for networked control systems with random time delay, *International Journal of Systems Science*, vol.42, no.3, pp.359-367, 2011.

[5] Y. Sun and S. Qin, Stability of networked control systems with packet dropout: An average dwell time approach, *IET Control Theory and Applications*, vol.5, no.1, pp.47-53, 2011.

[6] D. Wang, J. Wang and W. Wang, Output feedback control of networked control systems with packet dropouts in both channels, *Information Sciences*, vol.221, pp.544-554, 2013.

[7] J. L. Xiong and J. Lam, Stabilization of linear systems over networks with bounded packet loss, *Automatica*, vol.43, no.1, pp.80-87, 2007.

[8] K. You, M. Fu and L. Xie, Mean square stability for Kalman filtering with Markovian packet losses, *Automatica*, vol.47, no.12, pp.2647-2657, 2011.

[9] D. E. Quevedo, Robust stability of packetized predictive control of nonlinear systems with disturbances and Markovian packet losses, *Automatica*, vol.48, no.8, pp.1803-1811, 2012.

[10] P. Minero and M. Franceschetti, Finite-time stabilization via dynamic output feedback, *IEEE Trans. Autom. Control*, vol.58, no.2, pp.349-362, 2013.

[11] L. Weiss and E. F. Infante, Finite time stability under perturbing forces and on product spaces, *IEEE Trans. Autom. Control*, vol.12, no.1, pp.54-59, 1967.

[12] F. Amato and M. Ariola, Finite-time control of discrete-time linear systems, *IEEE Trans. Autom. Control*, vol.50, no.5, pp.724-729, 2005.

[13] F. Amato, A. Merola and C. Cosentino, Finite-time control of discrete-time linear systems: Analysis and design conditions, *Automatica*, vol.46, no.5, pp.919-924, 2010.

[14] Y. Sun and J. Xu, Finite-time boundedness and stabilization of networked control systems with time delay, *Mathematical Problems in Engineering*, 2012.

[15] S. Mastellone, C. T. Abdallah and P. Dorato, Model-based networked control for finite-time stability of nonlinear systems: The deterministic case, *Proc. of the 13th Mediterranean Conference on Control and Automation*, pp.1085-1090, 2005.

[16] S. Mastellone, C. T. Abdallah and P. Dorato, Model-based networked control for finite-time stability of nonlinear systems: The stochastic case, *Proc. of the 13th Mediterranean Conference on Control and Automation*, pp.1091-1096, 2005.

[17] S. Mastellone, C. T. Abdallah and P. Dorato, Stability and finite-time stability analysis of discrete-time nonlinear networked control systems, *Proc. of the American Control Conference*, pp.1239-1244, 2005.
[18] Y. Shang and L. Zhang, Finite-time stabilization of linear systems over networks with bounded packet dropout, *Journal of Convergence Information Technology*, vol.7, no.12, pp.762-769, 2012.
[19] Y. Sun and G. Li, Finite-time stability and stabilization of networked control systems with bounded Markovian packet dropout, *Discrete Dynamics in Nature and Society*, vol.2014, 2014.
[20] Y. Sun, Finite-time boundedness and stabilization of networked control systems with bounded packet dropout, *International Journal of Systems Science*, 2012.
[21] Y. Zhang, W. Cheng, X. Mu and C. Liu, Stochastic H_∞ finite-time control of discrete-time systems with packet loss, *Mathematical Problems in Engineering*, 2012.
[22] W. Xue and W. Mao, Asymptotic stability and finite-time stability of networked control systems: Analysis and synthesis, *Asian Journal of Control*, vol.15, no.5, pp.1376-1384, 2013.
[23] Y. Sun and S. Qin, Stability and stabilization of networked control systems with bounded packet dropout, *Acta Automatica Sinica*, vol.37, no.1, pp.113-117, 2011.