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SYNCHRONIZATION FOR A CLASS OF FRACTIONAL-ORDER NEURAL NETWORKS BY USING STATE FEED-BACK CONTROL

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Abstract. *This paper presents synchronization methods for a class of fractional-order neural networks by means of linear control. The proposed controller can guarantee the synchronization errors converge to an arbitrary small region of the origin. Of fundamental importance, the stability analysis for fractional-order neural networks in this paper can provide a framework for stability analysis in fractional-order systems. Finally, simulation results are given to show the effectiveness of the proposed methods.*

Keywords: Fractional-order system, Fractional-order neural network, Linear control

1. **Introduction.** Fractional-order calculus has been used in physics and engineering, and has more advantages comparing with integer-order systems in some fields [1, 2, 3, 4, 5]. For examples, stochastic extended fractional Kalman filter can be applied to stating reconstruction in some noisy environments [6], a fractional-order oscillator is able to produce better linear FM signals compared to the conventional oscillator [7], and a fractional-order operator can offer more accurate and elegant control performance than that of classical control methods [8, 9].

Due to the rapid development of the fractional-order calculus, lots of results on fractional-order neural networks have been achieved, for example, in [10, 11, 12, 13, 14]. The stability and asymptotical stability of fractional-order neural networks are studied by means of an energy-like function analysis [12]. It has been pointed out that fractional derivative offers a very good technique for the description of hereditary and memory properties of kinds of processes. If fact, fractional-order systems usually involve infinite memory terms. Taking these facts into consideration, the incorporation of a memory term into the classical neural network model can be seen as an extremely important improvement. In [15], the authors point out the use of developing and studying fractional-order mathematical models of neural networks because fractional differentiation provides neurons with a fundamental and general computation ability that can contribute to efficient information processing, stimulus anticipation and frequency-independent phase shifts of oscillatory neuronal firing. Besides, fractional-order neural networks are expected to be effective in applications of parameter estimations because they are characterized by infinite memory. Thus, it is meaningful to study fractional-order neural networks both in theory and in applications.

Synchronization for fractional-order neural networks has been studied by many authors [12, 16, 17, 18, 19, 20]. To the best of our knowledge, there are few studies consider the synchronization of fractional-order neural networks by using linear control methods up to now. Motivated by this, the synchronization of fractional-order neural networks is investigated in this paper. The remainder of this paper is organized as follows. Section 2 presents some preliminaries about the fractional calculus. Section 3 gives main results of this paper. Simulation results are presented in Section 4. Finally, Section 5 concludes this work.

2. **Preliminaries.** In this paper the lower limit of the fractional calculus is chosen as zero. The fractional-order integral with fractional order α can be described by

$$ {}_0D_t^{-\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)d\tau, \tag{1}$$

where $\Gamma(\cdot)$ represents the Euler Gamma function.

There are three definitions of fractional-order derivative which are often used in literature, i.e., Grunwald-Letnikov, Riemann-Liouville, and Caputo definitions. The initial conditions of Caputo derivative take on the same form as in integer-order differential equations. So Caputo derivative is the most frequently utilized in control engineering, and we will also use this definition. The Caputo fractional derivative is defined as

$$ {}_0^CD_t^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}f^{(n)}(\tau)d\tau, \tag{2}$$

where α is the fractional order, $n-1 \le \alpha < n$.

The Laplace transform of Caputo fractional derivative can be given as follows:

$$\int_0^{\infty} e^{-st}{}_0^CD_t^{\alpha}x(t)dt = s^{\alpha} - \sum_{k=0}^{n-1} s^{\alpha-k-1}x^{(k)}(0). \tag{3}$$

In this paper, the following definitions and lemmas will be used.

The Mittag-Leffler function with two parameters can be written as [3]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \tag{4}$$

where $\alpha, \beta > 0$ and $z \in C$. Note that $E_{1,1}(z) = e^z$.

The Laplace transform of Mittag-Leffler function is

$$\mathscr{L}\left\{t^{\beta-1}E_{\alpha,\beta}\left(-at^{\alpha}\right)\right\} = \frac{s^{\alpha-\beta}}{s^{\alpha}+a}. \tag{5}$$

Lemma 2.1. [4]. *If $A \in R^{n\times n}$, $0 < \alpha \le 1$, β is an arbitrary real number, and $b > 0$ is a real constant, then*

$$E_{\alpha,\beta}(A) \le \frac{b}{1+\|A\|}, \tag{6}$$

where $\mu \le |\arg(\mathrm{eig}(A))| \le \pi$ with $\mu \in R$ satisfies $\pi\alpha/2 < \mu < \min\{\pi, \pi\alpha\}$.

Lemma 2.2. [2]. *If $t \in [0, T]$ and*

$$x(t) \le h(t) + \int_0^t k(\tau)x(\tau)d\tau, \tag{7}$$

where $k(t) \ge 0$ and all the functions involved are continuous on the interval $[0, T]$. Then we can conclude

$$x(t) \le h(t) + \int_0^t k(\tau)h(\tau)\exp\left[\int_{\tau}^t k(u)du\right]d\tau. \tag{8}$$

Lemma 2.3. [3, 5]. *Let $0 < \alpha < 2$, β is a complex number, and μ is a real number. If*

$$\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}, \tag{9}$$

then for arbitrary integer $n \ge 1$ the following expansion holds:

$$E_{\alpha,\beta}(z) = -\sum_{j=1}^{n}\frac{1}{\Gamma(\beta-\alpha j)z^j} + \circ\left(\frac{1}{|z|^{n+1}}\right). \tag{10}$$

3. **Main Results.** The dynamic equation of a drive fractional-order neural network considered in this paper is described by

$$ {}_0^C D_t^\alpha x_i(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + I_i, \quad j = 1, 2, \cdots, n, \tag{11} $$

or equivalently

$$ {}_0^C D_t^\alpha x(t) = -Cx(t) + Af(x(t)) + I, \tag{12} $$

where $0 < \alpha < 1$ is the fractional order, n represents the number of units in a neural network; $x(t) = [x_1(t), x_2(t), \cdots, x_n(t)]^T$ is the state vector; $A = \{a_{ij}\}$, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, n$ corresponds to the connection of the ith neuron to the jth neuron; $f(x(t)) = [f_1(x_1(t)), f_2(x_2(t)), \cdots, f_n(x_n(t))]$ is the activation function of the neurons; $C = \text{diag}(c_i)$, $c_i > 0$ is the rate with which the ith neuron will reset its potential to the resting state in isolation when disconnected from external inputs as well as the network; $I = [I_1, I_2, \cdots, I_n]^T$ is an external bias vector.

The response fractional-order neural network combined with a synchronization controller $u(t)$ is designed as

$$ {}_0^C D_t^\alpha y(t) = -Cy(t) + Af(y(t)) + I + u(t). \tag{13} $$

To proceed, we need the following assumption.

Assumption 3.1. *We assume that the nonlinear functions $f_i(x(t))$, $i = 1, 2, \cdots, n$ are bounded, i.e., there exist some positive constants b_i such that $\|f_i(x(t))\| \le b_i$.*

Define the synchronization errors as

$$ e(t) = x(t) - y(t), \tag{14} $$

with $e(t) = [e_1(t), e_2(t), \cdots, e_n(t)]^T$, $e_i(t) = x_i(t) - y_i(t)$.

From (12) and (13) we have

$$ {}_0^C D_t^\alpha e_i(t) = -c_i e(t) + \sum_{j=1}^{n} a_{ij} (f_j(x_j(t)) - f_j(y_j(t))) - u_i(t). \tag{15} $$

Then we can design the following linear controller

$$ u_i(t) = k_i e_i(t). \tag{16} $$

We have the following results.

Theorem 3.1. *Consider the drive fractional-order neural network (12) and the response fractional-order neural network (13). If the controller is designed as (16) and the design parameters k_i are chosen large enough, then we have all signals in the closed-loop system will remain bounded and the synchronization errors $e_i(t)$ will converge to an arbitrary small neighbourhood of the origin.*

Proof: Substituting the synchronization controller (16) into (15) gives

$$ {}_0^C D_t^\alpha e_i(t) = -(c_i + k_i) e(t) + \sum_{j=1}^{n} a_{ij} \left(f_j(x_j(t)) - f_j(y_j(t))\right) - u_i(t). \tag{17} $$

Applying Laplace transform on (17), we can obtain

$$ E_i(s) = \frac{s^{\alpha-1}}{s^\alpha + c_i + k_i} e_i(0) + \frac{1}{s^\alpha + c_i} \sum_{j=1}^{n} a_{ij} \mathscr{L}(f_j(x_j(t)) - f_j(y_j(t))), \tag{18} $$

where $E_i(s)$ is the Laplace transform of $e_i(t)$, i.e., $E_i(s) = \mathscr{L}(e_i(t))$. According to the property (5), the solution of (11) is given by

$$\begin{aligned} e_i(t) &= e_i(0)E_{\alpha,1}(-(c_i+k_i)t^\alpha) \\ &+ \sum_{j=1}^n a_{ij} \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-(c_i+k_i)(t-\tau)^\alpha\right)(f_j(x_j(t)) - f_j(y_j(t)))d\tau. \end{aligned} \tag{19}$$

By using Assumption 3.1 we have

$$\begin{aligned} |e_i(t)| \leq& |e_i(0)|E_{\alpha,1}\left(-(c_i+k_i)t^\alpha\right) \\ &+ 2\sum_{j=1}^n |a_{ij}b_j| \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-c_i(t-\tau)^\alpha\right) d\tau. \end{aligned} \tag{20}$$

Noting that he Mittag-Leffler function with two parameters have the following property [3]:

$$\int_0^t \tau^{\beta-1} E_{\alpha,\beta}\left(-k\tau^\alpha\right) d\tau = t^\beta E_{\alpha,\beta+1}(-kt^\alpha), \tag{21}$$

we can obtain

$$|e_i(t)| \leq |e_i(0)|E_{\alpha,1}(-(c_i+k_i)t^\alpha) + M_i t^\alpha E_{\alpha,\alpha+1}\left(-(c_i+k_i)t^\alpha\right) \tag{22}$$

where $M_i = \sum_{j=1}^n |a_{ij}|b_j$.

From Lemma 2.3, we know there exists s positive constant t_0, for all $t > t_0$, the following inequalities hold:

$$M_i t^\alpha E_{\alpha,\alpha+1}(-(c_i+k_i)t^\alpha) \leq \frac{M_i}{c_i+k_i}. \tag{23}$$

Noting that

$$\lim_{t\to\infty} E_{\alpha,1}\left(-(c_i+k_i)t^\alpha\right) = 0, \tag{24}$$

we have a positive constant t_1 such that

$$|x_i(t)| \leq \frac{2M_i}{c_i+k_i} \tag{25}$$

holds for all $t > t_1$, $i = 1, 2, \cdots, n$. As a result we also know that all signals in the closed-loop systems will keep bounded. This ends the proof of Theorem 3.1.

4. **Simulation Results.** The following two-states fractional-order neural network is used as the drive system in the simulation:

$$\begin{cases} {}_0^C D_t^\alpha x_1(t) = -c_1 x_1(t) + 0.1\arctan x_1(t) + 7\sin x_2(t) - 1, \\ {}_0^C D_t^\alpha x_2(t) = -c_2 x_1(t) - 0.2\arctan x_1(t) - 10\sin x_2(t) + 0.7. \end{cases} \tag{26}$$

Let the initial conditions of the drive system be $x_1(0) = 3$, $x_2(0) = -4$. From the system model we known that Assumption 3.1 is satisfied.

The response fractional-order neural network is

$$\begin{cases} {}_0^C D_t^\alpha y_1(t) = -c_1 y_1(t) + 0.1\arctan x_1(t) + 7\sin x_2(t) - 1 + u_1(t), \\ {}_0^C D_t^\alpha y_2(t) = -c_2 y_1(t) - 0.2\arctan x_1(t) - 10\sin x_2(t) + 0.7 + u_2(t). \end{cases} \tag{27}$$

The initial condition of the response system is $y_1(0) = -2$, $y_2(0) = 2$.

In the simulation, the fractional order is $\alpha = 0.88$, the system parameters are chosen as $c_1 = 0.01$, $c_2 = 0.02$, and the controller design parameters are chosen as $k_1 = k_2 = 3$. The synchronization results are presented in Figure 1. From the simulation results, we can see that synchronization errors converge to zero rapidly. Since the controller is designed as $u_i(t) = k_i e_i(t)$, Figure 1 also means that the control input is continuous and bounded.

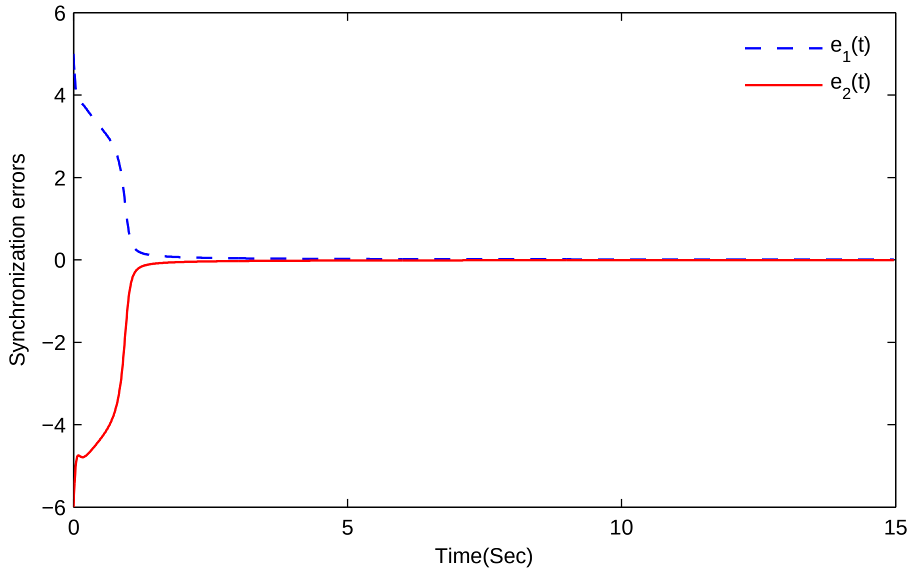

FIGURE 1. Time response of the synchronization errors $e_1(t)$ and $e_2(t)$

5. **Conclusion.** In this paper, linear control methods are used for synchronizing two identical fractional-order neural networks. It is known that linear controller is easy to construct and used in reality, thus the work of this paper is meaningful. The boundedness of signals in the closed-loop system and the convergence of synchronization error can be ensured. It should be pointed out that the mathematical model this paper used has a special structure (see Assumption 3.1). How to decrease this assumption thus design synchronization controller for fractional-order neural networks with more general structure is our future research direction.

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REFERENCES

[1] J. Cao and J. Liang, Boundedness and stability for Cohen-Grossberg neural network with time-varying delays, *J. Math. Anal. Appl.*, vol.296, pp.665-685, 2004.

[2] L. P. Chen, Y. Chai, R. C. Wu and J. Wang, Stability and stabilization of a class of nonlinear fractional-order systems with Caputo derivative, *IEEE Trans. Circuits Syst.*, vol.59, no.9, pp.602-606, 2012.

[3] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, CA, USA, 1999.

[4] J. H. Luo, State-feedback control for fractional-order nonlinear systems subject to input saturation, *Mathematical Problems in Engineering*, vol.2014, 2014.

[5] R. M. Martínez, J. L. M. Machuca, R. M. Guerra, J. A. León and G. F. Anaya, A new observer for nonlinear fractional order systems, *The 50th IEEE Conference on Decision and Control and European Control Conference*, Orlando, FL, USA, pp.3319-3324, 2011.

[6] K. B. Arman, F. Kia, P. Naser and L. Henry, A chaotic secure communication scheme using fractional chaotic systems based on an extended fractional Kalman filter, *Communications in Nonlinear Science and Numerical Simulation*, vol.14, pp.863-879, 2009.

[7] A. Oustaloup, Fractional order sinusoidal oscillators: Optimization and their use in highly linear FM modulation, *IEEE Trans. Circuits and Systems*, vol.28, no.10, pp.1007-1009, 1981.

[8] L. Song, S. Xu and J. Yang, Dynamical models of happiness with fractional order, *Communications in Nonlinear Science and Numerical Simulation*, vol.15, no.3, pp.616-628, 2010.

[9] S. Zhou, X. Lin and H. Li, Chaotic synchronization of a fractional-order system based on washout filter control, *Communications in Nonlinear Science and Numerical Simulation*, vol.16, no.3, pp.1533-1540, 2011.
[10] C. Song and J. Cao, Dynamics in fractional-order neural networks, *Neurocomputing*, vol.142, pp.494-498, 2014.
[11] E. Kaslika and S. Sivasundaram, Nonlinear dynamics and chaos in fractional-order neural networks, *Neural Networks*, vol.32, pp.245-256, 2012.
[12] H. Wang, Y. Yu and G. Wen, Stability analysis of fractional-order Hopfield neural networks with time delays, *Neural Networks*, vol.55, pp.98-109, 2014.
[13] J. Chen, Z. Zeng and P. Jiang, Global Mittag-Leffler stability and synchronization of memristor-based fractional-order neural networks, *Neural Networks*, vol.51, pp.1-8, 2014.
[14] L. Chen, Y. Chai, R. Wu, T. Ma and H. Zhai, Dynamic analysis of a class of fractional-order neural networks with delay, *Neurocomputing*, vol.111, pp.190-194, 2013.
[15] B. N. Lundstrom, M. H. Higgs, W. J. Spain et al., Fractional differentiation by neocortical pyramidal neurons, *Nature Neuroscience*, vol.11, no.11, pp.1335-1342, 2008.
[16] S. Zhou, H. Li and Z. Zhu, Chaos control and synchronization in a fractional neuron network system, *Chaos, Solitons & Fractals*, vol.36, no.4, pp.973-984, 2008.
[17] P. Arena, R. Caponetto, L. Fortuna et al., Bifurcation and chaos in noninteger order cellular neural networks, *International Journal of Bifurcation and Chaos*, vol.8, no.7, pp.1527-1539, 1998.
[18] J. Yu, C. Hu and H. Jiang, α-stability and α-synchronization for fractional-order neural networks, *Neural Networks*, vol.35, pp.82-87, 2012.
[19] G. Chen, J. Zhou and Z. Liu, Global synchronization of coupled delayed neural networks and applications to chaotic CNN models, *International Journal of Bifurcation and Chaos*, vol.14, no.7, pp.2229-2240, 2004.
[20] H. Zhang, T. Ma, G. B. Huang et al., Robust global exponential synchronization of uncertain chaotic delayed neural networks via dual-stage impulsive control, *IEEE Trans. Systems, Man, and Cybernetics, Part B: Cybernetics*, vol.40, no.3, pp.831-844, 2010.