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WEAK CONVERGENCE OF COLORING PROCESS ON SPARSE GRAPH

Lu Liu and Zhenting Hou

School of Mathematics and Statistics
Central South University
Shuli Building, Xiaoxiang Road, Changsha 410000, P. R. China
g.jiayi.liu@gmail.com; zthou@csu.edu.cn

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Abstract. *For a given sparse graph sequence G^N, we give definition of convergence of graph structure. We study a colouring process on a convergent graph sequence. At each time t, a yet uncoloured vertex, $\pi(t)$, is coloured by a state (color) in S. The state is drawn by distribution $v_t\left(\cdot\middle|G^{N,S,t}_{\pi(t),d}\right) \in \mathcal{P}(S)$, where $v_t\left(\cdot\middle|\pi(t), G^{N,S,t}_{\pi(t),d}\right)$ depends on time t, the subgraph (with colour) within distance d to vertex $\pi(t)$ at time t, denoted by $G^{N,S,t}_{\pi(t),d}$. Such coloring strategy is called local coloring strategy. We prove that for a convergent sequence of sparse graph, if the graphs are not too interactive, then any such colouring strategy, $v_t\left(\cdot\middle|G^{N,S,t}_{\pi(t),d}\right)$, induces a convergent sequence of empirical process on coloured graph structure. We point out the possible application of this weak convergence result to large deviation theorems on probability model defined by graph. Such models are natural generalization of, i.i.d. setting, discrete markov chain and mean field probability model.*
Keywords: Large deviation theory, Graph coloring, Local control, Random graph

1. **Introduction.** Large deviation principles (LDP) on dense graph have drawn large attention [1,3]. The kind of graph considered in these results are dense in the sense that each vertex of the graph is connected with a significant portion of all other vertexes. The focus of these studies is on graph (coloured or not) structure, i.e., "how many graphs are there with certain property P".

In this paper, we study the coloring process controlled by a local strategy. Establishing weak convergence property is a component of the proof of large deviation. For example, in the proof of Sanov theorem (LDP for i.i.d. model), weak convergence property is simply the law of large number for i.i.d. model. We pointed out in Section 3 that to establish LDP for probability model induced by sparse graph, it is necessary to study the coloring process on the graph. Our main result proves the convergence (as the graph size approaching infinity) of the empirical process on colored graph structure controlled by a local strategy under the condition that the graph is "not very interactive".

For a sparse graph, each vertex is connected with constantly many vertexes. Various notions of randomness and thus convergence on sparse graph are proposed [9]. Under Erdős-Rényi randomness, the rate function of any sparse graph would be $-\infty$, thus making LDP for sparse graph meaningless. What is also interesting is LDP for probability model defined (implicitly or explicitly) by a sparse graph; such models include i.i.d. setting, discrete time Markov process and mean field model. To see this, let us introduce some notations. A centered graph $G = (V, E, v)$ is a directed graph (V, E) with a specified center $v \in V$. (Some times we write G_v to indicate v as its center.) For a directed graph $G = (V, E)$, a vertex $i \in V$, let $G_{i,d}$ denote subgraph consisting of vertexes within distance d to i, which specify i as its center. Let S be a set. A coloured graph $G^S = (V, E, f)$ is a graph $G = (V, E)$ together with a *partial* function $f : V \to S$. The discrete homogeneous markov process model can be seen as a probability model of $X_0, X_1, \cdots, X_n$ with joint distribution $\prod_{i=1}^{n} q\left(G_{i,2}^S\right)$, where the graph G is a simple chain. And let $G_1 = (\{v_1, v_2\}, \{(v_1, v_2)\}, \mathit{center} = v_1)$, $G_2 = (\{v_1, v_2, v_3\}, \{(v_2, v_3), (v_1, v_2)\}, \mathit{center} = v_2)$, $G_3 = (\{v_1, v_2\}, \{(v_1, v_2)\}, \mathit{center} = v_2)$, then $q\left(G_1^S\right) = p(f(v_1))$, $q\left(G_2^S\right) = p(f(v_2)|f(v_1))$, $q\left(G_3^S\right) = p(f(v_2)|f(v_1))$, where p is transition probability of the process $X_0, X_1, \cdots, X_n$.

The problem on limit behavior and LDP of these models arise naturally. Take the discrete markov process model as an example. For a finite set S, a given p. What is the limit distribution of $(\{v_1, v_2\}, \{(v_1, v_2)\}, f)$, $f(v_1) = s$, $f(v_2) = s'$, as $n \to \infty$. What is the speed of convergence of empirical distribution of states to that limit (wrt time length). For discrete time markov process, the first problem is long known, and the second question is answered in [4,5]. [3] studied dynamic sparse graph with independent color. Research on graph coloring process also arises in ecology study. [7,8] study the limit behavior (wrt time) of various contact processes. The contact process simulating the survival of a species is a 2-colored graph process. The state of each vertex evolves according to the states of its neighbors. [2,6,10] study the limit behaviors of multitype contact process. The central problem is, when can two species coexist. However, research on relation of convergence of general graph coloring process with graph structure is vacant.

In Section 2 we introduce a class of probability model induced by graph. In Section 3 we outline a proof of large deviation of such models. And point out the relation to weak convergence theorem of local coloring strategy. In Section 4 we give our main result. And in Section 5 we prove 4.1. In Section 6 we conclude the paper.

2. **Sparse Graph Limit.** To discuss LDP, let us first define convergence for sparse graph sequence. We say a sequence of graphs converge if the empirical distribution of any given finite graph in that sequence has a limit. Details are as the following.

Let G, G^S, $G_{i,d}^S$ be directed graph, coloured directed graph, and centered coloured directed graph resp. $[G]$, $\left[G^S\right]$, $\left[G_i^S\right]$ denote their isomorphism equivalent class. When no ambiguity is made, we simply write G, G^S, $G_{i,d}^S$ for $[G]$, $\left[G^S\right]$, $\left[G_i^S\right]$ resp.

For a directed graph $G = (V, E)$, coloured directed graph $G^S = (V, E, f)$, let,

$$L_d^G = \frac{1}{|V|} \sum_{i=1}^{|V|} \delta_{[G_{i,d}]}; \quad L_d^{G^S} = \frac{1}{|V|} \sum_{i=1}^{|V|} \delta_{\left[G_{i,d}^S\right]} \tag{1}$$

$$\begin{aligned} \mathcal{G}_d &= \{[G] : G = (V, E, \mathit{center} = v), (\forall x \in V) d(x, v) \leq d\} \\ \mathcal{G}_d^S &= \left\{\left[G^S\right] : G^S = (V, E, f, \mathit{center} = v), (\forall x \in V) d(x, v) \leq d\right\} \end{aligned} \tag{2}$$

Clearly, $L_d^G \in \mathcal{P}(\mathcal{G}_d)$, $L_d^{G^S} \in \mathcal{P}\left(\mathcal{G}_d^S\right)$. The convergence of graph structure is defined by convergence of L_d^G. $\mathcal{G}_d$ is clearly a countable set, and $\mathcal{P}(\mathcal{G}_d)$ is clearly a subspace of $[0,1]^\infty$. So it is natural (and enough for this paper) to equip $\mathcal{P}(\mathcal{G}_d)$ with product topology. Under product topology, convergence of a sequence $L_d^{G^N}$ is equivalent to convergence of $L_d^{G^N}(G_d)$ for all $G_d \in \mathcal{G}_d$.

The probability model induced by a graph can usually be written as,

$$X_1, \cdots, X_N | G^N \sim \frac{\exp\left\{NF\left(L_d^{G^{N,S}}\right)\right\} \prod_{i \leq N} \rho(dX_i)}{\int_{S^N} \exp\left\{NF\left(L_d^{G^{N,S}}\right)\right\} \prod_{i \leq N} \rho(dX_i)} \tag{3}$$

where $X_1, \cdots, X_N | G^N$ is the joint distribution of $X_1, \cdots, X_N$ given graph G^N. And $F \in \mathbf{C}_b\left(\mathcal{P}\left(\mathcal{G}_d^S\right), \mathbb{R}\right)$. Take discrete time Markov process as an example, $F\left(L_d^{G^{N,S}}\right) = \int_{\mathcal{G}_d^S} \log\left(q\left(G^S\right)\right) L_d^{G^{N,S}}\left(dG^S\right)$, with $d = 1$.

It is natural to ask, for probability model (3), if the sequence of graph G^N converges in the sense that $L_d^{G^N}$ converges, then the coloured directed graph $L_d^{G^{N,S}}$ (whose law is determined in an obvious way through model (3)) also converges. And if $L_d^{G^{N,S}}$ converges, what is the speed?

3. **An Outline of LDP on Probability Model Induced by Sparse Graph.** The LDP for model (3) could be, for any given d, $F \in \mathbf{C}_b\left(\mathcal{P}\left(\mathcal{G}_d^S\right), \mathbb{R}\right)$, if $L_{d'}^{G^N}$ converges to some $L_{d'}^{\infty}$ for all d', then $L_d^{G^{N,S}}$ satisfies LDP with some rate function. LDP is usually obtained by transforming the problem into a control problem (see [5] Chapter 1,2). In our model, we study a random order coloring process. Random order instead of graph structure specific order, enables the application of this result to any convergent graph sequence. More specifically, let $\pi : N \to N$ be a uniformly random permutation of $\{1, 2, \cdots, N\}$. Let $v_t\left(\cdot \middle| G_{\pi(t)}^{N,S,t}\right) : t \times G_{\pi(t)}^{N,S,t} \mapsto \mathcal{P}(S)$, which can be viewed as a coloring strategy. We define the following stochastic process, $G^{N,S,v,t}$, $t \in [0,1]$.

Definition 3.1.

$$G^{N,S,v,0} = \left(V^N, E^N, f_0\right), \ dom(f_0) = \emptyset.$$

For $i = 1, 2, \cdots, N$

$$X_{\pi(i)} \sim v_{i/N}\left(\cdot \middle| G_{\pi(i)}^{N,S,v,(i-1)/N}\right)$$

Let $f_i(\pi(i)) = X_{\pi(i)}$, $f_i(k) = f_{i-1}(k)$ *if* $k \neq \pi(i)$.

Finally,

$$G^{N,S,v,t} = \left(V^N, E^N, f_{[tN]}\right) \tag{4}$$

Recall that $G_k^{N,S,v,t}$ *is coloured graph* $G^{N,S,v,t}$ *with specified center* k. *Note that the coloring strategy at time* t *does not use information of* $\pi(s)$, $s \geq t$. *The superscript* v *in* $G^{N,S,v,t}$ *indicates that law of* $G^{N,S,v,t}$ *is induced by* $v.(\cdot|\cdot)$. *Without ambiguity, we write* $\pi(t)$ *for* $\pi([tN])$.

Clearly, $G^{N,S,v,t}$, $t \in [0,1]$, induces a natural empirical process, $L_d^{G^{N,S,v,t}}$, $t \in [0,1]$. We sometimes write $L_{d'}^{G^{N,S,v,\cdot}}$ to denote stochastic process $L^{G^{N,S,v,t}}$, $t \in [0,1]$. The control problem is to minimize the following cost by choosing v,

$$cost(v) = \mathrm{E}\left[\int_0^1 R\left(v_t\left(\cdot \middle| G_{\pi(t)}^{N,S,v,t}\right) \middle\| \rho\right) dt + F\left(L_d^S\right)\right] \tag{5}$$

where $R(\mu||\rho)$ is Kullback-Leibler distance between μ, ρ.

The optimal strategy is actually conditional probability. So the optimal strategy colour is a vertex $\pi(i)$ at time i/N by a distribution on S depending on the current state of the whole graph $G^{N,S,v,i/N}$. However, it is reasonable to conjecture that local strategy can approximate performance of the optimal strategy. i.e.,

$$\mathcal{V}_d = \left\{v : [0,1] \to \left(\mathcal{G}_d^S \to \mathcal{P}(S)\right), \text{ i.e., } v_t\left(\cdot \middle| G_{c,d}^S\right) \in \mathcal{P}(S). \ \left(\forall G_{c,d}^S\right) \ v_t\left(\cdot \middle| G_{c,d}^S\right) \text{ is piecewise continuous wrt } t, \text{ and has finitely many discontinuity point.}\right\} \tag{6}$$

The strategy in $\mathcal{V}_{d'}$ is such a strategy that it colours a vertex with distribution depending on current local coloured subgraph $G_{\pi(t),d}^{N,S,v,t}$. We also write $v_{d,\cdot}$ to denote such local strategy as a stochastic process $v_{d,t}$, $t \in [0,1]$, and write $v_{d,t}$ to denote the strategy at time t which is a (random) function $v_{d,t} : \mathcal{G}_d^S \to \mathcal{P}(S)$. It is reasonable to conjecture,

$$\inf_{v_{d,\cdot} \in \mathcal{V}_d, d \in \omega} cost(v) = \inf_v cost(v) \tag{7}$$

To verify Equation (7), we can show that for any $\varepsilon > 0$, there exists sufficiently large M for all $N > M$, the optimal strategy v^* for colouring G^N is closed (within distance ϵ by certain metric) to some $v_{d',\cdot} \in \mathcal{V}_{d'}$ on a small (but constant) probability event, and the performance is close to the optimal performance. Using weak convergence result 4.1, we are able to show that on that small probability event, as $N \to \infty$, $L_d^{G^{N,S,v_{d',\cdot,\cdot}}}$ approaches to a fixed orbit, $L_d^{\infty,S,v_{d',\cdot,\cdot}}$. Therefore, the performance of $v_{d',\cdot}$ is always close to performance of the optimal strategy v^*; since $L_d^{G^{N,S,v_{d',\cdot,\cdot}}}$ approaches to $L_d^{\infty,S,v_{d',\cdot,\cdot}}$,

$$\inf_v cost(v) = \inf_{d',v_{d',\cdot}\in\mathcal{V}_{d'},L_d^{\infty,S,v_{d',\cdot,1}}=L_d^S} \left\{ \int_0^1 \mathrm{E}_{G^S_{c,d}\sim L^{\infty,S,v_{d',\cdot,t}}} \left[R\left(v_t\left(\cdot\middle|G^S_{c,d}\right)||\rho\right)\right] dt + F\left(L_d^S\right) \right\} \tag{8}$$

In this paper, we establish the weak convergence result. i.e., if $v, v_{d,\cdot}$ are close enough, then the empirical process $L_{d_0}^{G^{N,S,v,\cdot}}, L_{d_0}^{G^{N,S,v_{d,\cdot,\cdot}}}$ are also close.

4. **Main Result.** Our weak convergence result applies on graph that is slightly interactive. To characterize interactiveness of a graph, we need the following notions. For a graph G^N, a set of vertex A, let

$$\partial_k A = \left\{ i \in G^N : d^{G^N}(i, A) \le k \right\}, \quad \partial_{k_1,k_2} A = \left\{ i \in G^N : k_1 \le d^{G^N}(i, A) \le k_2 \right\}$$

i.e., the neighbor of A that is of distance less than k to A. And for a sequence of graph, G^N, let

$$\begin{aligned} N_k &= \max_{N,c\in V^N}\{|\partial_k\{c\}|\}, \quad D_k = \max_{N,c\in V^N}\{|\partial_k\{c\} - \partial_{k-1}\{c\}|\}, \\ D_{k_1,k_2} &= \max_{N,c\in V^N}\{|\partial_{k_2}\{c\} - \partial_{k_1}\{c\}|\} \end{aligned} \tag{9}$$

Fast growing N_k, D_k wrt k clearly means being more interactive. Our main result is as follows.

Theorem 4.1. *Let G^N be a sequence of convergent directed graph, i.e., for every d' there exists $L^\infty_{d'}$ s.t., $\lim_{N\to\infty} L_{d'}^{G^N} = L^\infty_{d'}$. Let $v_{d,\cdot} \in \mathcal{V}_d$, v be such that, $(\forall\omega, t)\left\| v\left(|G^{N,S,v,t}\right) - v_{d,\cdot}\left(|G^{N,S,v,t}_{\pi(t),d}\right)\right\|_1 \le \epsilon$.*

If there exists $0 = t_0 < t_1 < \cdots < t_K = 1 - \varepsilon$, let $\delta_k = t_{k+1} - t_k$ for $k \le K - 1$ s.t., $(\forall k \le K-1)\delta_k D_{d_0+(K-k)d+1,d_0+(K-k+1)d} \le \varepsilon$.

Then we have, for some function $C(\cdot)$, $\lim_{\varepsilon\to 0} C(\varepsilon) =$ some universal constant,

$$\begin{aligned} &\lim_{N\to\infty} \sup_{t\in[0,1]} \left\| L_{d_0}^{G^{N,S,v,t}} - L_{d_0}^{G^{N,S,v_{d,\cdot,t}}} \right\|_1 \\ &\le^{\mathcal{D}} 2\varepsilon N_{d_0} + \sum_{j\le K-1} \frac{C(\varepsilon)}{\varepsilon}\delta_j^2 D_{d_0+(K-j)d+1,d_0+(K-j+1)d} D_{d_0+(K-j-1)d+1,d_0+(K-j)d} \\ &\quad + \epsilon\delta_j N_{d_0+(K-j)d} \end{aligned} \tag{10}$$

From Theorem 4.1 we are able to derive condition on graph law ensuring convergence of empirical distribution, $L_{d_0}^{G^{N,S,v_{d,\cdot,\cdot}}}$, for all local coloring strategy.

Corollary 4.1. *Let G^N be a sequence of convergent directed graph, i.e., for any d' there exists some $L^\infty_{d'} \in \mathcal{P}\left(\mathcal{G}^S_{d'}\right)$, $(\forall d' \in \omega)$ $\lim_{N\to\infty} L_{d'}^{G^N} = L^\infty_{d'}$.*

Let D_k, N_k be defined as (9) for G^N. Let $\overline{D}_k = \max_{i\le k}\{D_i\}$, $\overline{D}_{k_1,k_2} = \max_{i\le k_1}\{D_{i,i+k_2-k_1}\}$.

Suppose, for all d, $d_0 \in \omega$ $\sum_{i=0}^{\infty} \frac{1}{\overline{D}^2_{d_0+id+1,d_0+(i+1)d}} = \infty$.

Then we have, for any local coloring strategy $v_{d,\cdot}$, any $d_0 \in \omega$, the empirical process sequence $L_{d_0}^{G^{N,S,v_{d,\cdot},\cdot}}$ converges, i.e., there exists $L_{d_0}^{\infty,S,v_{d,\cdot},\cdot}$, s.t.,

$$\lim_{N\to\infty} L_{d_0}^{G^{N,S,v_{d,\cdot},\cdot}} \to^{\mathcal{D}} L_{d_0}^{\infty,S,v_{d,\cdot},\cdot} \tag{11}$$

Furthermore, $L_{d_0}^{\infty,S,v_{d,\cdot},\cdot}$ is continuous in $v_{d,\cdot}$.

The proof is simple, simply let $\delta_i = O\left(1/\overline{D}^2_{d_0+(K-i)d+1,d_0+(K-i+1)d}\right)$.

5. **Proof of Theorem 4.1.** The proof is not hard but tedious. Within each time segment, $[t_k, t_k+\delta_k)$, and for every d'', the coloring strategy $v, v_{d,\cdot}$, each induces a transition probability on space $\mathcal{G}^S_{d''}$. To define such transition probability, for $t < t'$, $G^S_{c,d'}, G'^S_{c,d''}$, let,

$$L^{G^N,v,[t,t')}\left(G^S_{c,d'}, G'^S_{c,d''}\right) = \frac{\left|\left\{i \in V^N : G^{N,S,v,t}_{i,d'} \cong G^S_{c,d'} \wedge G^{N,S,v,t'}_{i,d''} \cong G'^S_{c,d''}\right\}\right|}{|V^N|} \tag{12}$$

For a fixed $G^{N,S,v,t}_{i,d'}$, the probability $G^{N,S,v,t}_{i,d''}$ transferred to $G'^{N,S,v,t'}_{i,d''}$ at time t' is induced by updating path chosen by π, and coloring strategy v. We write $L^{v,t}_{d''}$, $L^{v_{d,\cdot},t}_{d''}$ for $L^{G^{N,S,v,t}}_{d''}$, $L^{G^{N,S,v_{d,\cdot},t}}_{d''}$. Theorem 4.1 is due to spread of difference between $L^{v,t'}$, $L^{v_{d,\cdot},t'}$ during $[t_k, t_k+\delta_k))$ as Lemma 5.1 illustrates.

Lemma 5.1.

- *Given $d'' \geq d$, G^N, $\delta > 0$ s.t. $\delta D_{d''+1,d''+d}, \delta D_{d''-d+1,d''} \leq \varepsilon << 1$; and suppose $t+\delta \leq \varepsilon$;*
- *Let $v_{d,\cdot} \in \mathcal{V}_d$, v be such that, $(\forall \omega, t)$ $\left\|v\left(|G^{N,S,v,t}\right) - v_{d,\cdot}\left(|G^{N,S,v,t}_{\pi(t),d}\right)\right\|_1 \leq \epsilon$*

Then we have, for some function $C(\cdot)$, $\lim_{\varepsilon\to 0} C(\varepsilon) =$ some universal constant,

$$\begin{aligned}\lim_{N\to\infty}\left\|L^{G^{N,S,v,t+\delta}}_{d''} - L^{G^{N,S,v_{d,\cdot},t+\delta}}_{d''}\right\|_1 \leq^{\mathcal{D}} & \left\|L^{G^{N,S,v,t}}_{d''+d} - L^{G^{N,S,v_{d,\cdot},t}}_{d''+d}\right\|_1 + \epsilon\delta N_{d''} \\ & + \frac{C(\varepsilon)}{\varepsilon}\delta^2 D_{d''+1,d''+d} D_{d''-d+1,d''}\end{aligned} \tag{13}$$

Lemma 5.1 clearly implies Theorem 4.1.

5.1. **Proof of Lemma 5.1.** We begin by analyzing the transition probability induced by π, v.

Definition 5.1. *For $i \in V^N$, let $path(i, d', [t,t'))$ denote the vertex sequence within distance d' to i, chosen by π during $[t,t')$ with time point recorded, i.e.,*

$$path(i, d', [t,t')) = (n_1, \tau_1) \times (n_2, \tau_2) \times \cdots \times (n_l, \tau_l) \tag{14}$$

where $\{n_1, \cdots, n_l\} = \{j \in \partial_{d'}\{i\} : \exists \tau \in [t,t'), \pi(\tau) = j\}$, $t \leq \tau_1 < \tau_2 < \cdots < \tau_l < t'$, $\tau_j = \inf\{\tau : \pi(\tau) = n_j\}$. We write $PATH\left(G^S_{c,d'}\right)$ for the path space of $G^S_{c,d'}$, $PATH\left(\mathcal{G}^S_{d'}\right)$ for union of path space of $G^S_{c,d'} \in \mathcal{G}^S_{d'}$.

When no ambiguous is made, we always omit $G^S_{c,d'}$ and write $PATH$ instead. We write $\theta\left(G^S_{c,d'}\right) = G'^S_{c',d'}$ to denote θ is an isomorphism between $G^S_{c,d'}$, $G'^S_{c',d'}$. Note that an isomorphism between two graphs also induces an isomorphism between path space between two graphs, i.e., $\theta((n_1,\tau_1) \times \cdots \times (n_l,\tau_l)) = (\theta(n_1),\tau_1) \times \cdots \times (\theta(n_l),\tau_l)$.

The coloring process on G^N within time $[t,t')$ induces nature random measure on $\mathcal{G}^S_{d'} \times PATH\left(\mathcal{G}^S_{d'}\right) \times \mathcal{G}^S_{d''}$, with $d'' \leq d'$.

Definition 5.2. *For, any $Pa \subseteq PATH$, $G^{N,S}$,*

$$L^{G^{N,S},v,[t,t')}\left(G^S_{c,d'}, Pa, G^S_{c,d''}\right) = \frac{\left|\left\{i \in V^N : \exists\theta, \theta\left(G^S_{c,d'}\right) = G^{N,S,v,t}_{i,d'} \wedge \theta^{-1}(path(i,d',[t,t'))) \in Pa \wedge G^{N,S,v,t}_{i,d''} \cong G^S_{c,d''}\right\}\right|}{|V^N|} \tag{15}$$

$$L^{G^{N,S},v,[t,t')}\left(G^S_{c,d'}, Pa\right) = \frac{\left\{i \in V^N : \exists\theta, \theta\left(G^S_{c,d'}\right) = G^{N,S,v,t}_{i,d'} \wedge \theta^{-1}(path(i,d',[t,t'))) \in Pa\right\}}{|V^N|} \tag{16}$$

When no ambiguity is made, we write $L^{v,[t,t')}$, $L^{v_{d,\cdot},[t,t')}$ for $L^{G^{N,S},v,[t,t')}$, $L^{G^{N,S},v_{d,\cdot},[t,t')}$. By definition,

$$\left\|L^{v,t+\delta}_{d''} - L^{v_{d,\cdot},t+\delta}_{d''}\right\|_1 = \int_{\mathcal{G}^S_{d''}} \left| \int_{\mathcal{G}^S_{d''+d}\times PATH} \left[L^{v,[t,t+\delta)}\left(dG^S_{c_1,d''+d}, dpath, dG^S_{c_2,d''}\right) - L^{v_{d,\cdot},[t,t+\delta)}\left(dG^S_{c_1,d''+d}, dpath, dG^S_{c_2,d''}\right)\right]\right| \tag{17}$$

For every $G^S_{c,d''+2d'}$, let

$$PATH_1\left(G^S_{c,d''+d}\right) = \left\{path \in PATH\left(G^S_{c,d''+d}\right) : path \bigcap \partial_{d''+1,d''+d}\{c\} = \emptyset\right\}$$

$$PATH_2\left(G^S_{c,d''+d}\right) = \left\{path \in PATH\left(G^S_{c,d''+d}\right) : path \bigcap \partial_{d''-d+1,d''}\{c\} = \emptyset\right\}$$

By slightly abuse notations, we write $\mathcal{G}^S_{d''} \times PATH_1$ to denote $\left\{\left(G^S_{c,d''} \in \mathcal{G}^S_{c,d''}, path\right) : path \in PATH_1\left(G^S_{c,d''}\right)\right\}$, etc.

Clearly,

$$\begin{aligned}
&\int_{\mathcal{G}^S_{d''}} \left| \int_{\mathcal{G}^S_{d''+d}\times PATH} \left[L^{v,[t,t+\delta)}\left(dG^S_{c_1,d''+d}, dpath, dG^S_{c_2,d''}\right) - L^{v_{d,\cdot},[t,t+\delta)}\left(dG^S_{c_1,d''+d}, dpath, dG^S_{c_2,d''}\right)\right]\right| \\
&\le \int_{\mathcal{G}^S_{d''}} \left| \int_{\mathcal{G}^S_{d''+d}\times PATH_1} \left[L^{v,[t,t+\delta)}\left(dG^S_{c_1,d''+d}, dpath, dG^S_{c_2,d''}\right) - L^{v_{d,\cdot},[t,t+\delta)}\left(dG^S_{c_1,d''+d}, dpath, dG^S_{c_2,d''}\right)\right]\right| \\
&\quad + \int_{\mathcal{G}^S_{d''}} \left| \int_{\mathcal{G}^S_{d''+d}\times PATH_2} \left[L^{v,[t,t+\delta)}\left(dG^S_{c_1,d''+d}, dpath, dG^S_{c_2,d''}\right) - L^{v_{d,\cdot},[t,t+\delta)}\left(dG^S_{c_1,d''+d}, dpath, dG^S_{c_2,d''}\right)\right]\right| \\
&\quad + \int_{\mathcal{G}^S_{d''}} \left| \int_{\mathcal{G}^S_{d''+d}\times (PATH-PATH_1-PATH_2)} \left[L^{v,[t,t+\delta)}\left(dG^S_{c_1,d''+d}, dpath, dG^S_{c_2,d''}\right) - L^{v_{d,\cdot},[t,t+\delta)}\left(dG^S_{c_1,d''+d}, dpath, dG^S_{c_2,d''}\right)\right]\right| \\
&= I + II + III
\end{aligned} \tag{18}$$

We claim that, $\lim\limits_{N\to\infty} I + II \leq^{\mathcal{D}} \left\|L^{v,t}_{d''+d} - L^{v_{d,\cdot},t}_{d''+d}\right\|_1 + \epsilon\delta N_{d''}$ and

$$\lim_{N\to\infty} III \leq^{\mathcal{D}} \frac{C(\varepsilon)}{\varepsilon}\delta^2 D_{d''+d,d''+1} D_{d'',d''-d+1}$$

where $\lim\limits_{\varepsilon\to 0} C(\varepsilon) =$ some universal constant.

Claim. $\lim\limits_{N\to\infty} III \leq^{\mathcal{D}} \frac{C(\varepsilon)}{\varepsilon}\delta^2 D_{d''+d,d''+1} D_{d'',d''-d+1}$

It is easy to see,

$$\begin{aligned} III \leq & \int_{\mathcal{G}^S_{d''+d}\times(PATH-PATH_1-PATH_2)} L^{v,[t,t+\delta)}\left(dG^S_{c_1,d''+d}, dpath\right) \\ & + \int_{\mathcal{G}^S_{d''+d}\times(PATH-PATH_1-PATH_2)} L^{v_{d,\cdot},[t,t+\delta)}\left(dG^S_{c_1,d''+d}, dpath\right) \end{aligned} \tag{19}$$

Note that, since π is independent from $G^{N,S,v,t}$, for any $PATH' \subseteq PATH$, any coloring strategy v', any d', d'', let

$$L^{v',[t,t+\delta)}\left(path|G^S_{c,d''+d'}\right) = \frac{L^{v',[t,t+\delta)}\left(G^S_{c,d''+d'}, path\right)}{L^{v',t}\left(G^S_{c,d''+d'}\right)} \tag{20}$$

we have,

$$L^{v,[t,t+\delta)}\left(path|G^S_{c,d''+d}\right) =^{\mathcal{D}} L^{v,[t,t+\delta)}(path|G_{c,d''+d}) \tag{21}$$

i.e., $L^{v,[t,t+\delta)}\left(path|G^S_{c,d''+d}\right)$ depends only on graph structure of $G^S_{c,d''+d}$ and $path$ and is independent of v, state structure of $G^S_{c,d''+d}$. Furthermore, according to condition in Lemma 5.1, $t+\delta \leq \varepsilon << 1$, we have, for any $G^S_{c,d''+d}$,

$$\lim_{N\to\infty} \int_{PATH-PATH_1-PATH_2} L^{v,[t,t+\delta)}\left(dpath|G^S_{c,d''+d}\right) \leq^{\mathcal{D}} \frac{C(\varepsilon)}{\varepsilon}\delta^2 D_{d''+d,d''+1} D_{d'',d''-d+1} \tag{22}$$

$$\lim_{N\to\infty} \int_{PATH-PATH_1-PATH_2} L^{v_{d,\cdot},[t,t+\delta)}\left(dpath|G^S_{c,d''+d}\right) \leq^{\mathcal{D}} \frac{C(\varepsilon)}{\varepsilon}\delta^2 D_{d''+d,d''+1} D_{d'',d''-d+1} \tag{23}$$

Thus the claim follows.

Claim. $\lim\limits_{N\to} I + II \leq^{\mathcal{D}} \left\|L^{v,t}_{d''+d} - L^{v_{d,\cdot},t}_{d''+d}\right\|_1 + \epsilon\delta N_{d''}$

Similar to (20), for any $Pa \subseteq PATH$, let

$$L^{v,[t,t')}\left(G'^S_{c,d''}|G'^S_{c,d''+d}, Pa\right) = \frac{L^{v,[t,t')}\left(G^S_{c,d''+d}, Pa, G^S_{c,d''}\right)}{L^{v,[t,t')}\left(G^S_{c,d''+d}, Pa\right)} \tag{24}$$

It is easy to see that, for a certain path, $path$, we can define a nature conditional probability, $L^{v,[t,t')}\left(G'^S_{c,d''}|G^S_{c,d''+d}, path\right)$.

The key point is for i, s.t. $path(i, d''+d, [t, t+\delta)) \in PATH_1 \cup PATH_2$, the coloring distribution for X_{n_i} in $\partial_{d''}\{i\}$ during $[t, t+\delta)$ is not affected by state of X_{n_j} in $\partial_{d'',d''+d}\{i\}$ assigned during $[t, t+\delta)$. By induction on $|path \cap \partial_{d''}\{i\}|$ we have,

$$\int_{\mathcal{G}^S_{d''}} \left|L^{v,[t,t')}\left(G'^S_{c,d''}|G^S_{c,d''+d}, path\right) - L^{v_{d,\cdot},[t,t')}\left(G'^S_{c,d''}|G^S_{c,d''+d}, path\right)\right| \leq |path \cap \partial_{d''}\{c\}|\epsilon \tag{25}$$

Now using (25), we have,

$$\begin{aligned} II \leq & \int_{\mathcal{G}^S_{d''+d} \times PATH_2} L^{v,[t,t+\delta)} \left(dG^S_{c_1,d''+d}, dpath\right) \cdot |path \cap \partial_{d''-d}\{c_1\}| \cdot \epsilon \\ & + \int_{\mathcal{G}^S_{d''+d} \times PATH_2} \left| L^{v,[t,t+\delta)} \left(dG^S_{c_1,d''+d}, dpath\right) - L^{v_d,[t,t+\delta)} \left(dG^S_{c_1,d''+d}, dpath\right) \right| \\ = & \, II_1 + II_2 \end{aligned} \tag{26}$$

In exactly the same way, we also have,

$$\begin{aligned} I \leq & \int_{\mathcal{G}^S_{d''+d} \times PATH_1} L^{v,[t,t+\delta)} \left(dG^S_{c_1,d''+d}, dpath\right) \cdot |path \cap \partial_{d''}\{c_1\}| \cdot \epsilon \\ & + \int_{\mathcal{G}^S_{d''+d} \times PATH_1} \left| L^{v,[t,t+\delta)} \left(dG^S_{c_1,d''+d}, dpath\right) - L^{v_d,[t,t+\delta)} \left(dG^S_{c_1,d''+d}, dpath\right) \right| \\ = & \, I_1 + I_2 \end{aligned} \tag{27}$$

Clearly,

$$\begin{aligned} I_2 + II_2 \leq & \int_{\mathcal{G}^S_{d''+d} \times PATH} \Big| L^{v,t} \left(dG^S_{c_1,d''+d}\right) L^{v,[t,t+\delta)}(dpath|G^S_{c_1,d''+d}) \\ & - L^{v_d,t} \left(dG^S_{c_1,d''+d}\right) L^{v_{d,\cdot},[t,t+\delta)} \left(dpath|G^S_{c_1,d''+d}\right) \Big| \leq \left\| L^{v,t}_{d''+d} - L^{v_{d,\cdot},t}_{d''+d} \right\|_1 \end{aligned} \tag{28}$$

On the other hand,

$$\begin{aligned} I_1 + II_1 \leq & \int_{\mathcal{G}^S_{d''+d} \times PATH} L^{v,t} \left(dG^S_{c_1,d''+d}\right) L^{v,[t,t+\delta)} \left(dpath|G^S_{c_1,d''+d}\right) \cdot |path \cap \partial_{d''}\{c_1\}| \cdot \epsilon \\ & \leq {}^{\mathcal{D}}\epsilon\delta N_{d''} \end{aligned} \tag{29}$$

6. **Conclusion.** In this paper, we proved the weak convergence property for empirical process of subgraph with color controlled by local coloring strategy under the condition that the graph is not so interactive (see condition of Theorem 4.1). We left the problem of whether this weak convergence property holds for polynomial graph. We also pointed out the application of this weak convergence theorem on establishing LDP for probability model induced by graph.

REFERENCES

[1] S. Chatterjee and S. R. S. Varadhan, The large deviation principle for the erdős-rényi random graph, *European Journal of Combinatorics*, vol.32, no.7, pp.1000-1017, 2011.

[2] J. T. Cox, R. B. Schinazi et al., Survival and coexistence for a multitype contact process, *The Annals of Probability*, vol.37, no.3, pp.853-876, 2009.

[3] K. Doku-Amponsah, P. Mörters et al., Large deviation principles for empirical measures of colored random graphs, *The Annals of Applied Probability*, vol.20, no.6, pp.1989-2021, 2010.

[4] M. D. Donsker and S. R. S. Varadhan, Asymptotic evaluation of certain Markov process expectations for large time, I, *Communications on Pure and Applied Mathematics*, vol.28, no.1, pp.1-47, 1975.

[5] P. Dupuis and R. S. Ellis, *A Weak Convergence Approach to the Theory of Large Deviations*, John Wiley & Sons, 2011.

[6] R. Durrett, C. Neuhauser et al., Coexistence results for some competition models, *The Annals of Applied Probability*, vol.7, no.1, pp.10-45, 1997.

[7] T. E. Harris, Contact interactions on a lattice, *The Annals of Probability*, pp.969-988, 1974.

[8] T. M. Liggett, *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes*, Springer Science & Business Media, 2013.

[9] L. Lovász, *Large Networks and Graph Limits*, American Mathematical Soc., 2012.

[10] C. Neuhauser, Ergodic theorems for the multitype contact process, *Probability Theory and Related Fields*, vol.91, nos.3-4, pp.467-506, 1992.