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STATE ESTIMATION FOR SCALAR LINEAR SYSTEMS WITH THE DATA-RATE LIMITATION

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Abstract. *This paper investigates the state estimation problem for scalar linear continuous time-invariant systems over a stationary memoryless uncertain digital channel with the data-rate limitation. In particular, a coder-decoder pair is constructed to ensure observability of the system employing a constant data rate provided by such a channel. The conditions on the constant data rate for observability are derived, and a lower bound on the constant data rate is given. Furthermore, it is shown in our results that the disturbances have important effect on observability in the case with the data-rate limitation. An illustrative example is given to demonstrate the effectiveness of the lower bound given.*
Keywords: State estimation, Constant data rate, Observability, Scalar linear systems

1. **Introduction.** In many engineering applications, control systems employ multiple sensors and actuators communicating over digital channels. In this framework, the data rate theorem refers to the smallest feedback data rate above which an unstable dynamical system can be stabilized [1]. The problem of state estimation for networked control systems has received increasing interest in recent years [2, 3].

This result was generalized to different notions of stabilization and system models. The research on Gaussian linear systems was addressed in [4]. Information theory was employed in control systems as a powerful conceptual aid, which extended existing fundamental limitations of feedback systems, and was used to derive necessary and sufficient conditions for robust stabilization of uncertain linear systems, Markov jump linear systems and unstructured uncertain systems [5-7]. Control under communication constraints inevitably suffers signal transmission delay, data packet dropout and measurement quantization which might be potential sources of instability and poor performance of control systems [8]. [9] investigated the quantized feedback control problem for stochastic time-invariant linear control systems. A predictive control policy under data-rate constraints was proposed to stabilize the unstable plant in the mean square sense. [10] addressed LQ (linear quadratic) control of MIMO (multi-input multi-output), discrete-time linear systems, and gave the inherent tradeoffs between LQ cost and data rates. In [11], a quantized-observer based encoding-decoding scheme was designed, which integrated the state observation with encoding-decoding. [12] addressed some of the challenging issues on moving horizon state estimation for networked control systems in the presence of multiple packet dropouts.

In the literature, the data rate of the channel was defined as a time-varying variable. In the existing results, the lower bounds on the data rate for observability were suitable for the average data rate. However, control performance should be guaranteed at each time step. For the case with the time-varying data rate, good performance can be achieved only in an average or expected sense. The chief difference in our case is that, the data rate is an invariant constant, and control performance can be guaranteed at each time step.

This paper is concerned with the state estimation problem for scalar linear continuous time-invariant systems over a stationary memoryless uncertain digital channel with the data-rate limitation. Our purpose here is to construct a coder-decoder pair which can ensure observability of the system employing a constant data rate provided by such a channel. Our work here differs in that we present a lower bound on the constant data rate for observability of the system. Furthermore, we also examine the role that the disturbances have on state estimation in the case with the data-rate limitation.

The remainder of this paper is organized as follows: Section 2 introduces problem formulation; Section 3 deals with state estimation problem under the data-rate limitation; the results of numerical simulation are presented in Section 4; conclusions are stated in Section 5.

2. **Problem Formulation.**

Consider the following linear continuous time-invariant system

$$\dot{x}(t) = ax(t) + w(t), \tag{1}$$

where $x(t) \in \mathbb{R}$ is the state process, and $w(t) \in \mathbb{R}^m$ is the disturbance, assumed to be Lebesgue-measurable. The parameter a is a known constant. Let $B_l(z)$ denote the set $\{x : |x - z| \leq l\}$ centered at z. The initial state $x(0)$ and $w(t)$ are bounded, uncertain variables satisfying $x(0) \in B_{\phi_0}(0)$ and $w(t) \in B_{\phi_w}(0)$, respectively, where ϕ_0 and ϕ_w are two known constants.

In this paper, we assume that the information on the plant state that is provided by sensor data, would be encoded and transmitted over a stationary memoryless uncertain digital channel without data dropout and time delay. The data rate R provided by such a channel is an invariant constant. Here, we focus on the role that the data rate has on the state estimation problem. Let h denote the uniform sampling interval and k is an integer. The state of the plant evolves in discrete time according to

$$x(k+1) = \lambda x(k) + w(k), \tag{2}$$

where we set $x(k) := x(kh)$, $w(k) := w(kh)$, and $\lambda := e^{ah}$.

Let $\hat{x}(k)$ and $e(k)$ denote the state estimate and the estimation error, respectively. We define the estimation error as

$$e(k) := x(k) - \hat{x}(k).$$

The state $x(k)$ is causally encoded via an operator Θ as

$$\alpha(k) = \Theta(k, x(0), x(1), \cdots, x(k)),$$

where the codeword $\alpha(k)$ is transmitted over such a channel, and decoded via an operator Υ as

$$\hat{x}(k) = \Upsilon(k, \hat{\alpha}(0), \hat{\alpha}(1), \cdots, \hat{\alpha}(k)),$$

where $\hat{\alpha}(k)$ denotes the received symbol at the decoder. Then, one can compute the state estimate $\hat{x}(k)$ at the decoder.

The main task in this paper is to present the condition on the data rate for observability of the system (1). Here, we want to give a lower bound on the data rate, which can ensure observability of the system (1) in the sense

$$\limsup_{k\to\infty} |e(k)| < \infty$$

using the finite, constant data rate provided by the communication channel.

3. **Conditions on the Constant Data Rate for Observability.** This section deals with the state estimation problem for scalar linear time-invariant systems over a stationary memoryless uncertain digital channel with a constant data rate.

It is shown in our results that, the disturbances have important effect on observability of the system (1). Then, we first consider the case without the disturbances. In the case with $\lambda < 1$,

$$\limsup_{k\to\infty} |e(k)| < \infty$$

holds though no information on the plant state is sent to the decoder. Thus, we examine the case with $\lambda \geq 1$, and give the following result.

Theorem 3.1. *Consider the system (1) without the disturbances. The initial* $x(0)$ *is a bounded, uncertain variable with the range* $B_{\phi_0}(0)$*. Assume that* $\lambda = e^{ah} \geq 1$ *holds. The system (1) is asymptotically observable in the sense*

$$\limsup_{k\to\infty} |e(k)| = 0$$

if the constant data rate R *provided by the communication channel satisfies the following condition:*

$$R > \lceil a \log_2 e \rceil \text{ (bits/s)},$$

where $\lceil \cdot \rceil$ *represents the ceil function, and is defined as* $\lceil x \rceil := \min\{k \in \mathbb{Z} : k > x\}$*.*

Proof: Notice that, the initial $x(0) \in B_{\phi_0}(0) = [-\phi_0, \phi_0]$ holds. Then, we may set

$$\hat{x}(0) = 0,$$
$$e(0) = x(0) - \hat{x}(0) \in B_{\phi_0}(0) = [-\phi_0, \phi_0]. \tag{3}$$

For any time k, we assume that

$$x(k) \in B_{l(k)}(c(k)) = [-l(k) + c(k), l(k) + c(k)],$$
$$\hat{x}(k) = c(k),$$
$$e(k) = x(k) - \hat{x}(k) \in B_{l(k)}(0) = [-l(k), l(k)], \tag{4}$$

where $l(k)$ and $c(k)$ denote the radius and midpoint of the range of $x(k)$, respectively.

At time $k+1$, the range of $x(k+1)$ and $e(k+1)$ would grow by λ because of the system dynamics. Here, the value of the plant state is quantized, encoded, and transmitted over a digital communication channel in order to make the estimation error reduce. Thus, we divide the range $[-l(k) + c(k), l(k) + c(k)]$ into d equal intervals. Clearly, $x(k)$ falls into one of d equal intervals. The d indexes corresponding to the d equal intervals are encoded, and converted into the d codewords of R bits. The codeword corresponding to the interval that $x(k)$ falls into is sent to the decoder. Then, the decoder may know the interval that $x(k)$ falls into, and compute the estimation error.

It follows from [13] that the data rate R must satisfy the following inequality:

$$R \geq \lceil \log_2 d \rceil \text{ (bits/sample)}. \tag{5}$$

At time $k + 1$, it follows that

$$x(k+1) \in B_{l(k+1)}(c(k+1)) = [-l(k+1) + c(k+1), l(k+1) + c(k+1)],$$
$$\hat{x}(k+1) = c(k+1),$$
$$e(k+1) = x(k+1) - \hat{x}(k+1) \in B_{l(k+1)}(0) = [-l(k+1), l(k+1)],$$

where

$$l(k+1) = \frac{\lambda}{d} l(k). \tag{6}$$

Combined with Equations (3), (4) and (6), this implies that

$$l(k) = \left(\frac{\lambda}{d}\right)^k \phi_0. \tag{7}$$

If assume that

$$d > \lambda = e^{ah} \tag{8}$$

holds, it follows that

$$\lim_{k\to\infty} l(k) = 0.$$

It leads to

$$\limsup_{k\to\infty} |e(k)| = 0.$$

Substitute (8) into (5) and we have

$$R \geq \lceil \log_2 d \rceil > \lceil ah \log_2 e \rceil \text{ (bits/sample)}.$$

Namely, this means that

$$R > \lceil a \log_2 e \rceil \text{ (bits/s)}. \tag{9}$$

Thus, the system (1) is asymptotically observable if Inequality (9) holds. □

We next address the state estimation problem for the system (1) with the disturbances, and further examine the role that the disturbances have on state estimation under the data-rate limitation.

Then, we give the following result.

Theorem 3.2. *Consider the system (1) with the disturbances. The initial state $x(0)$ and $w(t)$ are bounded, uncertain variables satisfying $x(0) \in B_{\phi_0}(0)$ and $w(t) \in B_{\phi_w}(0)$, respectively. Assume that $\lambda = e^{ah} \geq 1$ holds. The system (1) is observable in the sense*

$$\limsup_{k\to\infty} |e(k)| < \infty$$

if the constant data rate R provided by the communication channel satisfies the following condition:

$$R > \lceil a \log_2 e \rceil \text{ (bits/s)}.$$

Proof: Similarly, we have

$$e(0) = x(0) - \hat{x}(0) \in B_{\phi_0}(0) = [-\phi_0, \phi_0], \tag{10}$$

$$w(k) \in B_{\phi_w}(0) = [-\phi_w, \phi_w]. \tag{11}$$

Using the same techniques as in the proof of Theorem 3.1, we can show that at time $k+1$, it follows that

$$x(k) \in B_{l(k)}(c(k)) = [-l(k) + c(k), l(k) + c(k)],$$
$$\hat{x}(k) = c(k),$$
$$e(k) = x(k) - \hat{x}(k) \in B_{l(k)}(0) = [-l(k), l(k)],$$

where

$$l(k) = \left(\frac{\lambda}{d}\right)^k \phi_0 + \left[1 + \frac{\lambda}{d} + \cdots + \left(\frac{\lambda}{d}\right)^{k-1}\right] \phi_w.$$

If assume that

$$d > \lambda = e^{ah} \tag{12}$$

holds, it follows that

$$\lim_{k\to\infty} l(k) = \frac{1}{1 - \frac{\lambda}{d}} \phi_w.$$

It leads to

$$\limsup_{k\to\infty} |e(k)| = \frac{1}{1 - \frac{\lambda}{d}} \phi_w < \infty.$$

Substitute (12) into (5) and we have

$$R > \lceil a \log_2 e \rceil \text{ (bits/s)}. \tag{13}$$

Thus, the system (1) is observable if Inequality (13) holds. □

Remark 3.1. *For the system (1) without the disturbances, we may obtain*

$$\limsup_{k\to\infty} e(k) = 0.$$

Conversely, for the system (1) with the disturbances, we have

$$\limsup_{k\to\infty} e(k) = \frac{1}{1-\frac{\lambda}{d}}\phi_w < \infty.$$

Thus, the disturbances have important effect on state estimation of networked control systems. It is possible to ensure observability of the system (1) in spite of the disturbances if the data rate is larger than the lower bound given in our results.

4. **Numerical Example.** In this section, we give a numerical example to illustrate the effectiveness of the lower bound on the data rate given in our results. We consider a discrete-time control system

$$x(k+1) = 4.5x(k) + w(k).$$

Let the initial $x(0) \in B_5(0)$ and $w(k) \in B_1(0)$. It follows from Theorem 3.1 and Theorem 3.2 that $R > 2.17$ (bits/sample). Here, we set $R = 3$ (bits/sample). For the case without the disturbances, the corresponding simulation is given in Figure 1. It is shown that the system is asymptotically observable. For the case with the disturbances, the corresponding simulation is given in Figure 2. It states that the estimation error is bounded. Boundability is a very weak notion of observability.

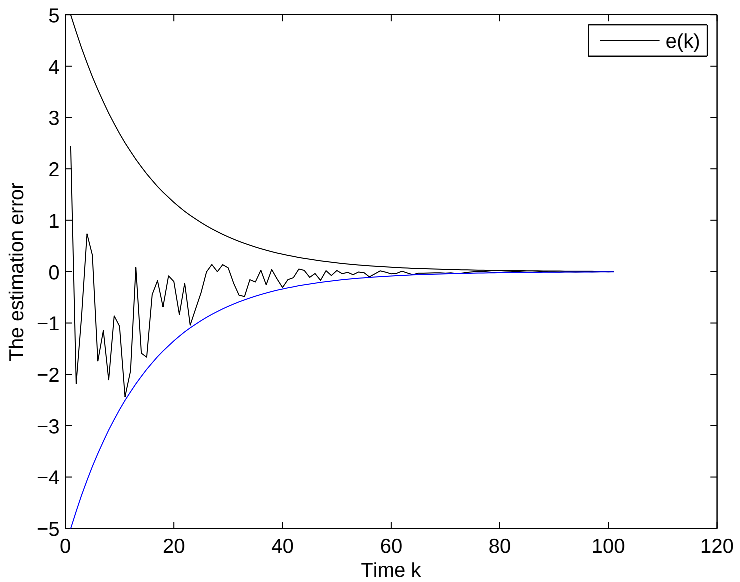

FIGURE 1. The estimation error responses in the case without the disturbances

5. **Conclusions.** In this paper, we addressed the state estimation problem for scalar linear continuous time-invariant systems. The information of the plant state was encoded and transmitted over a stationary memoryless uncertain digital channel with the limited data-rate. A coder-decoder pair was constructed to ensure observability of the system employing a constant data rate provided by such a channel. We derived the conditions on the constant data rate for observability. The simulation results have illustrated the effectiveness of the lower bound given. The study of nonlinear system with the limited data rate will be our future work.

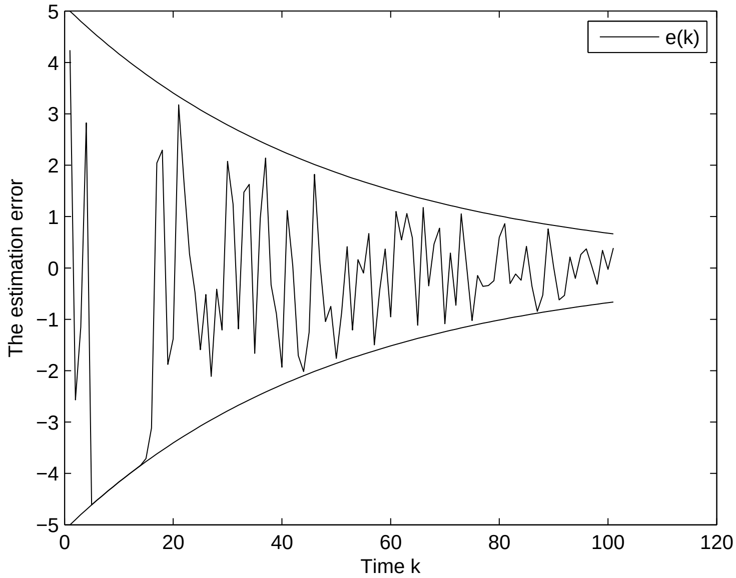

FIGURE 2. The estimation error responses in the case with the disturbances

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