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NUMERICAL SOLUTION OF VARIABLE ORDER FRACTIONAL DIFFERENTIAL EQUATION WITH CHEBYSHEV POLYNOMIALS

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Abstract. *Variable order calculus is a natural candidate to provide an effective mathematical framework for the description of complex dynamical problems. Therefore, we propose a numerical method to solve a kind of variable order nonlinear fractional differential equations with the second kind of Chebyshev polynomials. The main idea of this approach is that we derive two kinds of differential operational matrixes with Chebyshev polynomials. With the operational matrixes, the original equation is transformed into the products of several dependent matrixes, which can be viewed as an algebraic system after taking the collocation points. The numerical solution of the original equation is obtained by solving the algebraic system. Finally, the example shows that the numerical method is computationally efficient.*

Keywords: The second kind of Chebyshev polynomials, Variable order fractional differential equation, Differential operational matrixes

1. **Introduction.** Fractional differential equations (FDEs) generalized from classical integer order ones, which are obtained by replacing integer order derivatives by fractional ones. With further development of science research, more and more researchers have found that a variety of important dynamical problems exhibit fractional order behavior that may vary with time or space. This fact indicates that variable order calculus is a natural candidate to provide an effective mathematical framework for the description of complex dynamical problems. The modeling and application of variable order differential equations have been a front subject. However, since the kernel of the variable order operator has a variable exponent, analytical solutions of variable order FDEs are usually difficult to obtain. Therefore, the development of numerical methods in solving variable order FDEs is necessary.

Only a few authors studied numerical methods of variable order FDE. Coimbra [1] employed a consistent approximation for the solution of variable order FDE. Soon [2] proposed a second order Runge-Kutta method to numerically integrate the variable order FDE. Shen et al. [3] gave an approximate scheme for the variable order time fractional diffusion equation. Chen et al. [4, 5] paid their attention to Bernstein polynomials to solve variable order linear cable equation and variable order time fractional diffusion equation. An alternating direct method for the two dimensional variable order fractional percolation equation was proposed in [6]. Explicit and implicit Euler approximations for FDE were introduced in [7]. A numerical method with the Legendre polynomials is presented for a class of variable order FDEs in [8]. Chen et al. [9] introduced the numerical solution for a class of nonlinear variable order FDEs with Legendre wavelets. It is noteworthy that the Chebyshev polynomials family have beneficial properties so that they are widely used in approximation theory. For example, Sweilam et al. [10] used the second kind shifted Chebyshev polynomials for solving space fractional order diffusion equation. However, the second kind of Chebyshev polynomials have been paid less attention to solve variable order FDE. Accordingly, we will solve a kind of variable order nonlinear FDEs defined on the interval $[0, R]$, $R > 0$ with the second kind of Chebyshev polynomials. The FDE is shown as follows:

$$\begin{cases} D^{\alpha(t)}y^2(t) + a(t)D^{\beta(t)}y(t) + b(t)y'(t) = g(t), \quad (0 < \alpha(t), \beta(t) \le 1) \\ y(0) = y_0, \quad t \in [0, R], \end{cases} \tag{1}$$

where $D^{\alpha(t)}y^2(t)$ and $D^{\beta(t)}y(t)$ are the fractional derivatives in Caputo sense. $a(t)$ and $b(t)$ are assumed to be casual functions on $[0, R]$. $g(t)$ is known and $y(t)$ is unknown.

The basic idea of this approach is that we derive the kinds of differential operational matrixes with Chebyshev polynomials. With the operational matrixes, Equation (1) is transformed into the products of several dependent matrixes, which can be viewed as an algebraic system after taking the collocation points. By solving the algebraic system, the numerical solution of Equation (1) is acquired. Since the second kind of Chebyshev polynomials are orthogonal to each other, the operational matrixes with Chebyshev polynomials greatly reduce the size of computational work while accurately providing the series solution. From the example, we can see that the numerical solution gotten by our method is in good agreement with the exact solution, which demonstrates the correction and high efficiency of our method.

The paper is organized as follows. In Section 2, some necessary preliminaries are introduced. In Section 3, the basic definition and property of the second kind of Chebyshev polynomials are given. In Section 4, function approximation is given. In Section 5, two kinds of operational matrixes are derived and we applied the operational matrixes to solving the equation as given at beginning. In Section 6, we present the numerical example to demonstrate the efficiency of the algorithm. We presents conclusions resulting from the study in Section 7.

2. **Preliminaries.**

There are several definitions for variable order fractional derivatives, such as the one in Riemann-Liouville's sense and the one in Caputo's sense [11]. In this paper, the definition in Caputo's sense is considered.

Definition 2.1. *Captuo fractional derivate with order $\alpha(t)$ is defined by*

$$D^{\alpha(t)}y(t) = \frac{1}{\Gamma(1-\alpha(t))}\int_{0+}^{t}(t-\tau)y'(\tau)d\tau + \frac{y(0+)-y(0-)}{\Gamma(1-\alpha(t))}t^{-\alpha(t)}. \tag{2}$$

If we assume the starting time in a perfect situation, we can get Definition 2.2 as follows.

Definition 2.2.

$$D^{\alpha(t)}y(t) = \frac{1}{\Gamma(1-\alpha(t))}\int_{0}^{t}(t-\tau)y'(\tau)d\tau. \tag{3}$$

By Definition 2.2, we can get the following formula [4]

$$D^{\alpha(t)}t^n = \begin{cases} \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha(t))}t^{n-\alpha(t)}, \quad n = 1, 2, \cdots \\ 0, \quad n = 0 \end{cases} \tag{4}$$

3. **The Second Kind of Chebyshev Polynomials.**

The second kind of Chebyshev polynomials are defined on the interval $\mathrm{I} = [-1, 1]$. They are orthogonal to each other with respect to the weight function $\omega(x) = \sqrt{1-x^2}$. They satisfy the following formulas

$$U_0(x) = 1, \quad U_1(x) = 2x, \quad U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \quad n = 1, 2, \cdots ,$$

and

$$\int_{-1}^{1} \sqrt{1-x^2}U_n(x)U_m(x)dx = \begin{cases} \frac{\pi}{2}, & m = n, \\ 0, & m \neq n. \end{cases}$$

For $t \in [0, R]$, let $x = \frac{2t}{R} - 1$. We can get shifted second kind of Chebyshev polynomials $\tilde{U}_n(t) = U_n(\frac{2t}{R} - 1)$, which are also orthogonal with respect to the weight function $\omega(t) = \sqrt{tR - t^2}$ for $t \in [0, R]$. They satisfy the following formulas

$$\tilde{U}_0(t) = 1, \quad \tilde{U}_1(t) = \frac{4t}{R} - 2, \quad \tilde{U}_{n+1}(t) = 2\left(\frac{2t}{R} - 1\right)\tilde{U}_n(t) - \tilde{U}_{n-1}(t), \quad n = 1, 2, \cdots,$$

and

$$\int_{0}^{R} \sqrt{tR - t^2}\tilde{U}_n(t)\tilde{U}_m(t)dt = \begin{cases} \frac{\pi}{8}R^2, & m = n, \\ 0, & m \neq n. \end{cases}$$

The shifted second kind of Chebyshev polynomials can also be expressed as

$$\tilde{U}_n(t) = \begin{cases} 1, & n = 0, \\ \sum\limits_{k=0}^{[\frac{n}{2}]} (-1)^k \frac{(n-k)!}{k!(n-2k)!}(\frac{4t}{R} - 2)^{n-2k}, & n \geq 1, \end{cases} \tag{5}$$

where $[\frac{n}{2}]$ denotes the maximum integer which is no more than $\frac{n}{2}$.

Let $\Psi_n(t) = \left[\tilde{U}_0(t), \tilde{U}_1(t), \cdots, \tilde{U}_n(t)\right]^{\mathrm{T}}$ and $\mathbf{T}_n(t) = [1, t, \cdots, t^n]^{\mathrm{T}}$. Then

$$\Psi_n(t) = \mathbf{A}\mathbf{T}_n(t), \tag{6}$$

where

$$\mathbf{A} = \mathbf{B}\mathbf{C}. \tag{7}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ C_1^0\left(-\frac{R}{2}\right)^{1-0} & C_1^1\left(-\frac{R}{2}\right)^{1-1} & 0 & \cdots & 0 \\ C_2^0\left(-\frac{R}{2}\right)^{2-0} & C_2^1\left(-\frac{R}{2}\right)^{2-1} & C_2^2\left(-\frac{R}{2}\right)^{2-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ C_n^0\left(-\frac{R}{2}\right)^{n-0} & C_n^1\left(-\frac{R}{2}\right)^{n-1} & C_n^2\left(-\frac{R}{2}\right)^{n-2} & \cdots & C_n^n\left(-\frac{R}{2}\right)^{n-n} \end{bmatrix}.$$

If n is an even number, then

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & (-1)^0\frac{(1-0)!}{0!(1-0)!}(\frac{4}{R})^{1-0} & 0 & \cdots & 0 \\ (-1)^1\frac{(2-1)!}{1!(2-2)!}(\frac{4}{R})^{2-2} & 0 & (-1)^0\frac{(2-0)!}{0!(2-0)!}(\frac{4}{R})^{2-0} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^{\frac{n}{2}}\frac{(n-\frac{n}{2})!}{(\frac{n}{2})!(n-2\cdot\frac{n}{2})!}(\frac{4}{R})^{n-2\cdot\frac{n}{2}} & \cdots & (-1)^{\frac{n}{2}-1}\frac{(n-\frac{n}{2}+1)!}{(\frac{n}{2}-1)![n-2\cdot(\frac{n}{2}-1)]!}(\frac{4}{R})^{n-2\cdot(\frac{n}{2}-1)} & \cdots & (-1)^0\frac{(n-0)!}{0!(n-0)!}(\frac{4}{R})^{n-0} \end{bmatrix}.$$

If n is an odd number, then

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & (-1)^0\frac{(1-0)!}{0!(1-0)!}(\frac{4}{R})^{1-0} & 0 & \cdots & 0 \\ (-1)^1\frac{(2-1)!}{1!(2-2)!}(\frac{4}{R})^{2-2} & 0 & (-1)^0\frac{(2-0)!}{0!(2-0)!}(\frac{4}{R})^{2-0} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & (-1)^{\frac{n-1}{2}}\frac{\left(n-\frac{n-1}{2}\right)!}{\left(\frac{n-1}{2}\right)!\left(n-2\cdot\frac{n-1}{2}\right)!}\left(\frac{4}{R}\right)^{n-2\cdot\frac{n-1}{2}} & 0 & \cdots & (-1)^0\frac{(n-0)!}{0!(n-0)!}\left(\frac{4}{R}\right)^{n-0} \end{bmatrix}.$$

Therefore, we can easily gain Equation (8).

$$\mathbf{T}_n(t) = \mathbf{A}^{-1}\Psi_n(t). \tag{8}$$

4. Function Approximation.

Theorem 4.1. *Suppose that the function $y(t)$ is $n+1$ times continuously differentiable on the interval $[0, R]$. Let $u_n(t) = \sum_{i=0}^{n} \lambda_i \tilde{U}_i(t) = \Lambda^{\mathrm{T}} \Psi_n(t)$ be the best square approximation function of $y(t)$, where $\Lambda = [\lambda_0, \lambda_1, \cdots, \lambda_n]^{\mathrm{T}}$. Then*

$$\|y(t) - u_n(t)\|_2 \leq \frac{MS^{n+1}R}{(n+1)!}\sqrt{\frac{\pi}{8}},$$

where $M = \max_{t\in[0,R]} y^{n+1}(t)$ and $S = \max\{R - t_0, t_0\}$.

Proof: We consider the Taylor polynomials

$$y(t) = y(t_0) + y'(t-t_0) + \cdots + y^{(n)}(t_0)\frac{(t-t_0)^n}{n!} + y^{(n+1)}(\eta)\frac{(t-t_0)^{n+1}}{(n+1)!}, \quad t_0 \in [0, R]$$

where η is between t and t_0.

Let

$$p_n(t) = y(t_0) + y'(t-t_0) + \cdots + y^{(n)}(t_0)\frac{(t-t_0)^n}{n!}.$$

Then

$$|y(t) - p_n(t)| = \left| y^{(n+1)}(\eta)\frac{(t-t_0)^{n+1}}{(n+1)!} \right|.$$

Since $u_n(t) = \sum_{i=0}^{n} \lambda_i \tilde{U}_i(t) = \Lambda^{\mathrm{T}} \Psi_n(t)$ is the best square approximation function of $y(t)$, we can gain

$$\begin{aligned}
\|y(t) - u_n(t)\|_2^2 &\leq \|y(t) - p_n(t)\|_2^2 \\
&= \int_0^R \omega(t)\,[y(t) - p_n(t)]^2\,dt = \int_0^R \omega(t)\left[y^{(n+1)}(\eta)\frac{(t-t_0)^{n+1}}{(n+1)!}\right]^2 dt \\
&\leq \frac{M^2}{[(n+1)!]^2}\int_0^R \omega(t)(t-t_0)^{2n+2}dt \\
&= \frac{M^2}{[(n+1)!]^2}\int_0^R \sqrt{tR - t^2}(t-t_0)^{2n+2}dt.
\end{aligned}$$

Let $S = \max\{R - t_0, t_0\}$. Therefore,

$$\|y(t) - u_n(t)\|_2^2 \leq \frac{M^2 S^{2n+2}}{[(n+1)!]^2}\int_0^R \sqrt{tR - t^2}dt = \frac{M^2 S^{2n+2} R^2}{(n+1)!^2}\frac{\pi}{8}.$$

Finally, by taking the square root, Theorem 4.1 can be proved.

5. A Numerical Method for Solving Variable Order Nonlinear FDE.

5.1. Differential operational matrix of $D^{\alpha(t)}(u_n^2(t))$.

According to Equation (6) and Equation (4), we can get

$$\begin{aligned}
D^{\alpha(t)}(u_n^2(t)) &= D^{\alpha(t)}(\Lambda^{\mathrm{T}}\Psi_n\Psi_n^{\mathrm{T}}\Lambda) \\
&= D^{\alpha(t)}[\Lambda^{\mathrm{T}}\mathbf{A}\mathbf{T}_n(t)\mathbf{T}_n(t)^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\Lambda] \\
&= \Lambda^{\mathrm{T}}\mathbf{A}D^{\alpha(t)}[\mathbf{T}_n(t)\mathbf{T}_n(t)^{\mathrm{T}}]\mathbf{A}^{\mathrm{T}}\Lambda
\end{aligned}$$

$$
\begin{aligned}
&= \Lambda^{\mathrm{T}} \mathbf{A} D^{\alpha(t)} \begin{bmatrix} 1 & t & \cdots & t^n \\ t & t^2 & \cdots & t^{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ t^n & t^{n+1} & \cdots & t^{2n} \end{bmatrix} \mathbf{A}^{\mathrm{T}} \Lambda \\
&= \Lambda^{\mathrm{T}} \mathbf{A} D^{\alpha(t)} \begin{bmatrix} 0 & \frac{\Gamma(2)}{\Gamma(2-\alpha(t))} t^{1-\alpha(t)} & \cdots & \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha(t))} t^{n-\alpha(t)} \\ \frac{\Gamma(2)}{\Gamma(2-\alpha(t))} t^{1-\alpha(t)} & \frac{\Gamma(3)}{\Gamma(3-\alpha(t))} t^{2-\alpha(t)} & \cdots & \frac{\Gamma(n+2)}{\Gamma(n+2-\alpha(t))} t^{n+1-\alpha(t)} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha(t))} t^{n-\alpha(t)} & \frac{\Gamma(n+2)}{\Gamma(n+2-\alpha(t))} t^{n+1-\alpha(t)} & \cdots & \frac{\Gamma(2n+1)}{\Gamma(2n+1-\alpha(t))} t^{2n-\alpha(t)} \end{bmatrix} \mathbf{A}^{\mathrm{T}} \Lambda \\
&= \Lambda^{\mathrm{T}} \mathbf{A} \mathbf{M}_{\alpha(t)} \mathbf{A}^{\mathrm{T}} \Lambda,
\end{aligned} \tag{9}
$$

where

$$
\mathbf{M}_{\alpha(t)} = \begin{bmatrix} 0 & \frac{\Gamma(2)}{\Gamma(2-\alpha(t))} t^{1-\alpha(t)} & \cdots & \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha(t))} t^{n-\alpha(t)} \\ \frac{\Gamma(2)}{\Gamma(2-\alpha(t))} t^{1-\alpha(t)} & \frac{\Gamma(3)}{\Gamma(3-\alpha(t))} t^{2-\alpha(t)} & \cdots & \frac{\Gamma(n+2)}{\Gamma(n+2-\alpha(t))} t^{n+1-\alpha(t)} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha(t))} t^{n-\alpha(t)} & \frac{\Gamma(n+2)}{\Gamma(n+2-\alpha(t))} t^{n+1-\alpha(t)} & \cdots & \frac{\Gamma(2n+1)}{\Gamma(2n+1-\alpha(t))} t^{2n-\alpha(t)} \end{bmatrix}.
$$

$\mathbf{A}\mathbf{M}_{\alpha(t)}\mathbf{A}^{\mathrm{T}}$ is called the differential operational matrix of $D^{\alpha(t)}(u_n^2(t))$. Therefore,

$$
D^{\alpha(t)}(y_n^2(t)) \approx D^{\alpha(t)}(u_n^2(t)) = \mathbf{A}\mathbf{M}_{\alpha(t)}\mathbf{A}^{\mathrm{T}}. \tag{10}
$$

5.2. **Differential operational matrix of $D^{\alpha(t)}\Psi_n(t)$.** Let $D^{\alpha(t)}\Psi_n(t) = \mathbf{P}_{\alpha(t)}\Psi_n(t)$. $\mathbf{P}_{\alpha(t)}$ is called the differential operational matrix. The objective of this section is to generate this matrix.

According to Equation (6) and Equation (4), we have

$$
\begin{aligned}
D^{\alpha(t)}\Psi_n(t) &= D^{\alpha(t)}\left(\mathbf{A}\mathbf{T}_n(t)\right) = \mathbf{A} D^{\alpha(t)}\left([1, t, \cdots, t^n]^{\mathrm{T}}\right) \\
&= \mathbf{A}\left[0, \frac{\Gamma(2)}{\Gamma(2-\alpha(t))} t^{1-\alpha(t)}, \cdots, \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha(t))} t^{n-\alpha(t)}\right]^{\mathrm{T}} \\
&= \mathbf{A} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-\alpha(t))} t^{-\alpha(t)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha(t))} t^{-\alpha(t)} \end{bmatrix} \begin{bmatrix} 1 \\ t \\ \vdots \\ t^n \end{bmatrix} \\
&= \mathbf{A}\mathbf{N}_{\alpha(t)}\mathbf{A}^{-1}\Psi_n(t),
\end{aligned} \tag{11}
$$

where

$$
\mathbf{N}_{\alpha(t)} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-\alpha(t))} t^{-\alpha(t)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha(t))} t^{-\alpha(t)} \end{bmatrix}.
$$

Therefore,

$$
D^{\alpha(t)}\Psi_n(t) = \mathbf{A}\mathbf{N}_{\alpha(t)}\mathbf{A}^{-1}\Psi_n(t).
$$

$$
\mathbf{P}_{\alpha(t)} = \mathbf{A}\mathbf{N}_{\alpha(t)}\mathbf{A}^{-1}. \tag{12}
$$

$$
D^{\alpha(t)}u_n(t) = \Lambda^{\mathrm{T}}\mathbf{P}_{\alpha(t)}\Psi_n(t). \tag{13}
$$

In particular, for $\alpha(t) = 1$, we can get

$$
\Psi_n'(t) = \mathbf{A}\mathbf{N}_1\mathbf{A}^{-1} = \mathbf{P}_1\Psi_n(t). \tag{14}
$$

$$
u_n'(t) = \Lambda^{\mathrm{T}}\mathbf{P}_1\Psi_n(t). \tag{15}
$$

5.3. **The method in solving the variable order nonlinear FDE.** Let $y(t) \approx u_n(t) = \sum_{i=0}^{n} \lambda_i \tilde{U}_i(t) = \Lambda^{\mathrm{T}}\Psi_n(t)$. According to Equations (10), (13) and (15), the original Equation (1) is transformed into Equation (16) as follows:

$$\Lambda^{\mathrm{T}}\mathbf{A}\mathbf{M}_{\alpha(t)}\mathbf{A}^{\mathrm{T}}\Psi_n(t) + a(t)\Lambda^{\mathrm{T}}\mathbf{P}_{\alpha(t)}\Psi_n(t) + b(t)\Lambda^{\mathrm{T}}\mathbf{P}_1\Psi_n(t) = g(t), \quad t \in [0, R]. \tag{16}$$

Taking the collocation points $t_i = R\frac{2i+1}{2(n+1)}$, $i = 0, 1, \cdots, n$ to process Equation (16), we can gain Equation (17), i.e.,

$$\Lambda^{\mathrm{T}}\mathbf{A}\mathbf{M}_{\alpha(t_i)}\mathbf{A}^{\mathrm{T}}\Psi_n(t_i) + a(t)\Lambda^{\mathrm{T}}\mathbf{P}_{\alpha(t_i)}\Psi_n(t_i) + b(t_i)\Lambda^{\mathrm{T}}\mathbf{P}_1\Psi_n(t_i) = g(t_i). \tag{17}$$

Solving Equation (17) by Newton method [12], we can gain the vector $\Lambda = [\lambda_0, \lambda_1, \cdots, \lambda_n]^{\mathrm{T}}$. Subsequently, numerical solution $u_n(t) = \Lambda^{\mathrm{T}}\Psi_n(t)$ of Equation (1) is obtained.

6. **The Numerical Example and Result Analysis.** In this section, we verify the efficiency of our method to support above theoretical discussion. We compare the numerical solution with the analytical solution. The results indicate that our method is a powerful tool for solving variable order FDE. In this section, the notation

$$\varepsilon = \max_{i=0,1,\cdots,n} |y(t_i) - u_n(t_i)|$$

is used to show the precision of our proposed algorithm, where $t_i = R\frac{2i+1}{2(n+1)}$, $i = 0, 1, \cdots, n$.

Example 6.1. *Consider the following nonlinear variable order FDE:*

$$\begin{cases} D^{\frac{\sin t}{4}} y^2(t) + (t+1)D^{\frac{t}{4}} y(t) + t^2 y'(t) = g(t) \\ y(0) = 0, t \in [0, 3] \end{cases}$$

where

$$g(t) = \frac{128 t^{2-\frac{\sin t}{4}} \left[48(t+2)^2 - 4(3t+7)\sin t + (\sin t)^2)\right]}{(\sin t - 16)(\sin t - 12)(\sin t - 8)(\sin t - 4)\Gamma\left(1 - \frac{\sin t}{4}\right)} + \frac{8t^{1-\frac{t}{4}}(t+1)(3t+8)}{(t-8)(t-4)\Gamma\left(1-\frac{t}{4}\right)} + 2t^3 + 2t^2.$$

The exact solution is $y(t) = t^2 + 2t$. We find the numerical solution of Example 6.1 in MATLAB 2012 by our method.

Table 1. Computational results of Example 6.1 for different values of n

t	Λ	ε
$n = 3$	$[5.8125, 3.7500, 0.5625, 0.0000]^{\mathrm{T}}$	1.7764e-15
$n = 4$	$[5.8125, 3.7500, 0.5625, 0.0000, 0.0000]^{\mathrm{T}}$	7.1054e-15
$n = 5$	$[5.8125, 3.7500, 0.5625, -0.0000, 0.0000, -0.0000]^{\mathrm{T}}$	5.2958e-14
$n = 6$	$[5.8125, 3.7500, 0.5625, -0.0000, -0.0000, -0.0000, -0.0000]^{\mathrm{T}}$	1.6115e-13

The computational results are shown in Table 1 and Figure 1. As seen from Table 1, the vector Λ obtained is mainly composed of three terms, namely $\lambda_0 = 5.8125$, $\lambda_1 = 3.7500$, and $\lambda_2 = 0.5625$. This is in agreement with the exact solution $y(t) = t^2 + 2t$, because the exact solution is a polynomial of the 2nd degree. Furthermore, every value of ε is very small for different values of n. In addition, with n increasing, the values of ε are gradually bigger because of round-off error in MATLAB. Therefore, the approximation effect of $n = 3$ is best for different values of n. Therefore, we can conclude that only a small number of the second kind of Chebyshev polynomials needed can reach high precision.

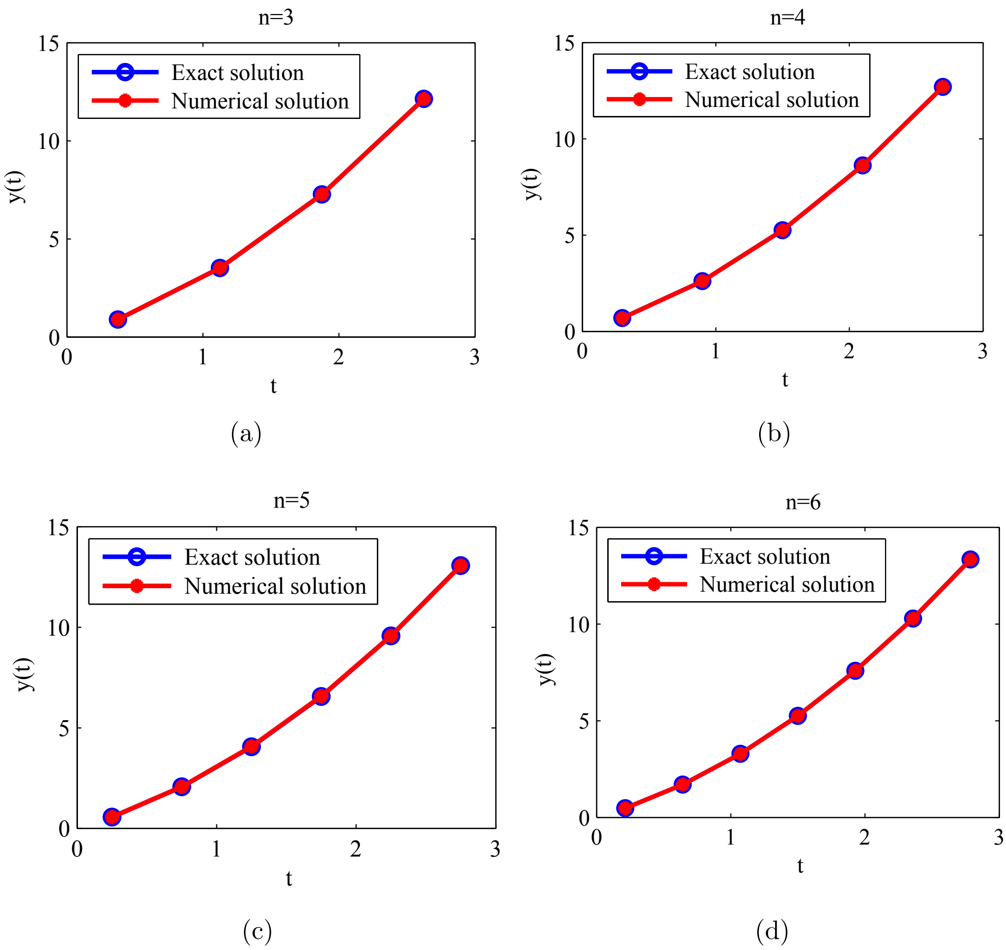

FIGURE 1. The exact solution and numerical solution of Example 6.1 for different values of n

In Figure 1, we have plotted the exact and the numerical solutions for different values of n respectively. It is evident that the numerical solution converges to the exact solution for each value of n. Furthermore, the numerical solutions are in good agreement with the exact solution. When $n = 3$, there has already been a good approximation effect, which verifies the correction and high efficiency of our method.

7. **Conclusions.** In this paper, we present a numerical method in solving the variable order nonlinear FDE with the second kind of Chebyshev polynomials. Taking advantage of the definition of the variable order fraction derivative and the simplicity of the second kind of Chebyshev polynomials, we transform the FDE into an algebraic system. By solving the algebraic system, the numerical solution is acquired. Numerical example shows that the numerical solution is in very good coincidence with the exact solution. The presented method can also be used to solve two-dimensional variable order fractional partial differential equations, which is left for further research.

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