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PREDICTION INTERVAL OF FUTURE WAITING TIMES OF THE EXPONENTIAL DISTRIBUTION UNDER MULTIPLY TYPE II CENSORING

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Abstract. *Wu (2015) proposed the general weighted moments estimators (GWMEs) of the scale parameter of the one-parameter exponential distribution based on a multiply type II censored sample and claimed that the proposed estimator outperforms the other 14 estimators in terms of the exact mean squared errors (MSEs) in most cases. We (2015) used the GWMEs to construct the pivotal quantities to construct the prediction intervals for future observations. In practice, users also want to know the waiting times between two consecutive future observations. Therefore, in this paper we propose the prediction interval for future waiting times or interarrival time. One real life example is given to illustrate the prediction intervals based on GWMEs.*
Keywords: Multiply type II censored sample, Exponential distribution, General weighted moments estimator, Prediction interval

1. **Introduction.** In most literature of reliability, the exponential distribution is widely used as a model of lifetime data. There are many applications of exponential distribution in the fields of analysis of reliability and the life test experiments. Please see some examples in Johnson et al. in 1994. The failure time Y follows a one-parameter exponential distribution if the probability density function (p.d.f.) of Y is given by $f(y) = \frac{1}{\theta}\exp(-\frac{y}{\theta})$, $y \geq 0$, $\theta > 0$, where θ is the scale parameter.

In life testing experiments, the experimenters may not be able to obtain the lifetimes of all items that are put on test due to the artificial mistakes or for implementing some purposes of experimental designs. Suppose that n items are put on the lifetest and the first r, middle l and the last s are unobserved or missing, this type of censoring is called the multiply type II censoring. Wu et al. proposed the simultaneous confidence intervals for all distances from the extreme populations for two-parameter exponential populations based on the multiply type II censored samples in 2011. Wu in 2015 proposed some weighted moments estimators (GWMEs) by assigning a single weight to each observation instead of considering only two weights in Wu and Yang in 2002 for exponential distribution under multiply type II censoring. The simulation comparison results show that the GWMEs outperform the 12 weighted moments estimators proposed by Wu and Yang in 2002 and approximate maximum likelihood estimator (AMLE) by Balakrishnan in 1990 and the best linear unbiased estimator (BLUE) by Balasubramanian and Balakrishnan in terms of the exact mean squared errors (MSEs) in most cases for exponential distribution in 1992. Wu proposed the prediction intervals of future observations based on the GWMEs in 2015. In practice, most users also want to know the waiting times between two consecutive future observations. Therefore, the purpose of this research is utilizing GWMEs to construct a pivotal quantity and use it to build the prediction interval of future waiting times or interarrival times between the two consecutive future observations. The structure of this research is organized as follows. In Section 2, the GWME for the one-parameter exponential distribution is introduced. In Section 3, the prediction intervals of future waiting times are introduced. One real life example to illustrate the proposed intervals is given in Section 4. At last, the conclusion is discussed in Section 5.

2. **The General Weighted Moments Estimation of the Scale Parameter of the One-Parameter Exponential Distribution.** Suppose that the lifetimes Y follows a one-parameter exponential distribution with p.d.f. given by $f(y) = \frac{1}{\theta}\exp(-\frac{y}{\theta})$, $y \geq 0$, $\theta > 0$, where θ is the scale parameter. Let $Y_{(r+1)} < \ldots < Y_{(r+k)} < Y_{(r+k+l+1)} < \ldots < Y_{(n-s)}$ be the available multiply type II censored sample from the above distribution.

The GWME to estimate the scale parameter θ is defined as $\hat{\theta} = W_{r+1}Y_{(r+1)} + \ldots + W_{r+k}Y_{(r+k)} + W_{r+k+l+1}Y_{(r+k+l+1)} + \ldots + W_{n-s}Y_{(n-s)} = \underset{\sim}{W}^T \underset{\sim}{Y}$, where $\underset{\sim}{W} = [W_{r+1}, \ldots, W_{r+k}, W_{r+k+l+1}, \ldots, W_{n-s}]^T$ and $\underset{\sim}{Y} = [Y_{(r+1)}, \ldots, Y_{(r+k)}, Y_{(r+k+l+1)}, \ldots, Y_{(n-s)}]^T$.

The weights $\underset{\sim}{W} = [W_{r+1}, \ldots, W_{r+k}, W_{r+k+l+1}, \ldots, W_{n-s}]^T$ are determined so that the MSE of the proposed GWME is minimized. The optimal weights are $\underset{\sim}{W} = A^{-1}\underset{\sim}{a}$, where

$$A = \begin{bmatrix} b_{r+1} + a_{r+1}^2 & \cdots & b_{r+1,r+k} + a_{r+1}a_{r+k} & \cdots & b_{r+1,n-s} + a_{r+1}a_{n-s} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ b_{r+1,r+k} + a_{r+1}a_{r+k} & \cdots & b_{r+k} + a_{r+k}^2 & \cdots & b_{r+k,n-s} + a_{r+k}a_{n-s} \\ b_{r+1,r+k+l+1} + a_{r+1}a_{r+k+l+1} & \cdots & b_{r+k,r+k+l+1} + a_{r+k}a_{r+k+l+1} & \cdots & \vdots \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ b_{r+1,n-s} + a_{r+1}a_{n-s} & \cdots & \vdots & \cdots & b_{n-s} + a_{n-s}^2 \end{bmatrix},$$

$\underset{\sim}{a} = [a_{r+1}, \ldots, a_{r+k}, a_{r+k+l+1}, \ldots, a_{n-s}]^T$ and $a_i = E(Y_{(i)})/\theta = \sum_{j=1}^{i} \frac{1}{n-j+1}$, $i = r+1, \ldots, r+k,\ r+k+l+1, \ldots, n-s$.

The GWME with minimum MSE is obtained as

$$\hat{\theta} = \underset{\sim}{W}^T \underset{\sim}{Y} = \underset{\sim}{a}^T A^{-1} \underset{\sim}{Y}. \tag{1}$$

The minimum MSE of GWME is

$$MSE\left(\hat{\theta}\right) = \left(\underset{\sim}{W}^T B \underset{\sim}{W} + \left(\underset{\sim}{W}^T \underset{\sim}{a} - 1\right)^2\right)\theta^2, \tag{2}$$

where $B = [b_{i,j}]_{i=r+1,\ldots,r+k,r+k+l+1,\ldots,n-s,j=r+1,\ldots,r+k,r+k+l+1,\ldots,n-s}$ are the mean vector and the covariance matrix of the random vector $\underset{\sim}{Y}$, $b_{i,i} = b_i = Var(Y_{(i)})/\theta^2 = \sum_{j=1}^{i} \frac{1}{(n-j+1)^2}$, $i = r+1, \ldots, r+k, r+k+l+1, \ldots, n-s$, and $b_{i,j} = Cov(Y_{(i)}, Y_{(j)})/\theta^2 = \sum_{j=1}^{i} \frac{1}{(n-j+1)^2}$, $i \neq j, (i,j) \in \{r+1, \ldots, r+k, r+k+l+1, \ldots, n-s\}$.

3. **Prediction Intervals of Waiting Time.** In order to predict the waiting time, the pivotal quantity is considered as $V = (Y_{(j)} - Y_{(j-1)})/\hat{\theta}$, $n-s < j \leq n$ based on the GWME $\hat{\theta}$ defined in (1). Since $\frac{Y_{(1)}}{\theta}, \ldots, \frac{Y_{(n)}}{\theta}$ are the n order statistics from a standard exponential distribution and $\frac{\hat{\theta}}{\theta} = \frac{\underset{\sim}{W}^T \underset{\sim}{Y}}{\theta}$ is a linear combination of n order statistics from a standard exponential distribution, the distribution of pivotal quantity $V = \left(\frac{Y_{(j)} - Y_{(j-1)}}{\theta}\right) \Big/ \frac{\hat{\theta}}{\theta}$ is independent of θ, $n-s < j \leq n$. Let $V(\delta; n, j, r, k, l, s)$ be the δ percentile of the distribution of V satisfying $P(V \leq V(\delta; n, j, r, k, l, s)) = \delta$.

Make use of the pivotal quantity, and the prediction interval of waiting time $Y_{(j)} - Y_{(j-1)}$, $n-s < j \leq n$ is proposed in the following theorem.

Theorem 3.1. *For multiply type II censored sample* $Y_{(r+1)} < \ldots < Y_{(r+k)} < Y_{(r+k+l+1)} < \ldots < Y_{(n-s)}$*, the prediction interval of waiting time* $Y_{(j)} - Y_{(j-1)}$*,* $n - s < j \leq n$ *is*

$$\left(V\left(\frac{\alpha}{2}; n, j, r, k, l, s\right)\hat{\theta}, V\left(1 - \frac{\alpha}{2}; n, j, r, k, l, s\right)\hat{\theta}\right).$$

Table 1. The δ percentile of the pivotal quantity $V = \left(Y_{(j)} - Y_{(j-1)}\right)/\hat{\theta}$ and $P(V \leq V(\delta; n, j, r, k, l, s)) = \delta$

						δ									
n	r	k	l	s	j	0.005	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990	0.995
12	2	5	2	2	11	0.0028	0.0056	0.0140	0.0283	0.0581	1.4295	1.9319	2.4659	3.2330	3.8490
12	2	5	2	2	12	0.0055	0.0111	0.0278	0.0566	0.1168	2.8593	3.8566	4.9312	6.4640	7.7520
12	3	5	2	1	12	0.0056	0.0111	0.0277	0.0562	0.1159	2.8082	3.7764	4.8062	6.2710	7.4750
12	3	5	1	2	11	0.0027	0.0055	0.0139	0.0282	0.0581	1.4264	1.9231	2.4591	3.2270	3.8730
12	3	5	1	2	12	0.0054	0.0110	0.0279	0.0562	0.1162	2.8513	3.8437	4.9165	6.4610	7.6960
12	2	5	3	1	12	0.0055	0.0110	0.0276	0.0561	0.1160	2.8168	3.7882	4.8284	6.3160	7.5120
12	2	5	1	3	10	0.0019	0.0038	0.0094	0.0192	0.0395	0.9747	1.3202	1.6906	2.2260	2.6780
12	2	5	1	3	11	0.0028	0.0056	0.0141	0.0286	0.0592	1.4599	1.9800	2.5364	3.3480	4.0200
12	2	5	1	3	12	0.0056	0.0113	0.0284	0.0573	0.1183	2.9192	3.9525	5.0721	6.6990	8.0710
12	1	5	3	2	11	0.0028	0.0057	0.0141	0.0285	0.0584	1.4302	1.9314	2.4721	3.2510	3.8840
12	1	5	3	2	12	0.0055	0.0110	0.0277	0.0567	0.1168	2.8630	3.8687	4.9420	6.4750	7.7570
12	1	5	2	3	10	0.0018	0.0037	0.0094	0.0191	0.0393	0.9733	1.3200	1.6948	2.2370	2.6880
12	1	5	2	3	11	0.0028	0.0056	0.0140	0.0285	0.0589	1.4605	1.9769	2.5358	3.3430	3.9960
12	1	5	2	3	12	0.0057	0.0114	0.0282	0.0573	0.1178	2.9172	3.9534	5.0759	6.7190	8.0030
24	2	5	2	2	23	0.0026	0.0052	0.0131	0.0268	0.0552	1.2691	1.6759	2.0990	2.6770	3.1280
24	2	5	2	2	24	0.0053	0.0105	0.0263	0.0534	0.1102	2.5407	3.3544	4.1980	5.3600	6.2720
24	3	5	2	1	24	0.0051	0.0104	0.0264	0.0539	0.1105	2.5273	3.3401	4.1808	5.3270	6.2330
24	3	5	1	2	23	0.0026	0.0053	0.0132	0.0267	0.0551	1.2686	1.6775	2.0991	2.6720	3.1280
24	3	5	1	2	24	0.0051	0.0106	0.0265	0.0536	0.1101	2.5398	3.3585	4.1996	5.3570	6.2430
24	2	5	3	1	24	0.0052	0.0105	0.0264	0.0535	0.1101	2.5258	3.3347	4.1728	5.3190	6.2090
24	2	5	1	3	22	0.0017	0.0035	0.0089	0.0180	0.0370	0.8490	1.1235	1.4075	1.8000	2.0980
24	2	5	1	3	23	0.0027	0.0053	0.0133	0.0270	0.0553	1.2745	1.6837	2.1126	2.6970	3.1530
24	2	5	1	3	24	0.0053	0.0106	0.0266	0.0539	0.1108	2.5562	3.3779	4.2238	5.3920	6.3160
24	1	5	3	2	23	0.0026	0.0052	0.0132	0.0267	0.0550	1.2690	1.6790	2.1015	2.6820	3.1450
24	1	5	3	2	24	0.0051	0.0105	0.0266	0.0536	0.1099	2.5324	3.3564	4.1955	5.3490	6.2460
24	1	5	2	3	22	0.0017	0.0034	0.0088	0.0179	0.0369	0.8509	1.1248	1.4077	1.7960	2.1040
24	1	5	2	3	23	0.0026	0.0052	0.0133	0.0267	0.0551	1.2753	1.6855	2.1144	2.6980	3.1560
24	1	5	2	3	24	0.0053	0.0105	0.0266	0.0542	0.1111	2.5521	3.3774	4.2261	5.4050	6.3210
36	2	5	2	2	35	0.0026	0.0052	0.0130	0.0263	0.0541	1.2274	1.6089	2.0047	2.5370	2.9460
36	2	5	2	2	36	0.0051	0.0102	0.0257	0.0529	0.1090	2.4496	3.2212	4.0022	5.0760	5.9030
36	3	5	2	1	36	0.0052	0.0105	0.0261	0.0527	0.1083	2.4490	3.2206	3.9991	5.0570	5.8690
36	3	5	1	2	35	0.0026	0.0052	0.0131	0.0265	0.0544	1.2262	1.6120	2.0036	2.5460	2.9520
36	3	5	1	2	36	0.0050	0.0101	0.0259	0.0527	0.1084	2.4503	3.2191	4.0088	5.0760	5.8930
36	2	5	3	1	36	0.0051	0.0103	0.0261	0.0530	0.1088	2.4430	3.2090	3.9900	5.0490	5.8540
36	2	5	1	3	34	0.0017	0.0035	0.0088	0.0177	0.0362	0.8183	1.0766	1.3388	1.6990	1.9800
36	2	5	1	3	35	0.0026	0.0051	0.0130	0.0265	0.0546	1.2293	1.6164	2.0157	2.5560	2.9640
36	2	5	1	3	36	0.0052	0.0103	0.0259	0.0529	0.1084	2.4600	3.2309	4.0266	5.1040	5.9350
36	1	5	3	2	35	0.0026	0.0052	0.0130	0.0263	0.0539	1.2278	1.6130	2.0122	2.5500	2.9640
36	1	5	3	2	36	0.0052	0.0103	0.0260	0.0526	0.1085	2.4530	3.2273	4.0141	5.0720	5.8830
36	1	5	2	3	34	0.0017	0.0034	0.0087	0.0176	0.0361	0.8195	1.0774	1.3415	1.6970	1.9750
36	1	5	2	3	35	0.0026	0.0052	0.0131	0.0265	0.0543	1.2295	1.6159	2.0124	2.5500	2.9630
36	1	5	2	3	36	0.0052	0.0102	0.0260	0.0530	0.1087	2.4598	3.2296	4.0192	5.0910	5.9310

Proof: Observe that $1-\alpha = P\left(V\left(\frac{\alpha}{2}; n, j, r, k, l, s\right) \le \left(\frac{Y_{(j)}-Y_{(j-1)}}{\hat{\theta}}\right) \le V\left(1-\frac{\alpha}{2}; n, j, r, k, l, s\right)\hat{\theta}\right) = P\left(V\left(\frac{\alpha}{2}; n, j, r, k, l, s\right)\hat{\theta} \le Y_{(j)} - Y_{(j-1)} \le V\left(1-\frac{\alpha}{2}; n, j, r, k, l, s\right)\hat{\theta}\right)$.

Since the exact distribution of V is too hard to derive algebraically, the δ percentile of the distributions of V is obtained based on Monte Carlo simulation. Moreover, all the simulations were run with the aid of AbSoft Fortran Inclusive of IMSL in 1999. In the simulation, 600,000 replicates are used to compute the percentiles of V for each combination of n, r, k, l, s, j, where $j = n-s+1, \ldots, n$. Due to the limitation of the number of pages, only a part of the percentiles of V are given in Table 1, for $\delta = 0.005, 0.010, 0.025, 0.050, 0.100, 0.900, 0.950, 0.975, 0.990, 0.995$ under $n = 12, 24, 36$ (see Table 1). Any specific percentile $V(\delta; n, j, r, k, l, s)$ for any censoring scheme (n, r, k, l, s) for the jth waiting time between the jth future observation and the previous one, $j = n-s+1, \ldots, n$, can be obtained by the software program provided by the author.

4. **Example.** An example of times to breakdown of an insulating fluid between electrodes recorded at five different voltages [6] is considered to illustrate the prediction interval of waiting time. Such a distribution of times to breakdown of an insulating fluid between electrodes is usually assumed to be exponentially distributed in engineering theory. We choose 35 kV, and suppose the data with $n = 12$, $r = 2$, $k = 3$, $l = 1$ and $s = 5$. The multiply type II censored failure times (seconds) are: –, –, 41, 87, 93, –, 116, –, –, –, –, –. The weights are 0.23568, 0.12544, 0.19776, 0.8058 and the estimated scale parameter is

$$\begin{aligned}
\hat{\theta} &= W_{r+1}Y_{(r+1)} + \ldots + W_{r+k}Y_{(r+k)} + W_{r+k+l+1}Y_{(r+k+l+1)} + \ldots + W_{n-s}Y_{(n-s)} \\
&= W_{(3)}Y_{(3)} + W_{(4)}Y_{(4)} + W_{(5)}Y_{(5)} + W_{(7)}Y_{(7)} \\
&= 41 * 0.23568 + 87 * 0.12544 + 93 * 0.19776 + 116 * 0.8058 = 132.4406.
\end{aligned}$$

Using Theorem 3.1, the 90% and 95% prediction intervals for waiting times $Y_{(8)} - Y_{(7)}$, $Y_{(9)} - Y_{(8)}$, $Y_{(10)} - Y_{(9)}$, $Y_{(11)} - Y_{(10)}$, $Y_{(12)} - Y_{(11)}$ are obtained in the following table.

TABLE 2. 90% and 95% prediction interval for waiting time $Y_{(j)} - Y_{(j-1)}$, $j = 8, \ldots, 12$

90%		
Future waiting time	$V(0.05; 12, j, 2, 3, 1, 5), V(0.95; 12, j, 2, 3, 1, 5)$	Prediction interval
$Y_{(8)} - Y_{(7)}$	0.0118, 0.8531	$(1.5628, 112.9851)$
$Y_{(9)} - Y_{(8)}$	0.0148, 1.0697	$(1.9601, 141.6717)$
$Y_{(10)} - Y_{(9)}$	0.0198, 1.4255	$(2.6223, 188.7941)$
$Y_{(11)} - Y_{(10)}$	0.0295, 2.1350	$(3.9070, 282.7607)$
$Y_{(12)} - Y_{(11)}$	0.0589, 4.2827	$(7.8007, 567.2034)$
95%		
Future waiting time	$V(0.025; 12, j, 2, 3, 1, 5), V(0.975; 12, j, 2, 3, 1, 5)$	Prediction interval
$Y_{(8)} - Y_{(7)}$	0.0058, 1.1098	$(0.76816, 146.9826)$
$Y_{(9)} - Y_{(8)}$	0.0073, 1.3875	$(0.9668, 183.7613)$
$Y_{(10)} - Y_{(9)}$	0.0098, 1.8605	$(1.2979, 246.4057)$
$Y_{(11)} - Y_{(10)}$	0.0146, 2.7797	$(1.9336, 368.1451)$
$Y_{(12)} - Y_{(11)}$	0.0292, 5.5634	$(3.8673, 736.8200)$

5. **Conclusion.** In this paper, the GWMEs are used to construct a pivotal quantity to build a prediction interval for future waiting times or interarrival times between two consecutive future observations. The percentiles of proposed pivotal quantity are obtained by Monte-Carlo method and tabulated in Table 1. Theorem 3.1 is proposed to build the prediction intervals based on GWME for multiply type II censored sample. At last, one real life example is given to illustrate the proposed prediction intervals for a specific structure of censoring scheme. In the future, this research can be extended to other location scale family, for example, extreme-value distribution or pareto distribution.

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