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PREMISE REDUCTION OF FUZZY INFERENCE BASED ON HIERARCHICAL VARIABLE WEIGHTS

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Abstract. *In this paper, we analyze some problems existing in rule premise reduction of multidimensional fuzzy inference, and propose a new premise reduction method based on hierarchical variable weights in factor space theory, which not only shows relative importance of antecedent components in fuzzy inference, but considers different preference requirements of decision makers. In addition, it is proved that the reduction of rule premises does not change the interpolation approximation of the fuzzy systems constructed by the improved CRI (Compositional Rule of Inference). An illustrative example shows that the proposed premise reduction method is practicable and effective.*
Keywords: Fuzzy inference, Factor space, Hierarchical variable weights, CRI, Fuzzy systems

1. **Introduction.** The structure of a fuzzy control system includes three parts, i.e., fuzzification, fuzzy inference and defuzzification, and the design of a fuzzy inference engine is one of the most critical parts. In 1973, Zadeh proposed CRI [1] such that the research of fuzzy inference and fuzzy systems develops rapidly. Mamdani used the CRI to fuzzy control and obtained successful application [2]. Furthermore, Buckley et al. [3-17] discussed the mechanism of fuzzy control systems and found some significant conclusions. Particularly, Li proved that a fuzzy control system is an interpolation function in mathematical essence [11], and proposed variable universe adaptive fuzzy control theory [12]. Based on the idea, an efficient fuzzy controller was designed, which was successfully applied to the quadruple inverted pendulum [13]. Obviously, the application of the fuzzy control theory based on CRI has achieved remarkable success in practice for the past few decades. However, the logical foundation and some details of fuzzy systems constructed by CRI are worthy to be further studied [14-19]. In particular, this paper will focus on the premise reduction of multidimensional fuzzy systems based on multiple premise CRI.

2. **Problem Description.** In multiple premise CRI, the general reasoning mode is as follows.

$$\begin{array}{ll}\text{Known } A_{11} \text{ and } A_{12} \text{ and } \cdots \text{ and } A_{1m} \longrightarrow B_1 \\ \quad\cdots\ \cdots \\ \quad A_{n1} \text{ and } A_{n2} \text{ and } \cdots \text{ and } A_{nm} \longrightarrow B_n \\ \text{given } \ A_1^* \text{ and } A_2^* \text{ and } \cdots \text{ and } A_m^* \\ \hline \text{To solve} \qquad\qquad\qquad\qquad B^*, \end{array} \tag{1}$$

where $A_{ij},\ A_j^* \in \mathcal{F}(X_j),\ B_i,\ B^* \in \mathcal{F}(Y)$ $(i = 1, \cdots, n;\ j = 1, \cdots, m)$. The solution procedure of (1) can be divided into three parts.

(P.1) Multiple premises need to be reduced to single premise in every rule by some reduction method. For example, taking

$$A_i = A_{i1} \times A_{i2} \times \cdots \times A_{im} \quad (i = 1, \cdots, n), \tag{2}$$

accordingly we have $A^* = A_1^* \times A_2^* \times \cdots \times A_m^*$. Then (1) can be reduced to the following form:

$$\begin{array}{ll} \text{Known } A_1 \longrightarrow B_1 & \\ \quad\cdots\ \cdots & \\ \quad A_n \longrightarrow B_n & \\ \text{given} \quad A^* & \\ \hline \text{To solve} \qquad B^*. & \end{array} \tag{3}$$

The procedure is called Premise Reduction.

(P.2) n fuzzy rules in (3) need to be reduced into one rule $A \longrightarrow B$. Generally, the operator "$\cup$" is used, and furthermore (3) is shown as follows:

$$\begin{array}{l} \text{Known } A \longrightarrow B \\ \text{given} \quad A^* \\ \hline \text{To solve} \qquad B^*. \end{array} \tag{4}$$

The procedure is called Rule Reduction.

(P.3) Solving (4) is called FMP (Fuzzy Modus Ponens).

It is necessary to point out that in (P.1), $A_i = A_{i1} \times A_{i2} \times \cdots \times A_{im} \in \mathcal{F}(X_1 \times X_2 \cdots \times X_m)$ $(i = 1, \cdots, n)$, and for $\forall x \triangleq (x_1, x_2, \cdots, x_m) \in X \triangleq X_1 \times X_2 \cdots \times X_m$, we can take $A_i(x) = \wedge_{j=1}^m A_{ij}(x_j)$, and we can also use other triangular norms to replace $\wedge$. In [14], Wang indicated that this method might lose $m-1$ pieces of information in m pieces of information obtained by m data. Therefore, vector-valued fuzzy sets can be adopted to overcome the limitation, and we take

$$A_i(x) = (A_{i1}(x_1), A_{i2}(x_2), \cdots, A_{im}(x_m)). \tag{5}$$

(5) is a vector that does not loss any information. However, the fuzzy inference based on vector-valued fuzzy sets is more complicated than the one based on real-valued fuzzy sets, and more difficult to deal with.

In fact, the value of $A_i(x)$ is obtained by comparison with the m values of rule antecedent components. We do not think that the $m-1$ pieces of information are lost. In addition, the premise reduction by $\wedge$ has been widely applied to practice, and it is easy to deal with in application. So we cannot negate it.

Wang and Li introduced a new theory named Factor Space Theory in [15]. They proposed that the state space $X(f)$ of multidimensional factor f can be obtained by synthesizing states of single factors into states of complex factors based on variable weighted synthesis functions. In fact, the operator "$\wedge$" is just a standard synthesis function.

On the basis of factor space theory, the aggregation procedure of antecedent components in fuzzy rules can be regarded as synthesis of multiple factors. So we put forward a premise reduction method based on weighted-balanced synthesis [16]. The method not only shows relative importance of every antecedent components in reasoning, but considers the balance influence of state combination. However, some defects still exist in this method. The method only reflects aggregation requirement of decision makers, which means the requirement of balancing states among antecedent components, but it does not include all the possible preferences of decision makers. In an actual inference problem, some antecedent components need to be encouraged; in other words, their weights should be enlarged when the input state values increase. On the other hand, when we consider these antecedent components separately, their state values need to be balanced, that is, their variable weights should be reduced when the input state values increase. Hierarchical variable weights in multifactor decision making was proposed in [20], which can solve the above problem, and some scholars applied the idea of hierarchical variable weights to practical models [21,22]. Therefore, based on hierarchical variable weights, a new premise reduction method of multidimensional fuzzy inference is given in this paper.

The paper is organized as follows. In Section 3, we propose the idea of hierarchical variable weights, and further we give an improved premise reduction method of multidimensional fuzzy inference. In Section 4, we discuss the interpolation approximation property of the fuzzy systems constructed by the improved CRI. An example is given to show the effectiveness of the new premise reduction method in Section 5, followed by conclusion in Section 6.

3. **Fuzzy Inference Based on Hierarchical Variable Weights.**

3.1. **Hierarchical variable weights.** Let $A_1, A_2, \cdots, A_m$ be m antecedent components in fuzzy inference. The constant weight vector and the state vector are denoted by $\omega = (\omega_1, \omega_2, \cdots, \omega_m)$ and $X = (x_1, x_2, \cdots, x_m)$ respectively. According to practical requirements, we divide $A_1, A_2, \cdots, A_m$ into p groups, and there are not the same components among the groups. Suppose that there are q_j antecedent components denoted by $A_{j_1}, A_{j_2}, \cdots, A_{j_{q_j}}$ in the jth group. We integrate them into a new antecedent component $\mathbb{A}_j$. Obviously, $\mathbb{A}_1, \mathbb{A}_2, \cdots, \mathbb{A}_p$ are independent of each other. The state value $\bar{x}_j$ of $\mathbb{A}_j$ is the variable weighted synthetic value of $A_{j_1}, A_{j_2}, \cdots, A_{j_{q_j}}$. The state vector and constant vector of $A_{j_1}, A_{j_2}, \cdots, A_{j_{q_j}}$ are denoted by $\left(x_{j_1}, x_{j_2}, \cdots, x_{j_{q_j}}\right)$ and $\left(\omega^0_{j_1}, \omega^0_{j_2}, \cdots, \omega^0_{j_{q_j}}\right)$ respectively. Then the state variable weight vector is generated as follows.

$$S_j\left(x_{j_1}, x_{j_2}, \cdots, x_{j_{q_j}}\right) = \left(S_{j_1}\left(x_{j_1}, x_{j_2}, \cdots, x_{j_{q_j}}\right), \cdots, S_{j_{q_j}}\left(x_{j_1}, x_{j_2}, \cdots, x_{j_{q_j}}\right)\right).$$

We use the idea of variable weighted synthesis [15] and obtain $\bar{x}_j = \sum_{i=1}^{q_j} \omega_{j_i}\Big(x_{j_1}, x_{j_2}, \cdots, x_{j_{q_j}}\Big) x_{j_i}$, where

$$\omega_{j_i}\left(x_{j_1}, x_{j_2}, \cdots, x_{j_{q_j}}\right) = \frac{\omega^0_{j_i} S_{j_i}\left(x_{j_1}, x_{j_2}, \cdots, x_{j_{q_j}}\right)}{\sum_{k=1}^{q_j} \omega^0_{j_k} S_{j_k}\left(x_{j_1}, x_{j_2}, \cdots, x_{j_{q_j}}\right)}.$$

Finally, we discuss the comprehensive value $M_p(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_p)$ of antecedent components. The specific construction process is as follows. Let $\left(\bar{\omega}^0_1, \bar{\omega}^0_2, \cdots, \bar{\omega}^0_p\right)$ be the constant weight vector of $\mathbb{A}_1, \mathbb{A}_2, \cdots, \mathbb{A}_p$. The state variable weight vector of $\mathbb{A}_1, \mathbb{A}_2, \cdots, \mathbb{A}_p$ is denoted by $\bar{S}(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_p) = \left(\bar{S}_1(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_p), \cdots, \bar{S}_p(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_p)\right)$. Then we have $M_p(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_p) = \sum_{j=1}^{p} \bar{\omega}_j(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_p)\bar{x}_j$, in which the variable weight $\bar{\omega}_j(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_p)$ is constructed in the following,

$$\bar{\omega}_j(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_p) = \frac{\bar{\omega}^0_j \bar{S}_j(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_p)}{\sum_{i=1}^{p} \bar{\omega}^0_i \bar{S}_i(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_p)}.$$

Remark 3.1. *According to different inference problems, $\mathbb{A}_1, \mathbb{A}_2, \cdots, \mathbb{A}_p$ can be further grouped and the construction method is similar to above.*

3.2. **Fuzzy inference procedure based on hierarchical variable weights.** As mentioned above, fuzzy inference can be divided into three parts, i.e., premise reduction, rule reduction and solving of FMP problem. We introduce three steps of fuzzy inference based on hierarchical variable weights.

(P.1′) Premise reduction. According to the hierarchical variable weights, the m antecedent components are synthesized, that is,

$$A_i(x) \triangleq A_i(x_1, x_2, \cdots, x_m) = M_p(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_p).$$

(P.2′) Rule reduction. The n rules are aggregated into $R = \bigcup_{i=1}^{n} R_i$, i.e.,

$$R(x, y) = \bigvee_{i=1}^{n} R_i(x, y) = \bigvee_{i=1}^{n} \theta(A_i(x), B_i(y)),$$

where θ is a fuzzy implication operator.

Remark 3.2. *The operation $\bigcup$ can be replaced by other operations as $\sum$. In* [23], *$\sum$ is revised as bounded sum $\oplus$.*

(P.3′) Solving (4). We use CRI method given by Zadeh and then get $B^* = A^* \wedge R$, i.e.,

$$B^*(y) = \bigvee_{x \in X} (A^*(x) \wedge R(x, y)).$$

4. **Interpolation Approximation of the Fuzzy Systems Constructed by the Fuzzy Inference Based on Hierarchical Variable Weights.** We discuss whether the fuzzy systems based on hierarchical variable weights have interpolation property or not from the point of view of function approximation.

Firstly, it is necessary to know several signs. Let $X_1, X_2, \cdots, X_m$ be m input universes, and Y is an output universe. $\mathcal{A}_1 = \{A_{i_1 1} \mid 1 \le i_1 \le n_1\}$, $\mathcal{A}_2 = \{A_{i_2 2} \mid 1 \le i_2 \le n_2\}$, $\cdots$, $\mathcal{A}_m = \{A_{i_m m} \mid 1 \le i_m \le n_m\}$, and $\mathcal{B} = \{B_{i_1 i_2 \cdots i_m} \mid 1 \le i_j \le n_j, j = 1, 2, \cdots, m\}$ are the fuzzy partitions to $X_1, X_2, \cdots, X_m, Y$, respectively. For convenience, We suppose $X_1, X_2, \cdots, X_m, Y$ are all $[0, 1]$, and $x_{i_j j}$, $y_{i_1 i_2 \cdots i_m}$ are the peak points of $A_{i_j j}$, $B_{i_1 i_2 \cdots i_m}$ $(1 \le i_j \le n_j,\ j = 1, 2, \cdots, m)$ respectively, where

$$0 \le x_{11} < \cdots < x_{n_1 1} \le 1, \cdots, 0 \le x_{1m} < \cdots < x_{n_m m} \le 1, 0 \le y_1 < \cdots < y_{n_1 n_2 \cdots n_m} \le 1.$$

$\mathcal{A}_1, \cdots, \mathcal{A}_m, \mathcal{B}$ can be regarded as linguistic variables. Then $n_1 n_2 \cdots n_m$ fuzzy rules are generated as follows:

If x_1 is $A_{i_1 1}$ and x_2 is $A_{i_2 2}$ and $\cdots$ and x_m is $A_{i_m m}$ then y is $B_{i_1 i_2 \cdots i_m}$.

For a given $x^* \triangleq (x_1^*, x_2^*, \cdots, x_m^*) \in X_1 \times X_2 \times \cdots \times X_m$, we use singleton fuzzification, i.e.,

$$A^*(x) = \begin{cases} 1, & x = x^*, \\ 0, & x \ne x^*. \end{cases}$$

Theorem 4.1. *On the basis of above-mentioned assumption, there exists a group of base elements $\Phi = \{\phi_{i_1 i_2 \cdots i_m} \mid 1 \le i_1 \le n_1, \cdots, 1 \le i_m \le n_m\}$ such that the fuzzy systems with m inputs and single output based on hierarchical variable weights can be expressed as some piecewise interpolation functions that take $\phi_{i_1 i_2 \cdots i_m}$ as their base functions, that is,*

$$f(x_1, x_2, \cdots, x_m) = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \cdots \sum_{i_m=1}^{n_m} \phi_{i_1 i_2 \cdots i_m}(x_1, x_2, \cdots, x_m) y_{i_1 i_2 \cdots i_m}. \tag{6}$$

Proof: According to the interpolation mechanism of fuzzy systems in [11], for the given $(x_1^*, x_2^* \cdots, x_m^*) \in X_1 \times X_2 \times \cdots \times X_m$, the output of the fuzzy systems should be

$$y^* = f(x_1^*, x_2^*, \cdots, x_m^*) = \frac{\sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \cdots \sum_{i_m=1}^{n_m} y_{i_1 i_2 \cdots i_m} M_p\left(\bar{x}_1^*, \bar{x}_2^*, \cdots, \bar{x}_p^*\right)}{\sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \cdots \sum_{i_m=1}^{n_m} M_p\left(\bar{x}_1^*, \bar{x}_2^*, \cdots, \bar{x}_p^*\right)}.$$

Let

$$\alpha_{i_1 i_2 \cdots i_m}(x_1^*, x_2^*, \cdots, x_m^*) = M_p\left(\bar{x}_1^*, \bar{x}_2^*, \cdots, \bar{x}_p^*\right),$$

$$\beta_{i_1 i_2 \cdots i_m}(x_1^*, x_2^*, \cdots, x_m^*) = \frac{1}{\sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \cdots \sum_{i_m=1}^{n_m} \alpha_{i_1 i_2 \cdots i_m}(x_1^*, x_2^*, \cdots, x_m^*)},$$

$$\phi_{i_1 i_2 \cdots i_m}(x_1^*, x_2^*, \cdots, x_m^*) = \beta_{i_1 i_2 \cdots i_m}(x_1^*, x_2^*, \cdots, x_m^*) \cdot \alpha_{i_1 i_2 \cdots i_m}(x_1^*, x_2^*, \cdots, x_m^*).$$

When $x_1^* = x_{i_1 1}, x_2^* = x_{i_2 2}, \cdots, x_m^* = x_{i_m m}$, then $A_{i_1 1}(x_{i_1 1}) = A_{i_2 2}(x_{i_2 2}) = \cdots = A_{i_m m}(x_{i_m m}) = 1$, $A_{k_1 1}(x_{i_1 1}) = A_{k_2 2}(x_{i_2 2}) = \cdots = A_{k_m m}(x_{i_m m}) = 0$ $(k \neq i)$.

Consequently,

$$y^* = \frac{y_{i_1 i_2 \cdots i_m} M_p\left(\bar{x}_1^*, \bar{x}_2^*, \cdots, \bar{x}_p^*\right)}{M_p\left(\bar{x}_1^*, \bar{x}_2^*, \cdots, \bar{x}_p^*\right)} = y_{i_1 i_2 \cdots i_m}.$$

If we put

$$f(x_1, x_2, \cdots, x_m) \triangleq \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \cdots \sum_{i_m=1}^{n_m} \phi_{i_1 i_2 \cdots i_m}(x_1, x_2, \cdots, x_m) y_{i_1 i_2 \cdots i_m},$$

then (6) is obtained.

Remark 4.1. *From Theorem 4.1, aggregation of antecedent components by hierarchical variable weights does not change the interpolation approximation of the fuzzy systems constructed by CRI.*

5. **Instance Analysis.** The prediction of time series is an important practical issue, which can be applied to many fields such as economics, control, and signal processing. We use the above idea to construct a multidimensional fuzzy system to predict Mackey-Glass chaotic time series generated by the differential equation with time delay

$$\frac{dx(t)}{dt} = \frac{0.2x(t-\tau)}{1 + x^{10}(t-\tau)} - 0.1x(t). \tag{7}$$

When $\tau > 17$, (7) shows a chaotic action. Take $\tau = 30$ (see Figure 1).

The prediction of time series utilizes known time series data at moment t to predict the value at a future moment $t + P$. In general, a mapping from q sampling points $(x(t-(q-1)I), \cdots, x(t-I), x(t))$ to $x(t+P)$ is constructed, where I is a time interval unit. We choose $q = 4$, $I = P = 6$. Then for every moment t, the input of the fuzzy system is a 4-dimensional vector $X(t) = (x(t-18), x(t-12), x(t-6), x(t))$, and the output is the predicted value $y(t) = x(t+6)$. Accordingly, we generate 1000 input-output data as the integer moment t varies from 118 to 1117. The first 500 data are used to generate fuzzy rules and a multidimensional fuzzy system is constructed on the basis of hierarchical variable weights. The rest 500 data are used to test the approximation effect of the improved fuzzy system.

We use the triangular membership functions for rule antecedents as follows:

$$A_1(x) = \begin{cases} \dfrac{x - x_2}{x_1 - x_2}, & x_1 \le x \le x_2, \\ 0, & \text{otherwise}, \end{cases}$$

$$A_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}}, & x_{i-1} \le x \le x_i, \\ \dfrac{x - x_{i+1}}{x_i - x_{i+1}}, & x_i \le x \le x_{i+1}, \\ 0, & \text{otherwise}, \end{cases} \qquad (i = 2, \cdots, m-1)$$

$$A_m(x) = \begin{cases} \dfrac{x - x_{m-1}}{x_m - x_{m-1}}, & x_{m-1} \le x \le x_m, \\ 0, & \text{otherwise}, \end{cases}$$

where x_i is the peak point of fuzzy set A_i $(1 \le i \le m)$.

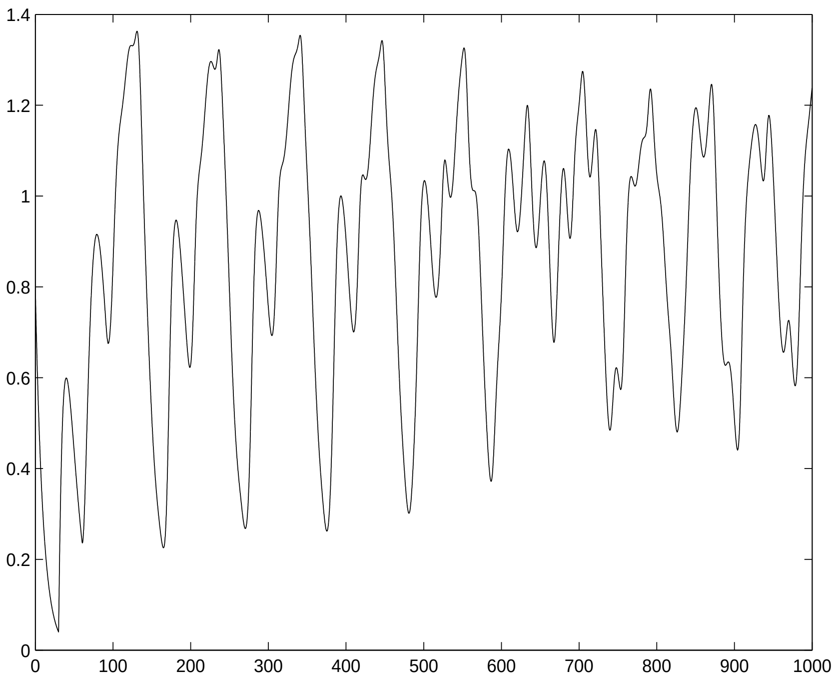

Figure 1. Mackey-Glass chaotic time series

It should be noted that there are four input variables in this experiment. For convenience, we can aggregate the four fuzzy inputs by using $A_i(x) = \sum_{k=1}^{4} 0.25 A_{ik}(x_{ik})$ in accordance with hierarchical variable weights.

Figure 2 gives the mean square errors (MSE) produced by the improved system based on hierarchical variable weights and the traditional Mamdani fuzzy system approximating a part of Mackey-Glass chaotic time series with varying numbers of fuzzy rules. It shows that the approximation degree of the two systems is improved as the numbers of fuzzy rules increase. Nevertheless, for the same number of fuzzy rules, the approximation degree of the improved fuzzy system is better than that of the traditional Mamdani inference system.

6. **Conclusion.** Through analyzing aggregation problem of antecedent components existing in multidimensional fuzzy inference, we reduced these antecedent components by hierarchical variable weights in factor space theory. This reduction method not only shows the relative importance of every antecedent component in fuzzy inference, but also considers the preference requirements of decision makers. In addition, the fuzzy systems constructed by improved fuzzy reasoning procedure still have the interpolation property in function approximation. An example was given to illustrate the effectiveness of the new idea. Therefore, the proposed method has theoretical and practical significance. In view of approximation properties of fuzzy systems to unknown functions, to study the

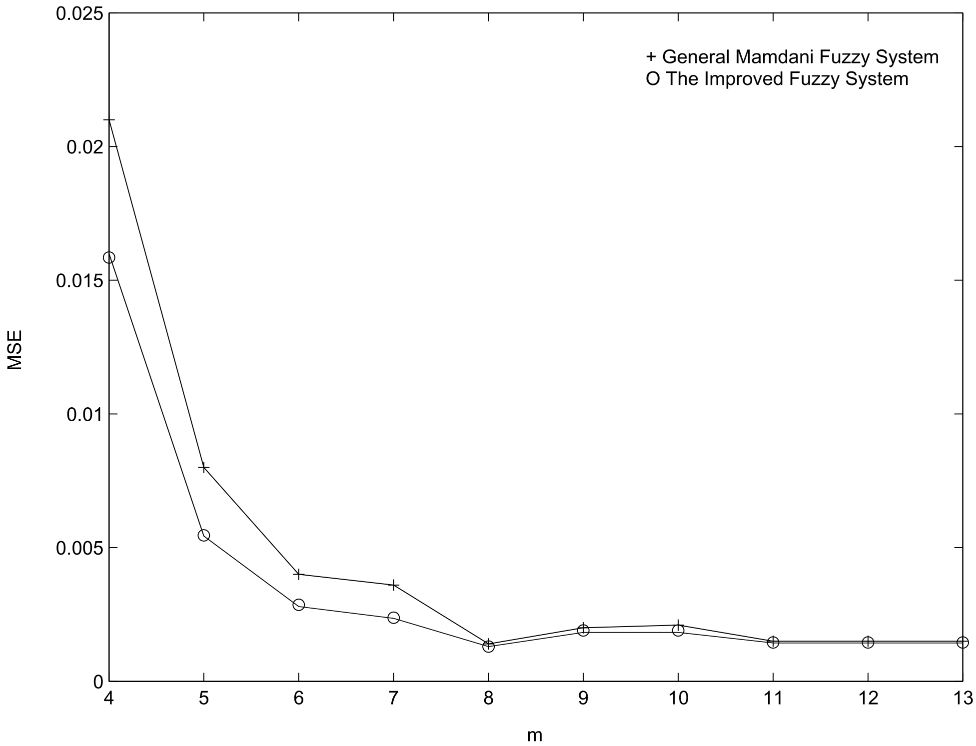

FIGURE 2. Comparison between mean square errors of the two fuzzy systems

approximation accuracy of the fuzzy systems based on hierarchical variable weights will be our next work in future.

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