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NEW METHODS OF CONSTRUCTING QUATERNARY SEQUENCE PAIRS

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Abstract. *In this letter, new methods of constructing quaternary sequence pairs are presented based on binary sequence pairs with two-level autocorrelation, almost perfect binary sequence pairs and cyclic shift sequences. The quaternary sequence pairs constructed by these methods are periodic autocorrelation signals. These methods are handy and flexible, and can greatly enrich the theory of quaternary sequence pairs.*
Keywords: Almost perfect binary sequence pairs, Quaternary sequence pairs, Correlation, Discrete correlation signals

1. **Introduction.** The perfect discrete signal plays an important role in the field of communication, radar, navigation and other wireless systems, and the design of perfect discrete signal is also an important subject [1]. The quaternary sequence is popular in many applications because of the simplicity of its achievement device [2, 3, 4]. [5] presented a new method of constructing quaternary sequence with even period based on cyclotomic classes of order two. However, so far, it is a great limitation that we only find perfect quaternary sequence with period 2, 4, 8 and 16 [6]. Perfect quaternary sequence pairs, a new periodic correlation signal, was presented recently. [7] had a study on the properties of perfect quaternary sequences and proposed three methods of constructing the perfect quaternary sequence pairs, which include direct construction, interleaving recursive construction and recursive construction. [8] presented almost perfect quaternary sequence pair which is a new discrete signal with good periodic correlation properties. In addition, Peng et al. studied the even period quaternary sequence pair with three-level autocorrelation in [9]. In this letter, new methods of constructing quaternary sequence pairs are presented based on binary sequence pairs with two-level autocorrelation [10], almost perfect binary sequence pairs [11] and cyclic shift sequences. These methods, which can greatly enrich the theory of quaternary sequence pairs, are more flexible than the methods mentioned in [5, 7, 8, 9]. The quaternary sequence pairs constructed by these new methods are periodic autocorrelation signals similarly.

2. **Preliminaries.**

Definition 2.1. *Let* $X = [x(0), x(1), \ldots, x(N-1)]$ *and* $Y = [y(0), y(1), \ldots, y(N-1)]$ *be two sequences of period* N*. If the element values for* X *and* Y *are* $\{1, -1, i, -i\}$*, where* $i = \sqrt{-1}$*, then* (X, Y) *is a quaternary sequence pair. The periodic correlation function of* (X, Y) *is* $R_{XY}(\tau) = \sum_{j=0}^{N-1} X(j)Y^*(j+\tau)$*, where* $j + \tau = (j + \tau) \bmod N$*.*

Definition 2.2. *If the periodic correlation function of the binary (quaternary) sequence pair* (X,Y) *at shift* $0 \le \tau < N$ *is*

$$R_{XY}(\tau) = \sum_{i=0}^{N-1} X(i)Y(i+\tau) = \begin{cases} E & \tau = 0 \\ F & \tau = a \\ 0 & else \end{cases} \tag{1}$$

where $i+\tau = (i+\tau) \bmod N$, $E > F \neq 0$, *and* $0 < a < N$, *then* (X,Y) *is called an almost perfect binary (quaternary) sequence pair.* E *is called main peak, and* F *is called side peak. Obviously, the out-of-phase correlation function of almost perfect binary (quaternary) sequence pair is zero everywhere except one point* $\tau = a$.

3. Construction Methods.

Theorem 3.1. *Let* (X,Y) *be a binary sequence pair with two-level autocorrelation of period* N, *and the cyclic shift sequence pair is* (X',Y'), $X'(j) = X(j+d)$, $Y'(j) = Y(j+d)$, $0 < d < N$, $j+d = (j+d) \bmod N$. *Let* (M,N) *be a sequence pair with period* N, $M(j) = \frac{1+i}{2}X(j) + \frac{1-i}{2}X'(j) = \frac{1+i}{2}X(j) + \frac{1-i}{2}X(j+d)$, $N(j) = \frac{1+i}{2}Y(j) + \frac{1-i}{2}Y'(j) = \frac{1+i}{2}Y(j) + \frac{1-i}{2}Y(j+d)$. *Then* (M,N) *is a quaternary sequence pair with two-level or four-level autocorrelation.*

Proof: The correlation function of (X,Y) is $R_{xy}(\tau) = \begin{cases} E & \tau = 0 \\ F & \tau \neq 0 \end{cases}$, where $E > F$. Then the correlation function of (M,N) is

$$\begin{aligned} & R_{MN}(\tau) \\ =& \sum_{j=0}^{N-1} M(j)N^*(j+\tau) \\ =& \sum_{j=0}^{N-1} \left[\frac{1+i}{2}X(j) + \frac{1-i}{2}X(j+d)\right]\left[\frac{1+i}{2}Y(j) + \frac{1-i}{2}Y(j+d)\right]^* \\ =& \frac{1}{4}\sum_{j=0}^{N-1} \begin{bmatrix} (1+i)X(j)(1-i)Y(j+\tau) + (1+i)X(j)(1+i)Y(j+d+\tau) \\ +(1-i)X(j+d)(1-i)Y(j+\tau) + (1-i)X(j+d)(1+i)Y(j+d+\tau) \end{bmatrix} \\ =& \frac{1}{4}\sum_{j=0}^{N-1} \left[X(j)Y(j+\tau) + iX(j)Y(j+d+\tau) - iX(j+d)Y(j+\tau)\right. \\ & \left. + X(j+d)Y(j+d+\tau)\right] \end{aligned}$$

When the period N is odd or N is even and $d \neq \frac{N}{2}$:
(1) For the case of $\tau = 0$:

$$R_{MN}(\tau) = \frac{1}{2}(E + iF - iF + E) = E$$

(2) For the case of $\tau = d$:

$$R_{MN}(\tau) = \frac{1}{2}(F + iF - iE + F) = F + \frac{iF - iE}{2}$$

(3) For the case of $\tau = N - d$:

$$R_{MN}(\tau) = \frac{1}{2}(F + iE - iF + F) = F + \frac{iE - iF}{2}$$

(4) If τ is the other value:

$$R_{MN}(\tau) = \frac{1}{2}(F + iF - iF + F) = F$$

Hence, (M,N) is a quaternary sequence pair with four-level autocorrelation when the period N is odd or N is even and $d \neq \frac{N}{2}$, and the correlation function is

$$R_{MN}(\tau) = \begin{cases} E & \tau = 0 \\ F + \frac{iF-iE}{2} & \tau = d \\ F + \frac{iE-iF}{2} & \tau = N-d \\ F & \text{else} \end{cases}$$

In the similar manner, when N is even and $d = \frac{N}{2}$, (M,N) is a quaternary sequence pair with two-level correlation function as the following.

$$R_{MN}(\tau) = \begin{cases} E & \tau = 0 \\ F & \tau \neq 0 \end{cases}$$

Concluding from the above, quaternary sequence pairs with two-level or four-level autocorrelation can be constructed by this method.

Example 3.1. *Let* $(X,Y) = (1,1,1,-1,-1,-1,-1,-1;1,-1,1,-1,-1,1,-1,-1)$ *be a binary sequence pair, the period* $N = 8$*, and the correlation function is*

$$R_{XY}(\tau) = \begin{cases} 4 & \tau = 0 \\ 0 & \tau \neq 0 \end{cases}$$

If $d = 2$*, the cyclic shift sequence pair is expressed as* $(X',Y') = (1,-1,-1,-1,-1,-1,1,1;1,-1,-1,1,-1,-1,1,-1)$*. We can get a quaternary sequence pair* $(M,N) = (1,i,i,-1,-1,-1,-i,-i;1,-1,i,-i,-1,i,-i,-1)$*, and its correlation function is*

$$R_{MN}(\tau) = \begin{cases} 4 & \tau = 0 \\ -2i & \tau = 2 \\ 2i & \tau = 6 \\ 0 & \textit{else} \end{cases}$$

If $d = 4$*, the cyclic shift sequence pair is expressed as* $(X',Y') = (-1,-1,-1,-1,1,1,1,-1;-1,1,-1,-1,1,-1,1,-1)$*. We can get a quaternary sequence pair* $(M,N) = (i,i,i,-1,-i,-i,-i,-1;i,-i,i,-1,-i,i,-i,-1)$*, and its correlation function is*

$$R_{MN}(\tau) = \begin{cases} 4 & \tau = 0 \\ 0 & \textit{else} \end{cases}$$

Theorem 3.2. *Let* (X,Y) *be a binary sequence pair with two-level autocorrelation of period* N*, and the cyclic shift sequence pair is* (X',Y')*,* $X'(j) = X(j+d)$*,* $Y'(j) = Y(j+d)$*,* $0 < d < N$*,* $j+d = (j+d) \bmod N$*. Let* (M,N) *be a sequence pair with period* N*,* $M(j) = \frac{1+i}{2}X(j) - \frac{1-i}{2}X'(j) = \frac{1+i}{2}X(j) - \frac{1-i}{2}X(j+d)$*,* $N(j) = \frac{1+i}{2}Y(j) - \frac{1-i}{2}Y'(j) = \frac{1+i}{2}Y(j) - \frac{1-i}{2}Y(j+d)$*. Then* (M,N) *is a quaternary sequence pair with two-level or four-level autocorrelation.*

Proof: Similar to Theorem 3.1, when the period N is odd or N is even and $d \neq \frac{N}{2}$, (M,N) is a quaternary sequence pair with four-level correlation function of

$$R_{MN}(\tau) = \begin{cases} E & \tau = 0 \\ F + \frac{iE-iF}{2} & \tau = d \\ F + \frac{iF-iE}{2} & \tau = N-d \\ F & \text{else} \end{cases}$$

When N is even and $d = \frac{N}{2}$, (M,N) is a quaternary sequence pair with two-level correlation function of

$$R_{MN}(\tau) = \begin{cases} E & \tau = 0 \\ F & \tau \neq 0 \end{cases}.$$

Example 3.2. *Let* $(X, Y) = (1, -1, -1, 1, 1, -1, -1; 1, -1, 1, 1, 1, 1, -1)$ *be a binary sequence pair, the period* $N = 7$*, and the correlation function is*

$$R_{XY}(\tau) = \begin{cases} 3 & \tau = 0 \\ -1 & \tau \neq 0 \end{cases}$$

If $d = 3$*, the cyclic shift sequence pair is expressed as* $(X', Y') = (1, 1, -1, -1, 1, -1, -1; 1, 1, 1, -1, 1, -1, 1)$*. We can get a quaternary sequence pair* $(M, N) = (i, -1, -i, 1, i, -i, -i; i, -1, i, 1, i, 1, -1)$*, and its correlation function is*

$$R_{MN}(\tau) = \begin{cases} 3 & \tau = 0 \\ -1 + 2i & \tau = 3 \\ -1 - 2i & \tau = 4 \\ -1 & \textit{else} \end{cases}$$

According to Theorem 3.1 and Theorem 3.2, it is significant that (M, N) is a perfect quaternary pair if (X, Y) is a perfect binary sequence pair. Similarly, (M, N) is a pseudorandom quaternary pair if (X, Y) is a pseudorandom binary sequence pair.

Theorem 3.3. *Let* (X, Y) *be an almost perfect binary sequence pair with period* N *(*N *is even), and the cyclic shift sequence pair is* (X', Y')*,* $X'(j) = X(j + a)$*,* $Y'(j) = Y(j + a)$*,* a *is a nonzero point where the out-of-phase correlation function of* (X, Y) *is nonzero,* $j + a = (j + a) \bmod N$*. Let* (M, N) *be a sequence pair with period* N*,* $M(j) = \frac{1+i}{2}X(j) + \frac{1-i}{2}X'(j) = \frac{1+i}{2}X(j) + \frac{1-i}{2}X(j + a)$*,* $N(j) = \frac{1+i}{2}Y(j) + \frac{1-i}{2}Y'(j) = \frac{1+i}{2}Y(j) + \frac{1-i}{2}Y(j + a)$*. Then* (M, N) *is a quaternary sequence pair with three-level, four-level or five-level autocorrelation.*

Proof: The correlation function of (M, N) is

$$\begin{aligned}
&R_{MN}(\tau) \\
=&\sum_{j=0}^{N-1} M(j)N^*(j + \tau) \\
=&\sum_{j=0}^{N-1} \left[\frac{1+i}{2}X(j) + \frac{1-i}{2}X(j + a)\right]\left[\frac{1+i}{2}Y(j) + \frac{1-i}{2}Y(j + a)\right]^* \\
=&\frac{1}{4}\sum_{j=0}^{N-1} \left[\begin{array}{l} (1 + i)X(j)(1 - i)Y(j + \tau) + (1 + i)X(j)(1 + i)Y(j + a + \tau) \\ +(1 - i)X(j + a)(1 - i)Y(j + \tau) + (1 - i)X(j + a)(1 + i)Y(j + a + \tau) \end{array}\right] \\
=&\frac{1}{4}\sum_{j=0}^{N-1} \big[X(j)Y(j + \tau) + iX(j)Y(j + a + \tau) - iX(j + a)Y(j + \tau) \\
&+ X(j + a)Y(j + a + \tau)\big]
\end{aligned}$$

Similar to the proof of Theorem 3.1, (M, N) is a quaternary sequence pair with five-level autocorrelation when $a \neq \frac{N}{2}$ and $a \neq \frac{kN}{3}$ $(k = 1, 2)$, and the correlation function is

$$R_{MN}(\tau) = \begin{cases} E + \frac{iF}{2} & \tau = 0 \\ F - \frac{iE}{2} & \tau = a \\ \frac{iE}{2} & \tau = N - a \\ -\frac{iF}{2} & \tau = 2a(\mathrm{mod} N) \\ 0 & \text{else} \end{cases}$$

(M, N) is a quaternary sequence pair with four-level autocorrelation when $a = \frac{kN}{3}$ $(k = 1, 2)$, and the correlation function is

$$R_{MN}(\tau) = \begin{cases} E + \frac{iF}{2} & \tau = 0 \\ F - \frac{iE}{2} & \tau = a \\ \frac{iE-iF}{2} & \tau = N - a = 2a(modN) \\ 0 & \text{else} \end{cases}$$

(M, N) is an almost perfect quaternary sequence pair when $a = \frac{N}{2}$, and the correlation function is

$$R_{MN}(\tau) = \begin{cases} E & \tau = 0 \\ F & \tau = a = \frac{N}{2} \\ 0 & \text{else} \end{cases}$$

Concluding from the above, quaternary sequence pairs with three-level, four-level or five-level autocorrelation can be constructed by this method.

Example 3.3. *Let* $(X, Y) = (-1, -1, -1, -1, 1, 1; -1, 1, -1, -1, 1, 1)$ *be an almost perfect binary sequence pair, the period* $N = 6$, $a = 4$, *and the correlation function is*

$$R_{XY}(\tau) = \begin{cases} 4 & \tau = 0 \\ -4 & \tau = 4 \\ 0 & else \end{cases}$$

The cyclic shift sequence pair is expressed as $(X', Y') = (1, 1, -1, -1, -1, -1; 1, 1, 1, -1, 1, -1, 1)$. *We can get a quaternary sequence pair* $(M, N) = (-i, -i, -1, -1, i, i; -i, 1, -1, -i, i, i)$, *and its correlation function is*

$$R_{MN}(\tau) = \begin{cases} 4 - 2i & \tau = 0 \\ 4i & \tau = 2 \\ -4 - 2i & \tau = 4 \\ 0 & else \end{cases}$$

Theorem 3.4. *Let* (X, Y) *be an almost perfect binary sequence pair with period* N *(*N *is even), and the cyclic shift sequence pair is* (X', Y'), $X'(j) = X(j + a)$, $Y'(j) = Y(j + a)$, a *is a nonzero point where the out-of-phase correlation function of* (X, Y) *is nonzero,* $j + a = (j + a) \bmod N$. *Let* (M, N) *be a sequence pair with period* N, $M(j) = \frac{1+i}{2}X(j) - \frac{1-i}{2}X'(j) = \frac{1+i}{2}X(j) - \frac{1-i}{2}X(j + a)$, $N(j) = \frac{1+i}{2}Y(j) - \frac{1-i}{2}Y'(j) = \frac{1+i}{2}Y(j) - \frac{1-i}{2}Y(j + a)$. *Then* (M, N) *is a quaternary sequence pair with three-level, four-level or five-level autocorrelation.*

Proof: Similar to Theorem 3.3, when $a \neq \frac{N}{2}$ and $a \neq \frac{kN}{3}$ $(k = 1, 2)$, (M, N) is a quaternary sequence pair with five-level autocorrelation of

$$R_{MN}(\tau) = \begin{cases} E - \frac{iF}{2} & \tau = 0 \\ F + \frac{iE}{2} & \tau = a \\ -\frac{iE}{2} & \tau = N - a \\ \frac{iF}{2} & \tau = 2a(\text{mod}N) \\ 0 & \text{else} \end{cases}$$

When $a = \frac{kN}{3}(k = 1, 2)$, (M, N) is a quaternary sequence pair with four-level autocorrelation of

$$R_{MN}(\tau) = \begin{cases} E - \frac{iF}{2} & \tau = 0 \\ F + \frac{iE}{2} & \tau = a \\ \frac{iF - iE}{2} & \tau = N - a = 2a(modN) \\ 0 & \text{else} \end{cases}$$

When $a = \frac{N}{2}$, (M, N) is an almost perfect quaternary sequence pair with three-level correlation function of

$$R_{MN}(\tau) = \begin{cases} E & \tau = 0 \\ F & \tau = a = \frac{N}{2} \\ 0 & \text{else} \end{cases}$$

Example 3.4. *Let* $(X, Y) = (-1, -1, 1, 1, -1, 1, 1, -1; -1, -1, -1, 1, -1, 1, -1, -1)$ *be an almost perfect binary sequence pair, the period* $N = 8$*,* $a = 4$*, and the correlation function is*

$$R_{XY}(\tau) = \begin{cases} 4 & \tau = 0 \\ -4 & \tau = 4 \\ 0 & else \end{cases}$$

The cyclic shift sequence pair is expressed as $(X', Y') = (-1, 1, 1, -1, -1, -1, 1, 1; -1, 1, -1, -1, -1, -1, -1, 1)$*. We can get a quaternary sequence pair* $(M, N) = (-i, -1, i, 1, -i, 1, i, -1; -i, -1, -i, 1, -i, 1, -i, -1)$*, and its correlation function is*

$$R_{MN}(\tau) = \begin{cases} 4 & \tau = 0 \\ -4 & \tau = 4 \\ 0 & else \end{cases}$$

4. **Conclusions.** The quaternary sequence pair with good periodic autocorrelation has a wide range of applications in data communication synchronous detection, code division multiple access and spectrum communication. In this letter, four new methods of constructing quaternary sequence pairs are based on binary sequence pairs with two-level autocorrelation, almost perfect binary sequence pairs and cyclic shift sequences. These methods are handy and flexible, and have high universality. The quaternary sequence pairs obtained by these methods can widen the existence space of the quaternary sequence pair and provide more choice for practical engineering applications.

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