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STATE ESTIMATION FOR MULTI-INPUT MULTI-OUTPUT CONTROL SYSTEMS WITH THE FIXED DATA RATE

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Abstract. *This paper addresses the state estimation problem for multi-input multi-output (MIMO) control systems where the sensors and controllers are connected via a stationary memoryless uncertain digital channel without data dropout and time delay. We focus on the argument on the inherent tradeoff between observability and the limited data rate. In particular, the fixed data rate is employed to ensure observability every time the plant is used, which is different from the time-varying data rate in the literature. An allocation algorithm is constructed to regulate the transmission of the information on the plant states in order to achieve minimum data rate for observability of MIMO control systems. The conditions on the fixed data rate are derived, and a lower bound on the fixed data rate for observability is given. An illustrative example is given to demonstrate the effectiveness of the proposed scheme.*
Keywords: State estimation, Fixed data rate, Observability, Multi-input multi-output systems

1. **Introduction.** In networked control systems, control loops are closed over a digital communication network. Sensors, controllers, and actuators are connected via communication channels. For such a system, there exist many advantages, such as increased system flexibility, decreased wiring and cost, and ease of installation [1]. In many applications, network resources (for example, network communication bandwidth) have to be shared by many systems at the same time. However, new issues arise when the sensors and the controllers are connected by communication channels with limited data rate [2,3].

This result was generalized to different notions of stabilization and system models. The research on Gaussian linear systems was addressed in [4]. Control under communication constraints inevitably suffers signal transmission delay, data packet dropout and measurement quantization which might be potential sources of instability and poor performance of control systems [5]. A predictive control policy under data-rate constraints was proposed to stabilize the unstable plant in the mean square sense. [6] addressed LQ (linear quadratic) control of MIMO discrete-time linear systems, and gave the inherent tradeoffs between LQ cost and data rates. In [7], a quantized-observer based encoding-decoding scheme was designed, which integrated the state observation with encoding-decoding. [8] addressed some of the challenging issues on moving horizon state estimation for networked control systems in the presence of multiple packet dropouts.

In this paper, we will extend the results in the literature from control under data-rate limitations to state estimation for MIMO control systems. We focus on the argument on the inherent tradeoff between observability and the limited data rate. In particular, the fixed data rate is employed to ensure observability at each time step. It is different from the case of the time-varying data rate, which only ensures control performance just in an average or expected sense. Thus, the fixed data rate is not viewed as a special case of the time-varying data rate. In MIMO control systems, an allocation algorithm must regulate the transmission of the information on the plant states, which is different from the scalar case.

Our purpose here is to construct an allocation algorithm to regulate the transmission of the information on the plant states in order to achieve minimum data rate for observability of MIMO control systems. Our work here differs in that we present a lower bound on the fixed data rate for observability of MIMO control system.

The remainder of this paper is organized as follows: Section 2 introduces problem formulation; Section 3 deals with the state estimation problem for MIMO control systems; the results of numerical simulation are presented in Section 4; conclusions are stated in Section 5.

2. **Problem Formulation.** In this paper, we consider the following linear continuous time multi-input multi-output (MIMO) system

$$\begin{aligned} \dot{X}(t) &= AX(t) + FV(t), \\ Y(t) &= CX(t), \end{aligned} \tag{1}$$

where $X(t) \in \mathbb{R}^n$ is the state process, $Y(t) \in \mathbb{R}^m$ is the measured output, and $V(t) \in \mathbb{R}^p$ is the process disturbance. A, C, and F are known constant matrices with appropriate dimensions. Here, it is assumed that the pair (A, C) is observable. Let $B_l(z)$ denote the set $\{x : |x-z| \le l\}$ centered at z. The initial state $X(0)$ and $V(t)$ are bounded, uncertain variables satisfying $\|X(0)\| \in B_{\phi_0}(0)$ and $\|V(t)\| \in B_{\phi_V}(0)$, respectively, where ϕ_0 and ϕ_V are two known constants. For the problem to be well-posed, it is assumed that there exists a nonsingular real matrix H that diagonalizes $A = H'\Lambda H$, where $\Lambda = diag[a_1, a_2, \cdots, a_n]$. Here, we also assume that each a_i is larger than 0 ($i = 1, 2, \cdots, n$) since the stable part does not play any key role on observability of the system (1). Consider this case avoids extraneous complexity. It makes our conclusions most transparent. Then, we may focus on the argument on the inherent tradeoff between observability and the limited data rate.

Let T_s be sample time. Then, the corresponding discrete-time system is

$$\begin{aligned} X(k+1) &= GX(k) + W(k), \\ Y(k) &= CX(k), \end{aligned} \tag{2}$$

where we define $X(k) := X(kT_s)$, $Y(k) := Y(kT_s)$, $W(k) := \int_{kT_s}^{(k+1)T_s} e^{A[(k+1)T_s - t]} FV(t)dt$, $G := e^{AT_s}$. Clearly, $W(k)$ is also bounded, uncertain variable. Assume that $\|W(k)\| \in B_{\phi_W}(0)$, where ϕ_W is known constant. Furthermore, it is assumed that the states of system (2) are measurable. The initial condition $X(0)$ and disturbance $W(0), \cdots, W(k)$ are mutually independent random variables.

In this paper, we consider the case where the sensors and controllers are geographically separated and connected by a stationary memoryless uncertain digital channel without data dropout and time delay. At each time step the channel can transmit without error R bits of the information on the plant states that are provided by the sensors. Specifically, we deal with the case where the data rate R provided by such a channel is an invariant constant.

Let $\hat{X}(k)$ and $E(k)$ denote the state estimate and the estimation error, respectively. We define the estimation error as

$$E(k) := X(k) - \hat{X}(k).$$

$X(k)$ is causally encoded via an operator Θ as

$$\alpha(k) = \Theta(k, X(0), X(1), \cdots, X(k)), \tag{3}$$

where the codeword $\alpha(k)$ is transmitted over such a channel, and decoded via an operator Υ as

$$\hat{X}(k) = \Upsilon(k, \hat{\alpha}(0), \hat{\alpha}(1), \cdots, \hat{\alpha}(k)), \tag{4}$$

where $\hat{\alpha}(k)$ denotes the received symbol at the decoder.

The multi-state case entails several difficult challenges. The first one is that, an allocation algorithm must regulate the transmission of the information on the plant states. Furthermore, it is more difficult to derive a lower bound on the fixed data rate for observability of a multi-state system than a scalar system. The encoding and decoding problems may get even more complicated for the multi-state case.

In this paper, we consider the MIMO case, argue the state estimation problem under the data-rate limitation, and present the inherent tradeoff between observability and the limited data rate. The main task here is to present the condition on the fixed data rate for observability of the MIMO system (1). Namely, we want to derive a lower bound on the fixed data rate, which can ensure observability of system (1) in the sense

$$\limsup_{k\to\infty} \|E(k)\| < \infty. \tag{5}$$

3. **State Estimation for the MIMO Control System.** This section deals with state estimation for the MIMO control system with the fixed data rate. In order to present an allocation algorithm on the data rate, we define a nonsingular real matrix by M that diagonalizes $G = M'\Delta M$ where $\Delta = diag\left[e^{a_1T_s}, e^{a_2T_s}, \cdots, e^{a_nT_s}\right]$. We define $\bar{X}(k) := MX(k)$, $\tilde{X}(k) := M\hat{X}(k)$, $\bar{W}(k) := MW(k)$, and $\bar{E}(k) := ME(k)$. Clearly, it holds that

$$\|\bar{W}(k)\| = \|MW(k)\| = \|W(k)\| \in B_{\phi_W}(0).$$

Then, the discrete-time system may be rewritten as

$$\bar{X}(k+1) = \Delta\bar{X}(k) + \bar{W}(k). \tag{6}$$

Here, we give an allocation algorithm on the basis of the eigenvalues of G in order to achieve minimum data rate for observability of system (6).

Then, the main result of the section is the following.

Theorem 3.1. *Consider system (1). Suppose that, the initial state $X(0)$ and disturbance $W(k)$ are bounded, uncertain variables satisfying $\|X(0)\| \in B_{\phi_0}(0)$ and $\|W(k)\| \in B_{\phi_W}(0)$, respectively. There exists a nonsingular real matrix H that diagonalizes $A = H'\Lambda H$, where $\Lambda = diag[a_1, a_2, \cdots, a_n]$. Then, system (1) is observable in the sense (5) if the fixed data rate R of the channel satisfies the following inequality:*

$$R \geq \left\lceil \log_2 \prod_{i=1}^{n} e^{a_iT_s} \right\rceil \text{ (bits/sample)},$$

where $\lceil\cdot\rceil$ represents the ceil function, and is defined as $\lceil x\rceil := \min\{k \in \mathbb{Z} : k > x\}$.

Proof: Let $\bar{X}(k) := [\bar{x}_1(k)\ \bar{x}_2(k)\ \cdots\ \bar{x}_n(k)]'$ and $\bar{W}(k) := [\bar{w}_1(k)\ \bar{w}_2(k)\ \cdots\ \bar{w}_n(k)]'$. Then, it follows from (6) that

$$\bar{x}_i(k+1) = e^{a_iT_s}\bar{x}_i(k) + \bar{w}_i(k), \quad i = 1, 2, \cdots, n.$$

Notice that

$$\|\bar{W}(k)\| \in B_{\phi_W}(0),$$
$$\|\bar{X}(0)\| = \|MX(0)\| = \|X(0)\| \in B_{\phi_0}(0).$$

Then, we have

$$\bar{x}_i(0) \in B_{\phi_0}(0), \tag{7}$$
$$\bar{w}_i(k) \in B_{\phi_W}(0). \tag{8}$$

Let $\bar{E}(k) := [\bar{e}_1(k)\ \bar{e}_2(k)\ \cdots\ \bar{e}_n(k)]'$ and $\tilde{X}(k) := [\tilde{x}_1(k)\ \tilde{x}_2(k)\ \cdots\ \tilde{x}_n(k)]'$. Here, we set

$$\tilde{X}(0) = M\hat{X}(0) = 0.$$

Then, we obtain

$$\bar{e}_i(0) = \bar{x}_i(0) - \tilde{x}_i(0) \in B_{\phi_0}(0).$$

For any time k, we assume that

$$\bar{x}_i(k) \in B_{l_i(k)}(c_i(k)),$$
$$\tilde{x}_i(k) = c_i(k),$$
$$\bar{e}_i(k) = \bar{x}_i(k) - \tilde{x}_i(k) \in B_{l_i(k)}(0),$$

where $l_i(k)$ and $c_i(k)$ denote the radius and midpoint of the range of $\bar{x}_i(k)$, respectively.

The plant states will be quantized, encoded, and transmitted over a digital communication channel in order to ensure observability of system (1). Thus, we divide the range $B_{l_i(k)}(c_i(k))$ into $n_i \in \mathbb{N}$ equal intervals. The n_i indexes are encoded, and converted into the n_i codewords. It follows from [9] that the data rate R must satisfy the following inequality:

$$R \geq \lceil \log_2 \textstyle\prod_{i=1}^{n} n_i \rceil \text{ (bits/sample)}. \tag{9}$$

At time $k+1$, it follows that

$$\bar{x}_i(k+1) \in B_{l_i(k+1)}(c_i(k+1)),$$
$$\tilde{x}_i(k+1) = c(k+1),$$
$$\bar{e}_i(k+1) = \bar{x}_i(k+1) - \tilde{x}_i(k+1) \in B_{l_i(k+1)}(0),$$

where

$$l_i(k+1) = \frac{e^{a_i T_s}}{n_i} l_i(k) + \phi_W. \tag{10}$$

Combined with Equalities (7), (8) and (10), this implies that

$$l_i(k) = \left(\frac{e^{a_i T_s}}{n_i}\right)^k \phi_0 + \left[1 + \frac{e^{a_i T_s}}{n_i} + \left(\frac{e^{a_i T_s}}{n_i}\right)^2 + \cdots + \left(\frac{e^{a_i T_s}}{n_i}\right)^{k-1}\right] \phi_W. \tag{11}$$

If it is assumed that

$$n_i > e^{a_i T_s} \tag{12}$$

holds, it follows that

$$\lim_{k\to\infty} l_i(k) = \frac{1}{1 - \frac{e^{a_i T_s}}{n_i}} \phi_W.$$

It leads to

$$\limsup_{k\to\infty} |\bar{e}_i(k)| \leq \frac{1}{1 - \frac{e^{a_i T_s}}{n_i}} \phi_W.$$

Thus, we obtain

$$\limsup_{k\to\infty} \|E(k)\| \leq \phi_W \sqrt{\sum_{i=1}^{n} \left(\frac{1}{1 - \frac{e^{a_i T_s}}{n_i}}\right)^2} < \infty.$$

Substituting (12) into (9), we give

$$R \geq \lceil \log_2 \textstyle\prod_{i=1}^{n} e^{a_i T_s} \rceil \text{ (bits/sample)}. \tag{13}$$

Thus, system (1) is observable if Inequality (13) holds. □

It is shown in Theorem 3.1 that, system (1) is observable if the fixed data rate is larger than the lower bound given. The lower bound is different from ones in the literature because the data rate in our results is fixed, but ones in the literature are time varying. Using the fixed data rate may lead to better control performances.

Furthermore, the estimation error is bounded in the presence of the disturbance. Boundability is a very weak notion of observability. Then, we also further argue state estimation for MIMO control systems without the disturbance, and give the following result.

Corollary 3.1. *Consider system (1) without the disturbance. Suppose that, the initial state $X(0)$ is a bounded, uncertain variable satisfying $\|X(0)\| \in B_{\phi_0}(0)$. There exists a nonsingular real matrix H that diagonalizes $A = H'\Lambda H$, where $\Lambda = diag[a_1, a_2, \cdots, a_n]$. Then, system (1) is asymptotically observable in the sense*

$$\limsup_{k\to\infty} \|E(k)\| = 0,$$

if the fixed data rate R of the channel satisfies the following inequality:

$$R \geq \left\lceil \log_2 \prod_{i=1}^{n} e^{a_i T_s} \right\rceil \text{ (bits/sample).}$$

Proof: Here, we set $\|W(k)\| = 0$. Namely, let $\phi_W = 0$. Then, it follows from (11) that

$$l_i(k) = \left(\frac{e^{a_i T_s}}{n_i}\right)^k \phi_0.$$

If it is assumed that Inequality (12) holds, it follows that

$$\lim_{k\to\infty} l_i(k) = 0.$$

It leads to

$$\limsup_{k\to\infty} |\bar{e}_i(k)| = 0.$$

Thus, we obtain

$$\limsup_{k\to\infty} \|E(k)\| = 0.$$

Thus, the proof is complete. □

4. **Numerical Example.** In this paper, we discuss a class of networked control problems which arises in the coordinated motion control of autonomous and semiautonomous mobile agents, e.g., unmanned air vehicles (UAVs), unmanned ground vehicles (UGVs), and unmanned underwater vehicles (UUVs). Here, we present a practical example, where three of all the states of an unmanned underwater vehicle evolve in discrete time according to

$$X(k+1) = \begin{bmatrix} 1.2312 & 1.5423 & -0.2543 \\ 0.2156 & -2.3247 & 0.6321 \\ 0.7834 & 0.2725 & 1.7517 \end{bmatrix} X(k) + 5W(k), \quad Y(k) = X(k).$$

Let $X(0) = [3.24\ -3.51\ 2.56]'$ and $\phi_W = 0.1$. Here, we set ϕ_W on the basis of the environmental noise. The quantization, coding and control scheme on the basis of the condition in Theorem 3.1 is employed. We set $R = 120$ (bits/s). For the above system

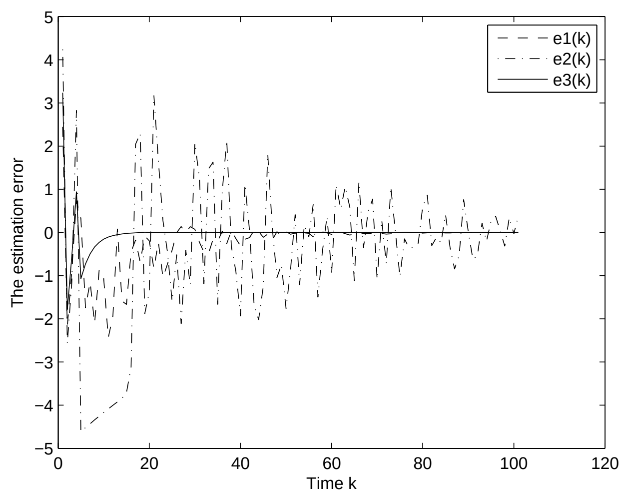

FIGURE 1. The estimation error responses with the disturbance

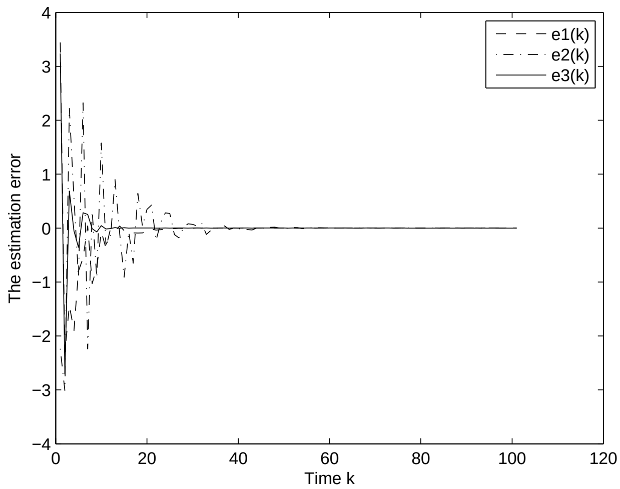

FIGURE 2. The estimation error responses without the disturbance

with the disturbance, the corresponding simulation is given in Figure 1. It is shown that the estimation error is bounded.

If we set $W(k) = 0$, the corresponding simulation is given in Figure 2. It is shown that the system is asymptotically observable if the data rate is larger than the lower bound given in Corollary 3.1. Clearly, the data rate and disturbance have important effects on observability. The fixed data rate is employed to ensure observability at each time step, and to achieve good performance not just on average.

5. **Conclusions.** In this paper, we considered MIMO control systems with the limited data rate, addressed the state estimation problem with the fixed data rate, and presented an allocation algorithm to regulate the transmission of the information on the plant states in order to achieve minimum data rate for observability. It was derived that the data rate and disturbance have important effects on observability of networked control systems. The simulation results have illustrated the effectiveness of the proposed scheme. The study of nonlinear system with limited data rate will be our future work.

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