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SOLVING THE INTERVAL LINEAR PROGRAMMING PROBLEMS BY A NEW APPROACH

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Abstract. *Uncertainties in many real-world problems mean that their parameters may be specified as lying between lower and upper bounds. To deal with such uncertainties, interval linear programming (ILP) problem is used which is a family of linear programming problems (namely, characteristic problem). The problem of determining optimal solution set of the ILP is a challenging problem. In most solving methods, given solution space is either infeasible such as the best and worst cases (BWC), ILP, and two-step method (TSM) or non-optimal such as modified ILP (MILP), enhanced ILP (EILP), and improved TSM (ITSM). Although in some methods such as improved ILP (IILP) and improved MILP (IMILP) given solution space is feasible and optimal, all feasible and optimal solutions are not obtained. In this paper, by using basis stability, we first find the constraints of the ILP which are active in all characteristic problems and then obtain the solution set of the ILP which is both feasible and optimal. The results of two numerical examples further indicated the feasibility and optimality of solutions of the ILP models.*

Keywords: Basis stability, Interval linear programming, Optimal solution set, Uncertainty

1. **Introduction.** Many problems in real world have uncertain coefficients. These uncertain coefficients have usually been converted to certain coefficients. Interval computations of Alefeld and Herzberger [1] is a way for treating uncertainties in data measurements. Instead of exact values we deal with real intervals.

An interval matrix is defined as $A^I = \left[\underline{A}, \overline{A}\right] = \left\{A \in \mathbb{R}^{m\times n} |\ \underline{A} \leq A \leq \overline{A}\right\}$, where $\underline{A} \leq \overline{A}$. Center and radius matrices are defined as $A^c = \frac{1}{2}\left(\underline{A} + \overline{A}\right)$, $\Delta_{\mathbf{A}} = \frac{1}{2}\left(\overline{A} - \underline{A}\right)$. A special case of an interval matrix is an interval vector which is a one-column interval matrix $\mathbf{X} = \left\{X |\ \underline{X} \leq X \leq \overline{X}\right\}$, where $\underline{X}$, $\overline{X} \in \mathbb{R}^m$. Interval arithmetic is defined in [1, 2].

We review some definitions [2]. A system $A^I\mathbf{x} = \mathbf{b}^I$ is said to be weakly (strongly) feasible if some (each) system $A\mathbf{x} = \mathbf{b}$ with data satisfying $A \in A^I, \mathbf{b} \in \mathbf{b}^I$ is feasible. In the same way we define weak and strong feasibility of a system of interval linear inequalities $A^I\mathbf{x} \leq \mathbf{b}^I$. A square interval matrix A^I is called regular if each $A \in A^I$ is nonsingular. A family of the ILP problems is defined as min $\{\mathbf{c}^T\mathbf{x} : A\mathbf{x} \leq (=\geq)\mathbf{b}, \mathbf{x} \geq 0\}$, where $\mathbf{c} \in \mathbf{c}^I \subseteq \mathbb{R}^n$, $A \in \mathbf{A}^I \subseteq \mathbb{R}^{n\times m}$, $\mathbf{b} \in \mathbf{b}^I \subseteq \mathbb{R}^m$ and $\mathbf{c}^I$, $\mathbf{A}^I$ and $\mathbf{b}^I$ are interval sets.

Many researchers worked on the ILP problems. Some solving methods of interval linear programming problems consist of BWC, ILP, MILP, EILP, ITSM, IILP, and IMILP.

In the best and worst cases (BWC) method proposed by Tong the ILP model has been converted into two sub-models [3], the best and the worst sub-models which have the largest and the smallest feasible spaces, respectively. A given point is feasible for the ILP model if it satisfies the constraints of the best problem and it is optimal for the ILP model if it is optimal for at least one characteristic model. The BWC method has been developed by Chinneck and Ramadan when the ILP model includes equality constraints [4]. Huang and Moore proposed a new interval linear programming method (ILP) [5].

Although the BWC method introduces exact bounds for the values of the objective function, it may result in infeasible decision variable spaces. Also, solution space given by the ILP method is not absolutely feasible, so the modified ILP (MILP) and enhanced ILP (EILP) methods have been proposed. In the MILP and EILP, by adding extra constraints, we can obtain absolutely feasible solution space [6, 7]. The solution space obtained through ITSM proposed by Wang and Huang [8] does not violate any constraints by introducing extra constraints in the solving process.

In [9], the authors improved the MILP method (IILP and IMILP methods) such that the obtained solution space is both feasible and optimal. In [10], the authors obtained the optimal solution set for the ILP when the feasible solution components of the best problems which have the largest feasible space, are positive. Therefore, if at least one of the components of the feasible solutions of the best problem is zero, then using this method for determining the optimal solution set of the ILP is useless.

In this paper, we find constraints of the ILP which are active in all characteristic problems and hence we can obtain the optimal solution set of the ILP. Firstly, we convert the ILP problem to a convex combination problem with coefficients $0 \leq \lambda_j \leq 1$, $0 \leq \mu_{ij} \leq 1$ and $0 \leq \mu_i \leq 1$ for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$. If for all i, j, $\mu_{ij} = \mu_i = \lambda_j = 0$, then the best problem has been obtained (in case of minimization problem). We move from the best problem towards the worst problem by tiny variations of λ_j, μ_{ij} and μ_i from 0 to 1. Then we solve each of the obtained problems. All the optimal solutions form a space that we call the optimal solution set of the ILP.

2. ILP, Convex Combination Problem and B-Stability.

In this section, we review the optimal value bounds of the ILP problem and then introduce the convex combination problem. Also, we define basis stability and its conditions, under which, we confidently determine the optimal solution set of the ILP.

2.1. ILP and convex combination problems.

In this section, we define ILP and convex combination problems.

Definition 2.1. *An ILP problem is defined as*

$$\begin{aligned} \min\ & z = \textstyle\sum_{j=1}^{n} [\underline{c}_j, \overline{c}_j] x_j \\ s.t.\ & \textstyle\sum_{j=1}^{n} [\underline{a}_{ij}, \overline{a}_{ij}] x_j \leq [\underline{b}_i, \overline{b}_i],\ i = 1, 2, \dots, m \\ & x_j \geq 0,\ j = 1, 2, \dots, n. \end{aligned} \tag{1}$$

We call problems (2) and (3) as characteristic version and convex combination problems of (1):

$$\begin{aligned} \min\ & z_1 = \textstyle\sum_{j=1}^{n} c_j x_j \\ \text{s.t.}\ & \textstyle\sum_{j=1}^{n} a_{ij} x_j \leq b_i,\ i = 1, 2, \dots, m \\ & x_j \geq 0,\ j = 1, 2, \dots, n, \end{aligned} \tag{2}$$

$$\begin{aligned} \min\ & z_2 = \textstyle\sum_{j=1}^{n} \left[(1-\lambda_j)\underline{c}_j + \lambda_j \overline{c}_j\right] x_j \\ \text{s.t.}\ & \textstyle\sum_{j=1}^{n} \left[(1-\mu_{ij})\underline{a}_{ij} + \mu_{ij}\overline{a}_{ij}\right] x_j \leq (1-\mu_i)\overline{b}_i + \mu_i \underline{b}_i,\ i = 1, 2, \dots, m \\ & x_j \geq 0,\ j = 1, 2, \dots, n, \end{aligned} \tag{3}$$

where $c_j \in \left[\underline{c}_j, \overline{c}_j\right]$, $a_{ij} \in \left[\underline{a}_{ij}, \overline{a}_{ij}\right]$, $b_i \in \left[\underline{b}_i, \overline{b}_i\right]$ and $0 \leq \lambda_j \leq 1$, $0 \leq \mu_i \leq 1$, $0 \leq \mu_{ij} \leq 1$ for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$.

Lemma 2.1. *Two problems (2) and (3) are the same.*

Proof: The proof is straightforward.

Theorem 2.1. [2] *An interval system* $A^I\mathbf{x} \le \mathbf{b}^I$, $\mathbf{x} \ge 0$ *is strongly feasible if and only if the system* $\overline{A}\mathbf{x} \le \underline{\mathbf{b}}$, $\mathbf{x} \ge 0$ *is feasible.*

Theorem 2.2. [2] *An interval system* $A^I\mathbf{x} = \mathbf{b}^I$, $\mathbf{x} \ge 0$ *is weakly feasible if and only if the system* $\underline{A}\mathbf{x} \le \overline{\mathbf{b}}$, $\overline{A}\mathbf{x} \ge \underline{\mathbf{b}}$, $\mathbf{x} \ge 0$ *is feasible.*

Theorem 2.3. [2] *An interval system* $A^I\mathbf{x} \le \mathbf{b}^I$, $\mathbf{x} \ge 0$ *is weakly feasible if and only if the system* $\underline{A}\mathbf{x} \le \overline{\mathbf{b}}$, $\mathbf{x} \ge 0$ *is feasible.*

Theorem 2.4. [3] *Let* $x_j \ge 0$ *for all* j*, then for the interval inequality* $\sum_{j=1}^n \left[\underline{a}_{ij}, \overline{a}_{ij}\right] x_j \le \left[\underline{b}_i, \overline{b}_i\right]$, $\sum_{j=1}^n \underline{a}_{ij}x_j \le \overline{b}_i$ *and* $\sum_{j=1}^n \overline{a}_{ij}x_j \le \underline{b}_i$ *are the largest and smallest feasible spaces respectively.*

Theorem 2.5. [3] *For ILP problem (1), the best and worst optimal values of the objective function are obtained by solving the following problems, respectively.*

$$\min\ \underline{z} = \sum_{j=1}^n \underline{c}_j x_j$$

$$s.t.\ \sum_{j=1}^n \underline{a}_{ij}x_j \le \overline{b}_i,\ i = 1, 2, \ldots, m,\ x_j \ge 0,\ j = 1, 2, \ldots, n,$$

$$\min\ \overline{z} = \sum_{j=1}^n \overline{c}_j x_j$$

$$s.t.\ \sum_{j=1}^n \overline{a}_{ij}x_j \le \underline{b}_i,\ i = 1, 2, \ldots, m,\ x_j \ge 0,\ j = 1, 2, \ldots, n.$$

The first and second sub-models are the best and worst cases (namely, BWC) respectively.

Let $\mathbf{A}^I$ and $\mathbf{b}^I$ be an interval matrix and vector, respectively. For an interval linear system $\mathbf{A}^I\mathbf{x} \le \mathbf{b}^I$, we define $S = \{\mathbf{x} | A\mathbf{x} \le \mathbf{b}, A \in \mathbf{A}^I, \mathbf{b} \in \mathbf{b}^I\}$.

Definition 2.2. *The narrowest interval vector containing the set* S *is called the interval hull of* S*.*

S is generally not an interval vector. It is usually difficult to describe S. Since S is generally so complicated in shape, it is usually impractical to use it. Instead, it is a common practice to seek the interval vector $\mathbf{x}^I$ containing S which has the narrowest possible interval components [11].

2.2. **Basis stability.** One of the essential points in the ILP problems is basis stability. Fundamental questions with ILP such as calculating the optimal solution set may be computationally very expensive. However, by exploiting basis stability, the question becomes much easier to solve, so we obtain all possible solutions and optimal values range; hence the optimal solution set of the ILP problems will be a subset of the optimal solutions generated by B-stability [12, 13].

Definition 2.3. [13] *The problem* $\min\{\mathbf{c}^T\mathbf{x} : A\mathbf{x} = \mathbf{b}, \mathbf{x} \ge 0\}$ *where* $\mathbf{c} \in \mathbf{c}^I \subseteq \mathbb{R}^n$, $A \in \mathbf{A}^I \subseteq \mathbb{R}^{m\times n}$ *and* $\mathbf{b} \in \mathbf{b}^I \subseteq \mathbb{R}^m$*, is called B-stable with basis* B *if* B *is an optimal basis for each characteristic problem. The ILP problem is called [unique] non-degenerate B-stable if each characteristic model has a [unique] non-degenerate optimal basic solution with the basis* B*.*

Let $B \subseteq \{1, 2, \ldots, n+m\}$ be an index set such that $\mathbf{A}_B$ is non-singular, where $\mathbf{A}_B$ denotes the restriction of A to the columns indexed by B. Similarly, $N = \{1, 2, \ldots, n+$

$m\}\setminus B$ stands for non-basic variables and $\mathbf{A}_N$ denotes the restriction to non-basic indices. The problem with center values is quite specific and not an arbitrary characteristic model.

In general, B-stability conditions have been presented in [12, 13]. These conditions are as follows:

(1) (regularity) $\mathbf{A}_B$ is regular.

(2) (feasibility) The solution set of the interval system $\mathbf{A}_B\mathbf{x}_B = \mathbf{b}$ is non-negative.

(3) (optimality) $\mathbf{A}_B$ is optimal, i.e., $\mathbf{c}_N^T - \mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{A}_N \geq 0^T$.

Theorem 2.6. [14] *If $\rho\left(|(A_B^c)^{-1}|\Delta_{A_B}\right) < 1$, then $\mathbf{A}_B$ is regular, where $\rho(.)$ denotes the spectral radius and Δ_{A_B} and A_B^c are the radius and center of the interval matrix $\mathbf{A}_B$, respectively.*

Theorem 2.7. [14] *If $\max_{1\leq i\leq n}\left(|(A_B^c)^{-1}|\Delta_{A_B}\right)_{ii} \geq 1$, then $\mathbf{A}_B$ is not regular.*

Theorem 2.8. [15] *If the interval vector $\mathbf{r}$ is an enclosure to the solution set of system $\mathbf{A}_B\mathbf{x}_B = \mathbf{b}$, then:*

$$\underline{r}_i = \min\left\{-\mathbf{x}_i^* + (\mathbf{x}_i^c + |\mathbf{x}_i^c|)\,M_{ii}, \frac{1}{2M_{ii}-1}\left(-\mathbf{x}_i^* + (\mathbf{x}_i^c + |\mathbf{x}_i^c|)M_{ii}\right)\right\},$$

$$\overline{r}_i = \max\left\{\mathbf{x}_i^* + (\mathbf{x}_i^c - |\mathbf{x}_i^c|)\,M_{ii}, \frac{1}{2M_{ii}-1}(\mathbf{x}_i^* + (\mathbf{x}_i^c - |\mathbf{x}_i^c|)M_{ii})\right\},$$

where

$$M = \left(I - |(A_B^c)^{-1}|\Delta_{A_B}\right)^{-1},\ \mathbf{x}^c = (A_B^c)^{-1}b^c,\ \mathbf{x}^* = M\left(|\mathbf{x}^c| + |(A_B^c)^{-1}|\Delta_b\right),$$

and A_B^c is non-singular and $\rho\left(|(A_B^c)^{-1}|\Delta_A\right) < 1$.

Theorem 2.9. [12] *Let diag(q) denote the diagonal matrix with entries $q_1, \ldots, q_m$. If for each $q \in \{\pm 1\}^m$, the solution set of system*

$$\begin{cases} \left((A_B^c)^T - (\Delta_{A_B})^T diag(q)\right)\mathbf{y} \leq \overline{\mathbf{c}}_B \\ -\left((A_B^c)^T + (\Delta_{A_B})^T diag(q)\right)\mathbf{y} \leq -\underline{\mathbf{c}}_B \\ diag(q)\mathbf{y} \geq 0 \end{cases} \tag{4}$$

lies in the solution set of system

$$\begin{cases} \left((A_N^c)^T + (\Delta_{A_N})^T diag(q)\right)\mathbf{y} \leq \underline{\mathbf{c}}_N \\ diag(q)\mathbf{y} \geq 0. \end{cases} \tag{5}$$

then, the optimality condition holds.

3. Optimal Solution Set of the ILP with a New Approach.

An optimal solution set of the ILP is defined as the set of all optimal solutions of the characteristic problems.

In [10], the authors prove that the optimal solution set of the ILP is equal to the intersection of the space generated by the best problem constraints and the worst problem constraints with the inverse sign, when all feasible solution components of the best problem are positive. Since checking this assumption is difficult, then we use basis stability to determine the optimal solution set of the ILP problem. Firstly, by using basis stability, we find the constraints of the ILP which are active in all characteristic problems and then we obtain the optimal solution set of the ILP.

Remark 3.1. *Since there are no dependencies, the constraints $A\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq 0$ are equivalent to $A\mathbf{x} + I\mathbf{y} = \mathbf{b}$, $\mathbf{x}, \mathbf{y} \geq 0$. The unwanted extra interval width is called the dependence problem or simply dependence* [11].

Definition 3.1. *For each $i = 1, 2, \ldots, m$, we define $S_i = \left\{\mathbf{x} : \sum_{j\in B} \overline{a}_{ij}x_j \geq \underline{b}_i\right\}$, and $T_i = \left\{\mathbf{x} : \sum_{j\in B} \underline{a}_{ij}x_j \leq \overline{b}_i\right\}$, where B is an index set corresponding to basis variables.*

Theorem 3.1. *Let ILP model (1) be unique non-degenerate stable with basis B and $N = \{1, 2, \ldots, n+m\} \setminus B$. The optimal solution set of the ILP model is equal to*

$$\begin{cases} \left(\bigcap_{i=1}^m T_i\right) \bigcap \left(\bigcap_{i=1}^m S_i\right), & \\ x_j > 0, & j \in B \\ x_j = 0, & j \in N \end{cases}$$

Proof: Let $\mathbf{x}^*$ be optimal solution of model (2) such that $c_j \in \left[\underline{c}_j, \overline{c}_j\right]$, $a_{ij} \in \left[\underline{a}_{ij}, \overline{a}_{ij}\right]$ and $b_i \in \left[\underline{b}_i, \overline{b}_i\right]$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Since the ILP model is unique non-degenerate B-stable, then $\sum_{j \in B} a_{ij} x_j^* = b_i$, $x_j^* > 0$ for $j \in B$ and $x_j^* = 0$ for $j \in N$. Therefore, for each $i = 1, 2, \ldots, m$,

$$\sum_{j \in B} \underline{a}_{ij} x_j^* \leq \sum_{j \in B} a_{ij} x_j^* = b_i \leq \overline{b}_i,$$
$$\sum_{j \in B} \overline{a}_{ij} x_j^* \geq \sum_{j \in B} a_{ij} x_j^* = b_i \geq \underline{b}_i.$$

So, for each $i = 1, 2, \ldots, m$, $\mathbf{x}^* \in T_i$ and $\mathbf{x}^* \in S_i$. Therefore, $\mathbf{x}^* \in \bigcap_{i=1}^m T_i$ and $\mathbf{x}^* \in \bigcap_{i=1}^m S_i$, and hence $\mathbf{x}^* \in \left(\bigcap_{i=1}^m T_i\right) \bigcap \left(\bigcap_{i=1}^m S_i\right)$. Conversely, suppose

$$\begin{cases} \mathbf{x}^* \in \left(\bigcap_{i=1}^m T_i\right) \bigcap \left(\bigcap_{i=1}^m S_i\right), & \\ x_j^* > 0, & j \in B \\ x_j^* = 0, & j \in N \end{cases}$$

so for each $i = 1, 2, \ldots, n$, $\mathbf{x}^* \in S_i$ and $\mathbf{x}^* \in T_i$; therefore, for $i = 1, 2, \ldots, m$, $\sum_{j \in B} \overline{a}_{ij} x_j^* \geq \underline{b}_i$, $\sum_{j \in B} \underline{a}_{ij} x_j^* \leq \overline{b}_i$, which gives,

$$\sum_{j \in B} \left(a_{ij}^c + \Delta_{a_{ij}}\right) x_j^* \geq b_i^c - \Delta_{b_i}, \quad \sum_{j \in B} \left(a_{ij}^c - \Delta_{a_{ij}}\right) x_j^* \leq b_i^c + \Delta_{b_i},$$

therefore

$$-\left(\sum_{j \in B} \Delta_{a_{ij}} x_j^* + \Delta_{b_i}\right) \leq \sum_{j \in B} a_{ij}^c x_j^* - b_i^c \leq \sum_{j \in B} \Delta_{a_{ij}} x_j^* + \Delta_{b_i},$$

then

$$\left|\sum_{j \in B} a_{ij}^c x_j^* - b_i^c\right| \leq \sum_{j \in B} \Delta_{a_{ij}} x_j^* + \Delta_{b_i} \quad i = 1, 2, \ldots, m.$$

We define $\mathbf{y} \in \mathbb{R}^m$ by

$$y_i = \begin{cases} \frac{\sum_{j \in B} a_{ij}^c x_j^* - b_i^c}{\sum_{j \in B} \Delta_{a_{ij}} x_j^* + \Delta_{b_i}} & \sum_{j \in B} \Delta_{a_{ij}} x_j^* + \Delta_{b_i} > 0 \\ 1 & \sum_{j \in B} \Delta_{a_{ij}} x_j^* + \Delta_{b_i} = 0, \end{cases}$$

for $i = 1, 2, \ldots, n$. Then $|y_i| \leq 1$ and for $i = 1, 2, \ldots, m$, $\sum_{j \in B} a_{ij}^c x_j^* - b_i^c = y_i \big(\sum_{j \in B} \Delta_{a_{ij}} x_j^* + \Delta_{b_i}\big)$; therefore, $\sum_{j \in B} \left(a_{ij}^c - y_i \Delta_{a_{ij}}\right) x_j^* = b_i^c + y_i \Delta_{b_i}$. Since $|y_i| \leq 1$, $a_{ij}^\circ = a_{ij}^c - y_i \Delta_{a_{ij}} \in \left[\underline{a}_{ij}, \overline{a}_{ij}\right]$ and $b_i^\circ = b_i^c + y_i \Delta_{b_i} \in \left[\underline{b}_i, \overline{b}_i\right]$. Therefore, $\mathbf{x}^*$ is a feasible solution of model (2) such that $a_{ij}^\circ \in \left[\underline{a}_{ij}, \overline{a}_{ij}\right]$ and $b_i^\circ \in \left[\underline{b}_i, \overline{b}_i\right]$. Now, consider the following characteristic model:

$$\begin{aligned} \min \ & \sum_{j=1}^n c_j^\circ x_j \\ \text{s.t.} \ & \sum_{j \in B} a_{ij}^\circ x_j \leq b_i^\circ, \ i = 1, 2, \ldots, m, \\ & x_j \geq 0, \ j \in B, \end{aligned} \tag{6}$$

where $c_j^\circ \in [\underline{c}_j, \overline{c}_j]$ is arbitrary, for $j = 1, 2, \ldots, n$. Note that $\mathbf{x}^*$ is a feasible solution of the model (6), since for $i = 1, 2, \ldots, m$, $\sum_{j=1}^n a_{ij}^\circ x_j^* = b_i^\circ$. For optimality $\mathbf{x}^*$, consider the dual of the model (6):

$$\begin{aligned} &\max \sum_{i=1}^m b_i^\circ u_i \\ &\text{s.t.} \sum_{i=1}^m a_{ij}^\circ u_i \geq c_j^\circ, \; j \in B, \\ &\qquad u_i \geq 0, \; i = 1, 2, \ldots, m. \end{aligned}$$

Since the ILP model is non-degenerate B-stable, then the optimal solution components of model (6) are positive, and for each $j \in B$, $c_j^\circ = \sum_{i=1}^m a_{ij}^\circ u_i^*$, where u_i^* is the optimal solution of the dual model. Therefore, $\mathbf{c}^\circ = \sum_{i=1}^m u_i^* \mathbf{a}_i^\circ$, and the objective function gradient $\mathbf{c}^\circ$ has been presented as a non-negative linear combination of the gradients of the active constraints at $\mathbf{x}^*$ (i.e., $\mathbf{a}_1^\circ, \mathbf{a}_2^\circ, \ldots, \mathbf{a}_m^\circ$), and hence, in view of the Karush-Kuhn-Tucker optimality conditions, $\mathbf{x}^*$ is the optimal solution of model (6).

Corollary 3.1. *If $m = n$ and all feasible solution components of the best problem which has the largest feasible space are positive, then $\mathbf{x}^*$ is an optimal solution of the ILP if and only if $\sum_{j=1}^n \overline{a}_{ij} x_j^* \geq \underline{b}_i$, $\sum_{j=1}^n \underline{a}_{ij} x_j^* \leq \overline{b}_i$, which has been proved in* [10].

4. **Examples.** In this section, we solve two examples. Firstly, all active constraints in ILP model are found, and then the optimal solution set is obtained.

Example 4.1. *Consider the following ILP problem:*

$$\begin{aligned} \min \;\; & z = [1, 5]x_1 + [3, 4]x_2 \\ s.t. \;\; & [0.5, 1]x_1 + [-2, -1]x_2 \leq [-1, 0] \\ & [-4, -3]x_1 + [0, 1]x_2 \leq [-3, -2] \\ & x_1, x_2 \geq 0. \end{aligned}$$

Let x_3 and x_4 be the slack variables of the constraints. A candidate basis for B-stability is $B = (1, 2)$, since it is optimal for characteristic problem with center values.
1) According to Theorem 2.6, the spectral radius is 0.57, so $\mathbf{A}_B = \begin{pmatrix} [0.5, 1] & [-2, -1] \\ [-4, -3] & [0, 1] \end{pmatrix}$ is regular.
2) According to Theorem 2.8, an enclosure to the solution set of $\mathbf{A}_B \mathbf{x}_B = \mathbf{b}$ is $\mathbf{x}_B = \begin{pmatrix} [0.15, 2] \\ [0, 3] \end{pmatrix}$ which is non-negative.
3) According to Theorem 2.9, the solution set of system (4) lies in the solution set of system (5) for each $q \in \{\pm 1\}^2$. Thus the problem is B-stable.

The best and worst values of the objective function is $\frac{7}{8}$ and 22, respectively. The optimal solutions are $\underline{\mathbf{x}}^* = \begin{pmatrix} 0.5 \\ 0.125 \end{pmatrix}$ and $\overline{\mathbf{x}}^* = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

We solve the convex combination problem for some values of λ_j, μ_i and μ_{ij} for $i, j = 1, 2$ in $[0, 1]$. All optimal solution points lie in solution space of the ILP where some of the points have been shown in Figure 1. Since $m = n = 2$ and for each $j = 1, 2$, $\underline{x}_j^*$, $\overline{x}_j^* > 0$, then in view of Corollary 3.1, the optimal solution set is shown in Figure 2 which is the same as Figure 1. The solution spaces obtained through BWC and our method have been shown in Figure 3. By comparing these methods, we conclude that in the solution space resulting from BWC, there are infeasible solutions.

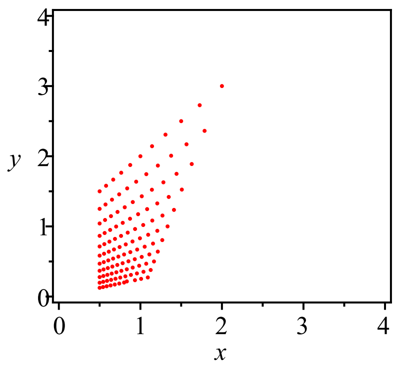

FIGURE 1. Approxination of the optimal solution space for the ILP problem of Example 4.1

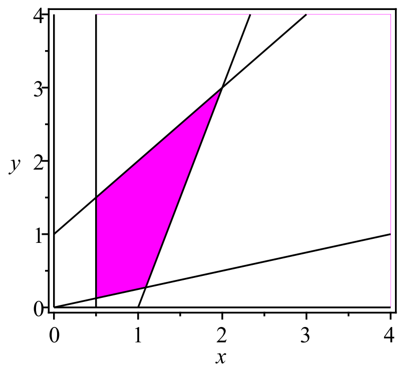

FIGURE 2. The optimal solution set for the ILP problem of Example 4.1

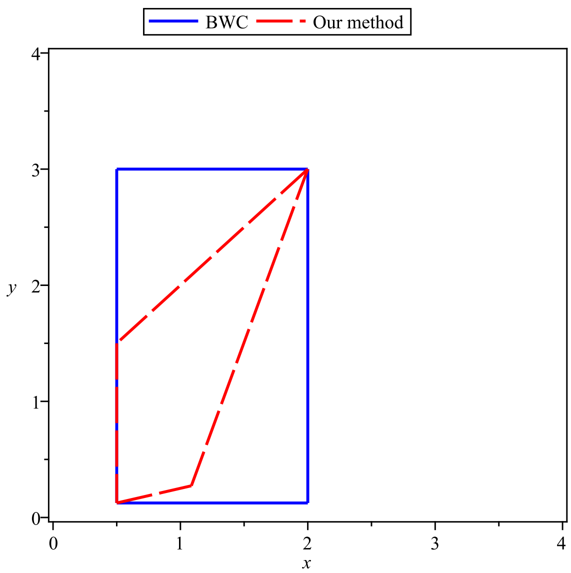

FIGURE 3. The solution spaces of BWC and our methods for Example 4.1

Example 4.2. *Consider the ILP problem as follows:*

$$\min \ \ z = [-2, -1]x_1 + x_2$$

$$s.t. \ [2, 3]x_1 + x_2 \leq [3, 4]$$

$$x_1 + [1, 2]x_2 \leq [2, 3]$$

$$x_1, x_2 \geq 0.$$

Let x_3 and x_4 be the slack variables of the constraints. A candidate basis for B-stability is $B = (1, 4)$, since it is optimal for the best problem as characteristic problem.

1) According to Theorem 2.6, the spectral radius is 0.2, so $\mathbf{A}_B = \begin{pmatrix} [2,3] & 0 \\ 1 & 1 \end{pmatrix}$ is regular.

2) According to Theorem 2.8, an enclosure to the solution set of $\mathbf{A}_B \mathbf{x}_B = \mathbf{b}$ is $\mathbf{x}_B = \begin{pmatrix} [0,3] \\ [0,4] \end{pmatrix}$ which is non-negative.

3) According to Theorem 2.9, the solution set of system (4) lies in the solution set of system (5) for each $q \in \{\pm 1\}^2$. Thus the problem is B-stable.

The best and worst values of the objective function are -4 and -1 respectively. The optimal solutions are $\underline{\mathbf{x}}^* = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\overline{\mathbf{x}}^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. If we solve above problem for different values of λ_j, μ_i and μ_{ij} in $[0, 1]$, then all optimal solution points form solution space of

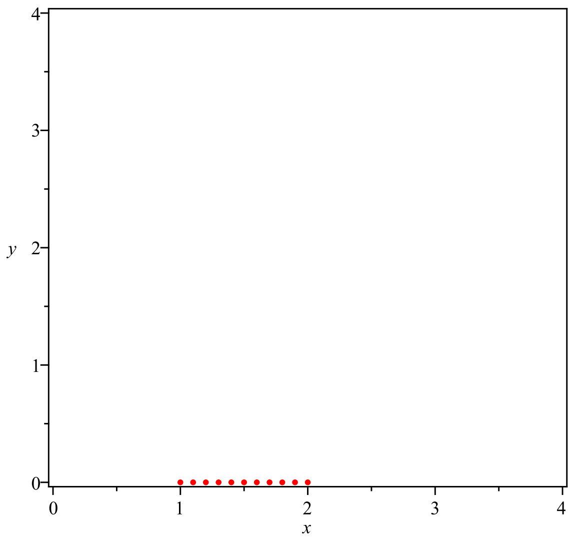

FIGURE 4. Some solution points for the ILP problem of Example 4.2

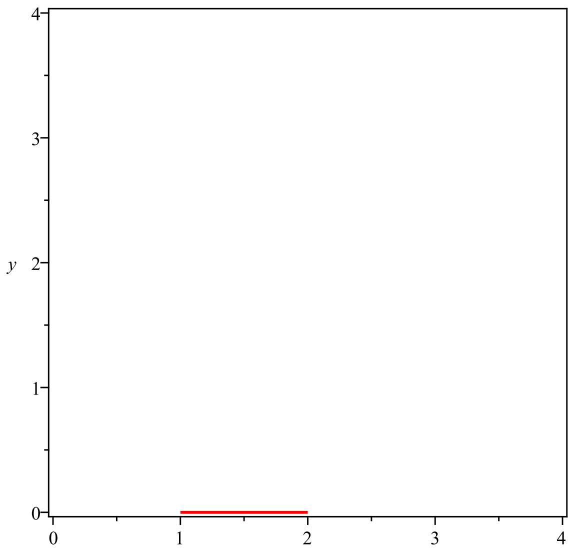

FIGURE 5. The optimal solution set for the ILP problem of Example 4.2

the ILP. Some of the points are shown in Figure 4. According to Theorem 3.1, we obtain the optimal solution set by solving the system

$$\begin{cases} 2x_1 + 0x_4 \le 4, & x_1 + x_4 \le 3 \\ 3x_1 + 0x_4 \ge 3, & x_1 + x_4 \ge 2 \\ x_1, x_4 > 0, & x_2 = x_3 = 0, \end{cases}$$

to be $1 \le x_1^* \le 2$, and $x_2^* = 0$, which is shown in Figure 5.

5. **Conclusion.** In this study, solving interval linear programming (ILP) problems is considered. In all solving methods of the ILP, optimal solutions cannot be completely obtained, such as BWC, ILP, MILP, and ITSM. Although the best and worst cases (BWC) method produces the best and worst optimal values, it results in infeasible space. In the BWC, the ILP model converts to two submodels, the best and worst submodels. In the ILP method, it is possible that the problem is not absolutely feasible, so the MILP, EILP and ITSM methods have been proposed. In these methods, by adding extra constraints, absolutely feasible solutions are obtained, but we lose some optimal solutions.

We first find the active constraints in all characteristic models of the ILP, and then obtain the optimal solution set of the ILP by using the constraints of the best and worst submodels. Two numerical examples further verified the effectiveness of the proposed method.

Further research will focus on other methods for obtaining the optimal solution set of the ILP models without considering the basis stability.

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