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A BATCH ARRIVAL UNRELIABLE RETRIAL G-QUEUE WITH WORKING VACATIONS AND VACATION INTERRUPTION

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Abstract. *This paper considers an $M^{[X]}/G/1$ retrial G-queue with general retrial times, in which the server is subject to breakdown and repair. In a normal service period, we assume that a breakdown is caused by the arrival of a negative customer, no matter whether the server is busy or free. When the orbit becomes empty at the time of service completion, the server goes for a working vacation. Applying the matrix-analytic method, we obtain the necessary and sufficient condition for the system to be stable. Using the supplementary variable method, we deal with the generating function of the number of customers in the orbit. Finally, some numerical examples are presented.*
Keywords: G-queue, Retrial, Breakdown, Working vacation, Vacation interruption

1. **Introduction.** In the past years, many researchers have discussed retrial queues with the concept of positive and negative customers. Yang et al. [1] studied an $M^{[X]}/G/1$ retrial G-queue with single vacation, where the breakdown of the server is represented by the arrival of a negative customer. Gao and Wang [2] analyzed an M/G/1 retrial G-queue with orbital search and non-persistent customers, where the server is subject to failure due to the negative customers arrival.

In 2002, Servi and Finn [3] first introduced working vacation, where the server provides service at a lower speed in the vacation period rather than stopping service completely. Murugan and Santhi [4] studied an M/G/1 queue with working vacations and server breakdown. Zhang and Liu [5] discussed an M/G/1-G queue with working vacations and vacation interruption, where the breakdown of the server is caused by the arrival of a negative customer. Do [6] first studied an M/M/1 retrial queue with working vacations. Retrial queueing systems with working vacations have been investigated extensively; readers can refer to Gao and Wang [7] and Gao et al. [8]. Recently, Rajadurai et al. [9] considered an M/G/1 retrial G-queue with working vacations and vacation interruption, where the server is subject to failure due to the negative customers arrival.

In this paper, using the method of supplementary variable technique, we consider an $M^{[X]}/G/1$ retrial queue with the combination of general retrial times, negative customers, working vacations, vacation interruption, breakdown and repair. Our work is a generalization of the well-known model discussed by Zhang and Liu [5], Gao et al. [8] and Rajadurai et al. [9]. In a regular busy period, we assume that negative customers can arrive and cause the server break down no matter whether the server is busy or free, and this case has been considered by Wang and Zhang [10], which is different from the situation discussed by Rajadurai et al. [9]. To the authors' best of knowledge, there is no research work investigating such a queueing model. Let parameters in this paper take proper values; many queueing systems with working vacations and vacation interruption will be the special cases of the model we consider. The mathematical results and theory of this model can be applied in the computer processing system. Our model is also helpful to managers who can design a system with economic management.

This paper is organized as follows. The detailed mathematical form of our model is given in Section 2. Using the matrix-analytic method, the stable condition is obtained in Section 3. Using the supplementary variable method, we deal with the steady state joint distribution of the server state and the number of customers in the retrial orbit. In Section 4, the effects of various parameters on the mean orbit length are analyzed numerically. Finally, Section 5 concludes the paper.

2. **System Model.** Positive customers arrive in batches according to a compound Poisson process with rate λ. The batch size X is a random variable with a common distribution $P(X = n) = g_n$, $n = 1, 2, \cdots$, and the probability generating function is defined as $g(z) = \sum_{n=1}^{\infty} g_n z^n$. If an arriving batch finds the server free, one of the customers from the batch begins his service and rest of them join a retrial orbit. We assume that only the customer at the head of the orbit queue is allowed to the server, and the retrial time R has a distribution function $R(x) = 1 - \exp\left\{-\int_0^x r(t)dt\right\}$. In a normal busy period, if the system becomes empty before the arrival of negative customers, the server begins a working vacation, and the vacation time V follows an exponential distribution with parameter θ. At a service completion instant in the vacation period, if there are customers in the system at that moment, the server will stop the vacation and come back to the normal working level. If the system is empty, on the other hand, the vacation continues. At the end of each vacation, the server only takes another new vacation if the system is empty. We assume that negative customers arrive according to a Poisson process with rate δ and only arrive when the sever is in a normal busy period. If a negative customer arrives, the server will break down no matter whether the server is busy or free. The positive customer being in service, if any, will also be removed. When the server fails, a repair procedure starts immediately and the server stops providing service. The repair time S has a distribution function $S(x) = 1 - \exp\left\{-\int_0^x s(t)dt\right\}$. At the end of the repair, if the system is empty, the server begins a working vacation. If the system is not empty, on the other hand, the server begins to provide normal service. Moreover, we assume that the normal service time S_b and the lower service time S_v have distribution functions $G_b(x) = 1 - \exp\left\{-\int_0^x \mu(t)dt\right\}$ and $G_v(x) = 1 - \exp\left\{-\int_0^x \eta(t)dt\right\}$, respectively.

Throughout the rest of the paper, for a distribution function $F(x)$, we define $\overline{F}(x) = 1 - F(x)$, $\widetilde{F}(s) = \int_0^{\infty} e^{-sx}dF(x)$, $\overline{F}^*(s) = \int_0^{\infty} e^{-sx}\overline{F}(x)dx$. Clearly, we have $\overline{F}^*(s) = \frac{1-\widetilde{F}(s)}{s}$.

Let $N(t)$ represent the number of customers in the retrial orbit at time t, and $I(t)$ denote the server state, defined as follows

$$I(t) = \begin{cases} 0, & \text{the server is in a working vacation period at time } t \text{ and the server is free,} \\ 1, & \text{the server is in a working vacation period at time } t \text{ and the server is busy,} \\ 2, & \text{the server is during a normal service period at time } t \text{ and the server is free,} \\ 3, & \text{the server is during a normal service period at time } t \text{ and the server is busy,} \\ 4, & \text{the server is under repair period at time } t \text{ .} \end{cases}$$

At time $t \geq 0$, we define the random variable $\xi(t)$ as follows: if $I(t) = 1$, $\xi(t)$ denotes the elapsed lower service time; if $I(t) = 2$, $\xi(t)$ represents the elapsed retrial time; if $I(t) = 3$, $\xi(t)$ stands for the elapsed normal service time; if $I(t) = 4$, $\xi(t)$ denotes the elapsed repair time. Therefore, the system can be described by a Markov process $X(t) = \{I(t), N(t), \xi(t)\}$ with state space

$$\Omega = \{(0, 0)\} \cup \{(i, 0, x), i = 1, 3, 4, x \geq 0\} \cup \{(i, n, x), i = 1, 2, 3, 4, n \geq 1, x \geq 0\}.$$

Let $\{t_n; n = 1, 2, \cdots\}$ be the sequence of epochs at which a service completion occurs or a breakdown occurs or a repair time ends. The sequence of random variables $Y_n =$

$\{I(t_n^+), N(t_n^+)\}$ forms an embedded Markov chain with state space $\{(0,0)\} \cup \{(4,0)\} \cup \{(i,k), i = 2,4, k \geq 1\}$.

3. **Stable Condition and Steady State Analysis.** We first introduce a few definitions:

$$a_n = \sum_{k=0}^{n} g_n^{(k)} \int_0^\infty \frac{(\lambda x)^k}{k!} e^{-\lambda x} e^{-\delta x} dG_b(x), \quad b_n = \sum_{k=0}^{n} g_n^{(k)} \int_0^\infty \frac{(\lambda x)^k}{k!} e^{-\lambda x} \delta e^{-\delta x} \overline{G}_b(x) dx,$$

$$c_n = \sum_{k=0}^{n} g_n^{(k)} \int_0^\infty \frac{(\lambda x)^k}{k!} e^{-\lambda x} e^{-\theta x} dG_v(x), \quad d_n = \sum_{k=0}^{n} g_n^{(k)} \int_0^\infty \frac{(\lambda x)^k}{k!} e^{-\lambda x} \theta e^{-\theta x} \overline{G}_v(x) dx,$$

$$u_n = \sum_{k=0}^{n} g_n^{(k)} \int_0^\infty \frac{(\lambda x)^k}{k!} e^{-\lambda x} dS(x), \quad v_n = \sum_{j=0}^{n} d_j a_{n-j}, \quad w_n = \sum_{j=0}^{n} d_j b_{n-j}, \quad n \geq 0.$$

After some computations, we can obtain

$$A(z) = \sum_{n=0}^{\infty} a_n z^n = \widetilde{G}_b\Big(\lambda - \lambda g(z) + \delta\Big), \quad B(z) = \sum_{n=0}^{\infty} b_n z^n = \delta \overline{G}_b^*\Big(\lambda - \lambda g(z) + \delta\Big),$$

$$C(z) = \sum_{n=0}^{\infty} c_n z^n = \widetilde{G}_v\Big(\lambda - \lambda g(z) + \theta\Big), \quad D(z) = \sum_{n=0}^{\infty} d_n z^n = \theta \overline{G}_v^*\Big(\lambda - \lambda g(z) + \theta\Big),$$

$$U(z) = \sum_{n=0}^{\infty} u_n z^n = \widetilde{S}\Big(\lambda - \lambda g(z)\Big), \quad V(z) = \sum_{n=0}^{\infty} v_n z^n = D(z)A(z),$$

$$W(z) = \sum_{n=0}^{\infty} w_n z^n = D(z)B(z).$$

The transition probability matrix of $\{Y_n; n \geq 1\}$ can be written as the block-Jacobi matrix

$$P = \begin{pmatrix} W_0 & W_1 & W_2 & \cdots \\ A_0 & A_1 & A_2 & \cdots \\ & A_0 & A_1 & \cdots \\ & & A_0 & \cdots \end{pmatrix},$$

where

$$W_0 = \begin{pmatrix} \frac{\theta}{\lambda+\theta} + \frac{\lambda g_1}{\lambda+\theta}(c_0 + v_0) & \frac{\lambda g_1}{\lambda+\theta} w_0 \\ u_0 & 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} \widetilde{R}(\lambda+\delta)a_0 & \widetilde{R}(\lambda+\delta)b_0 \\ 0 & 0 \end{pmatrix},$$

$$W_n = \begin{pmatrix} \sum\limits_{k=1}^{n+1} \frac{\lambda g_k}{\lambda+\theta}(c_{n-k+1} + v_{n-k+1}) & \sum\limits_{k=1}^{n+1} \frac{\lambda g_k}{\lambda+\theta} w_{n-k+1} \\ u_n & 0 \end{pmatrix}, \quad n \geq 1,$$

$$A_1 = \begin{pmatrix} \widetilde{R}(\lambda+\delta)a_1 + \lambda \overline{R}^*(\lambda+\delta)g_1 a_0 & \widetilde{R}(\lambda+\delta)b_1 + \lambda \overline{R}^*(\lambda+\delta)g_1 b_0 + \delta \overline{R}^*(\lambda+\delta) \\ u_0 & 0 \end{pmatrix},$$

$$A_n = \begin{pmatrix} \widetilde{R}(\lambda+\delta)a_n + \sum\limits_{k=1}^{n} \lambda \overline{R}^*(\lambda+\delta) g_k a_{n-k} & \widetilde{R}(\lambda+\delta)b_n + \sum\limits_{k=1}^{n} \lambda \overline{R}^*(\lambda+\delta) g_k b_{n-k} \\ u_{n-1} & 0 \end{pmatrix},$$

$n \geq 2$.

Theorem 3.1. *The embedded Markov chain $\{Y_n; n \geq 1\}$ is ergodic if and only if $\rho < 1$, where $\rho = \delta \overline{R}^*(\lambda+\delta) + \lambda \overline{R}^*(\lambda+\delta) g'(1) + U'(1) + \Big(\widetilde{R}(\lambda+\delta) + \lambda \overline{R}^*(\lambda+\delta)\Big)\Big(A'(1) + B'(1) - A(1)U'(1)\Big)$.*

Proof: It is not difficult to see that $\{Y_n; n \geq 1\}$ is an irreducible and aperiodic Markov chain, so we just need to prove that $\{Y_n; n \geq 1\}$ is positive recurrent if and only if $\rho < 1$. We have

$$\begin{aligned} A &= \sum_{k=0}^{\infty} A_k \\ &= \begin{pmatrix} \left(\widetilde{R}(\lambda+\delta) + \lambda\overline{R}^*(\lambda+\delta)\right)A(1) & \left(\widetilde{R}(\lambda+\delta) + \lambda\overline{R}^*(\lambda+\delta)\right)B(1) + \delta\overline{R}^*(\lambda+\delta) \\ 1 & 0 \end{pmatrix}, \end{aligned}$$

and the invariant probability vector of matrix A is given by

$$\begin{aligned} \pi &= (\pi_1, \pi_2) \\ &= \left(\frac{1}{2 - \left(\widetilde{R}(\lambda+\delta) + \lambda\overline{R}^*(\lambda+\delta)\right)A(1)}, \frac{1 - \left(\widetilde{R}(\lambda+\delta) + \lambda\overline{R}^*(\lambda+\delta)\right)A(1)}{2 - \left(\widetilde{R}(\lambda+\delta) + \lambda\overline{R}^*(\lambda+\delta)\right)A(1)} \right). \end{aligned}$$

The vector β is defined as $\beta = \sum_{n=0}^{\infty} nA_n e$, and β is explicitly given by

$$\begin{aligned} \beta = \Big(&\delta\overline{R}^*(\lambda+\delta) + \lambda\overline{R}^*(\lambda+\delta)g'(1) \\ &+ \left(\widetilde{R}(\lambda+\delta) + \lambda\overline{R}^*(\lambda+\delta)\right)\left(A'(1) + B'(1)\right),\ 1 + U'(1)\Big)^T. \end{aligned}$$

The embedded Markov chain $\{Y_n; n \geq 1\}$ is positive recurrent if and only if $\pi\beta < 1 \Longleftrightarrow \rho < 1$. □

From Burke's theorem and PASTA property, we know that the steady state probabilities of $X(t)$ exist if and only if $\rho < 1$, and define

$$\begin{aligned} P_{0,0} &= \lim_{t\to\infty} P(I(t) = 0, N(t) = 0), \\ P_{i,n}(x)dx &= \lim_{t\to\infty} P\left(I(t) = i, N(t) = n, x \leq \xi(t) < x + dx\right), \\ & i = 1, 3, 4, n \geq 0;\ i = 2, n \geq 1. \end{aligned}$$

By the method of supplementary variable technique, we can obtain the following equations

$$\frac{d}{dx}P_{1,n}(x) = -\left(\lambda + \theta + \eta(x)\right)P_{1,n}(x) + (1 - \delta_{n,0})\sum_{k=1}^{n} \lambda g_k P_{1,n-k}(x), \quad n \geq 0, \tag{1}$$

$$\frac{d}{dx}P_{2,n}(x) = -\left(\lambda + \delta + r(x)\right)P_{2,n}(x), \quad n \geq 1, \tag{2}$$

$$\frac{d}{dx}P_{3,n}(x) = -\left(\lambda + \delta + \mu(x)\right)P_{3,n}(x) + (1 - \delta_{n,0})\sum_{k=1}^{n} \lambda g_k P_{3,n-k}(x), \quad n \geq 0, \tag{3}$$

$$\frac{d}{dx}P_{4,n}(x) = -\left(\lambda + s(x)\right)P_{4,n}(x) + (1 - \delta_{n,0})\sum_{k=1}^{n} \lambda g_k P_{4,n-k}(x), \quad n \geq 0, \tag{4}$$

where $\delta_{n,0}$ is the Kronecker's symbol. The boundary conditions are

$$\lambda P_{0,0} = \int_0^\infty P_{1,0}(x)\eta(x)dx + \int_0^\infty P_{3,0}(x)\mu(x)dx + \int_0^\infty P_{4,0}(x)s(x)dx, \tag{5}$$

$$P_{1,n}(0) = \lambda g_{n+1} P_{0,0}, \quad n \geq 0, \tag{6}$$

$$P_{2,n}(0) = \int_0^\infty P_{1,n}(x)\eta(x)dx + \int_0^\infty P_{3,n}(x)\mu(x)dx + \int_0^\infty P_{4,n}(x)s(x)dx, \quad n \geq 1, \tag{7}$$

$$P_{3,n}(0) = \theta\int_0^\infty P_{1,n}(x)dx + \int_0^\infty P_{2,n+1}(x)r(x)dx$$

$$+ (1-\delta_{n,0})\sum_{k=0}^{n-1}\lambda g_{k+1}\int_0^\infty P_{2,n-k}(x)dx, \quad n\geq 0, \tag{8}$$

$$P_{4,n}(0) = (1-\delta_{n,0})\delta\int_0^\infty P_{2,n}(x)dx + \delta\int_0^\infty P_{3,n}(x)dx, \quad n\geq 0, \tag{9}$$

and the normalization condition is

$$P_{0,0} + \sum_{n=1}^{\infty}\int_0^\infty P_{2,n}(x)dx + \sum_{n=0}^{\infty}\left(\int_0^\infty P_{1,n}(x)dx + \int_0^\infty P_{3,n}(x)dx + \int_0^\infty P_{4,n}(x)dx\right) = 1. \tag{10}$$

Define $P_i(x,z) = \sum_{n=b}^{\infty} P_{i,n}(x)z^n$, $i=1,3,4$, $b=0$; $i=2$, $b=1$; from Equations (1)-(4), we have

$$P_1(x,z) = P_1(0,z)e^{-(\lambda-\lambda g(z)+\theta)x}\overline{G}_v(x), \tag{11}$$

$$P_2(x,z) = P_2(0,z)e^{-(\lambda+\delta)x}\overline{R}(x), \tag{12}$$

$$P_3(x,z) = P_3(0,z)e^{-(\lambda-\lambda g(z)+\delta)x}\overline{G}_b(x), \tag{13}$$

$$P_4(x,z) = P_4(0,z)e^{-(\lambda-\lambda g(z))x}\overline{S}(x). \tag{14}$$

From Equations (5)-(9), after some computations, we can obtain

$$\lambda P_{0,0} = P_{1,0}(0)c_0 + P_{3,0}(0)a_0 + P_{4,0}(0)u_0, \tag{15}$$

$$zP_1(0,z) = g(z)\lambda P_{0,0}, \tag{16}$$

$$P_2(0,z) = C(z)P_1(0,z) + A(z)P_3(0,z) + U(z)P_4(0,z) - \lambda P_{0,0}, \tag{17}$$

$$zP_3(0,z) = zD(z)P_1(0,z) + \left(\widetilde{R}(\lambda+\delta) + \lambda\overline{R}^*(\lambda+\delta)g(z)\right)P_2(0,z), \tag{18}$$

$$P_4(0,z) = \delta\overline{R}^*(\lambda+\delta)P_2(0,z) + B(z)P_3(0,z). \tag{19}$$

Using Equation (16), from (17)-(19), after some tedious algebraic manipulations, we can get

$$P_2(0,z) = \frac{K_2(z)}{M(z)}\lambda P_{0,0}, \quad P_3(0,z) = \frac{K_3(z)}{zM(z)}\lambda P_{0,0}, \quad P_4(0,z) = \frac{K_4(z)}{zM(z)}\lambda P_{0,0},$$

where

$$M(z) = z - \delta\overline{R}^*(\lambda+\delta)U(z)z - \left(\widetilde{R}(\lambda+\delta) + \lambda\overline{R}^*(\lambda+\delta)g(z)\right)\left(A(z) + B(z)U(z)\right),$$

$$K_2(z) = g(z)C(z) - z + g(z)V(z) + g(z)W(z)U(z),$$

$$K_3(z) = \left(1 - \delta\overline{R}^*(\lambda+\delta)U(z)\right)g(z)D(z)z + \left(\widetilde{R}(\lambda+\delta) + \lambda\overline{R}^*(\lambda+\delta)g(z)\right)\left(g(z)C(z) - z\right),$$

$$K_4(z) = \delta\overline{R}^*(\lambda+\delta)\left(g(z)C(z) - z + g(z)V(z)\right)z + g(z)W(z)z + \left(\widetilde{R}(\lambda+\delta) + \lambda\overline{R}^*(\lambda+\delta)g(z)\right)\left(g(z)C(z) - z\right)B(z).$$

In order to get generating functions, we introduce a lemma here. The proof can be obtained by some computations, and we omit it here.

Lemma 3.1.

$$M'(1) = 1-\rho, \quad K_2'(1) = C'(1) + g'(1) - 1 + V'(1) + W'(1) + W(1)U'(1),$$

$$M''(1) = -\delta\overline{R}^*(\lambda+\delta)\left(U''(1) + 2U'(1)\right) - 2\lambda\overline{R}^*(\lambda+\delta)g'(1)\left(A'(1) + B'(1) + B(1)U'(1)\right) - \lambda\overline{R}^*(\lambda+\delta)g''(1)$$

$$
\begin{aligned}
&-\Big(\widetilde{R}(\lambda+\delta)+\lambda\overline{R}^*(\lambda+\delta)\Big)\Big(A''(1)+B''(1)+2B'(1)U'(1)+B(1)U''(1)\Big),\\
K_2''(1) &= g''(1)+2g'(1)\Big(C'(1)+V'(1)+W'(1)+W(1)U'(1)\Big)+C''(1)+V''(1)\\
&\quad+W''(1)+2W'(1)U'(1)+W(1)U''(1),\\
K_3'(1) &= -\overline{R}^*(\lambda+\delta)D(1)\Big(\delta U'(1)+\lambda g'(1)\Big)\\
&\quad+\Big(1-\delta\overline{R}^*(\lambda+\delta)\Big)\Big(g'(1)+D'(1)-C(1)+C'(1)\Big),\\
K_3''(1) &= -\overline{R}^*(\lambda+\delta)D(1)\Big[\delta U''(1)+2\delta\Big(g'(1)+1\Big)U'(1)+\lambda g''(1)\Big]\\
&\quad+\Big(1-\delta\overline{R}^*(\lambda+\delta)\Big)\Big[g''(1)+2g'(1)\Big(D'(1)+D(1)+C'(1)\Big)\\
&\quad+D''(1)+2D'(1)+C''(1)\Big]\\
&\quad-2\delta\overline{R}^*(\lambda+\delta)U'(1)D'(1)+2\lambda\overline{R}^*(\lambda+\delta)g'(1)\Big(C'(1)+g'(1)C(1)-1\Big),\\
K_4'(1) &= \delta\overline{R}^*(\lambda+\delta)\Big[C'(1)+\Big(g'(1)+1\Big)\Big(C(1)+V(1)\Big)+V'(1)-2\Big]\\
&\quad+\Big(g'(1)+1\Big)W(1)+W'(1)+\lambda\overline{R}^*(\lambda+\delta)g'(1)\Big(C(1)-1\Big)B(1)\\
&\quad+\Big(\widetilde{R}(\lambda+\delta)+\lambda\overline{R}^*(\lambda+\delta)\Big)\Big[\Big(C'(1)+g'(1)C(1)-1\Big)B(1)\\
&\quad+\Big(C(1)-1\Big)B'(1)\Big],\\
K_4''(1) &= \delta\overline{R}^*(\lambda+\delta)\Big[C''(1)+2g'(1)\Big(C'(1)+V'(1)\Big)+g''(1)\Big(C(1)+V(1)\Big)+V''(1)\Big]\\
&\quad+2\delta\overline{R}^*(\lambda+\delta)\Big[C'(1)+g'(1)\Big(C(1)+V(1)\Big)-1+V'(1)\Big]+g''(1)W(1)\\
&\quad+2g'(1)\Big(W'(1)+W(1)\Big)+W''(1)+2W'(1)\\
&\quad+\lambda\overline{R}^*(\lambda+\delta)\Big[g''(1)\Big(C(1)-1\Big)B(1)+2g'(1)\Big(C'(1)+g'(1)C(1)-1\Big)B(1)\Big]\\
&\quad+2\lambda\overline{R}^*(\lambda+\delta)g'(1)\Big(C(1)-1\Big)B'(1)+\Big(\widetilde{R}(\lambda+\delta)+\lambda\overline{R}^*(\lambda+\delta)\Big)\Big(C''(1)\\
&\quad+2g'(1)C'(1)+g''(1)C(1)\Big)B(1)+\Big(\widetilde{R}(\lambda+\delta)\\
&\quad+\lambda\overline{R}^*(\lambda+\delta)\Big)\Big[2\Big(C'(1)+g'(1)C(1)-1\Big)B'(1)+\Big(C(1)-1\Big)B''(1)\Big],\\
P_2(0,1) &= \frac{K_2'(1)}{1-\rho}\lambda P_{0,0},\ P_2'(0,1)=\lim_{z\to1}P_2'(0,z)=\frac{K_2''(1)}{2(1-\rho)}\lambda P_{0,0}-\frac{K_2'(1)M''(1)}{2(1-\rho)^2}\lambda P_{0,0},\\
P_3(0,1) &= \frac{K_3'(1)}{1-\rho}\lambda P_{0,0},\\
P_3'(0,1) &= \lim_{z\to1}P_3'(0,z)=\frac{K_3''(1)}{2(1-\rho)}\lambda P_{0,0}-\frac{K_3'(1)}{1-\rho}\lambda P_{0,0}-\frac{K_3'(1)M''(1)}{2(1-\rho)^2}\lambda P_{0,0},\\
P_4(0,1) &= \frac{K_4'(1)}{1-\rho}\lambda P_{0,0},\\
P_4'(0,1) &= \lim_{z\to1}P_4'(0,z)=\frac{K_4''(1)}{2(1-\rho)}\lambda P_{0,0}-\frac{K_4'(1)}{1-\rho}\lambda P_{0,0}-\frac{K_4'(1)M''(1)}{2(1-\rho)^2}\lambda P_{0,0}.
\end{aligned}
$$

Define the marginal generating functions $\Phi_i(z)=\int_0^\infty P_i(x,z)dx,\ i=1,2,3,4$. From Equations (11)-(14), we have the following theorem.

Theorem 3.2.

$$\Phi_1(z) = P_1(0,z)\frac{D(z)}{\theta} = \frac{g(z)D(z)}{\theta z}\lambda P_{0,0}, \quad \Phi_2(z) = P_2(0,z)\overline{R}^*(\lambda+\delta),$$
$$\Phi_3(z) = P_3(0,z)\frac{B(z)}{\delta}, \quad \Phi_4(z) = P_4(0,z)\frac{1-U(z)}{\lambda-\lambda g(z)},$$

where $P_{0,0}$ can be determined by the normalization condition

$$P_{0,0} + \Phi_1(1) + \Phi_2(1) + \Phi_3(1) + \Phi_4(1) = 1,$$

which leads to

$$P_{0,0} = \left(1 + \frac{D(1)}{\theta}\lambda + \frac{K_2'(1)}{1-\rho}\overline{R}^*(\lambda+\delta)\lambda + \frac{K_3'(1)}{1-\rho}\frac{B(1)}{\delta}\lambda + \frac{K_4'(1)}{1-\rho}\frac{U'(1)}{g'(1)}\right)^{-1}.$$

Clearly, the probability generating function of the number of customers in the orbit is given by

$$\Phi(z) = P_{0,0} + \Phi_1(z) + \Phi_2(z) + \Phi_3(z) + \Phi_4(z).$$

After some calculations, the mean orbit length is given by

$$\begin{aligned} E[L] = \lim_{z\to 1}\Phi'(z) &= \frac{\left(g'(1)-1\right)D(1) + D'(1)}{\theta}\lambda P_{0,0} + P_2'(0,1)\overline{R}^*(\lambda+\delta) \\ &+ P_3'(0,1)\frac{B(1)}{\delta} + P_3(0,1)\frac{B'(1)}{\delta} + P_4'(0,1)\frac{U'(1)}{\lambda g'(1)} \\ &+ P_4(0,1)\frac{g'(1)U''(1) - g''(1)U'(1)}{2\lambda\left(g'(1)\right)^2}. \end{aligned}$$

4. **Numerical Results.** In this section, we present some numerical examples to illustrate the effects of the varying parameters on the mean orbit length $E[L]$. For the simplicity purpose, we assume that the normal service time, the lower service time, the retrial time and the repair time are governed by exponential distribution with parameters μ, η, α and β, respectively. Moreover, we assume that the arrival batch size X follows a geometric distribution with parameter p, that is, $P(X = n) = p(1-p)^{n-1}$. Under the stable condition, the values of parameters are arbitrarily chosen as $\lambda = 1.2$, $p = 0.8$, $\delta = 0.5$,

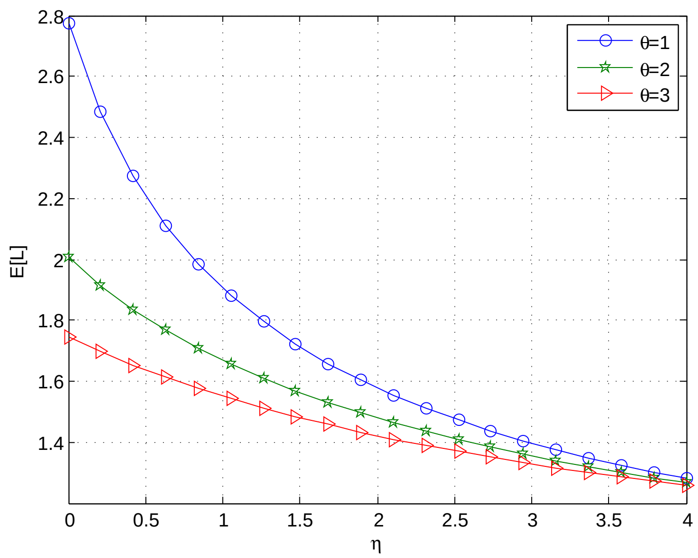

FIGURE 1. The effect of η on $E[L]$ for different values of θ

$\beta = 2$, $\mu = 6$, $\eta = 3$, $\theta = 1$ and $\alpha = 4$, unless they are considered as variables in the respective figures.

Figure 1 presents the mean orbit length $E[L]$ versus η for different values of θ, and it is obvious that $E[L]$ decreases with increasing values of η. As expected, increasing θ decreases the value of $E[L]$, and this dependency becomes smaller for larger values of η. An especial case is $\eta = 0$, i.e., the server cannot provide service in a vacation period, and θ has a noticeable effect on $E[L]$ which cannot be ignored.

In our model, only the arrival of a negative customer can cause the server break down, and there is no service in a breakdown period. Clearly, Figure 2 indicates that $E[L]$ increases evidently with increasing values of δ, especially when β is small. We can also find that $E[L]$ decreases as β increases, which agrees with the intuitive expectation. An extreme case is $\delta = 0$, i.e., the server cannot break down, and Figure 2 shows that β has no effect on $E[L]$ when $\delta = 0$.

Figure 3 illustrates that $E[L]$ decreases with increasing values of α; the reason is that the mean retrial time decreases as α increases. And the smaller the mean retrial time is,

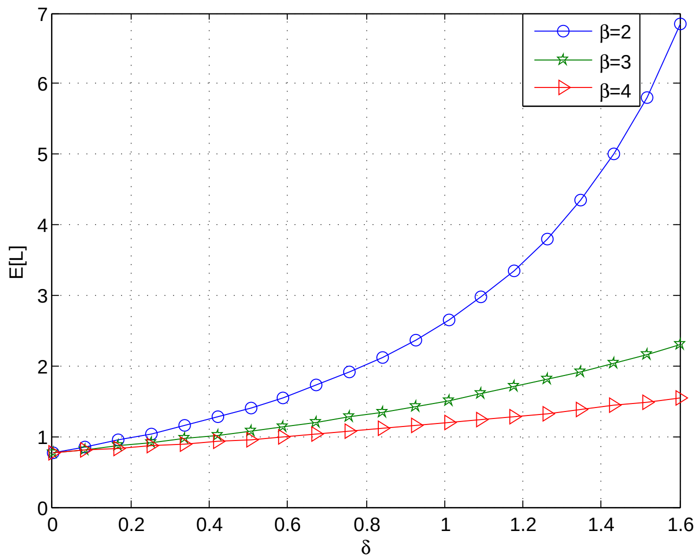

FIGURE 2. The effect of δ on $E[L]$ for different values of β

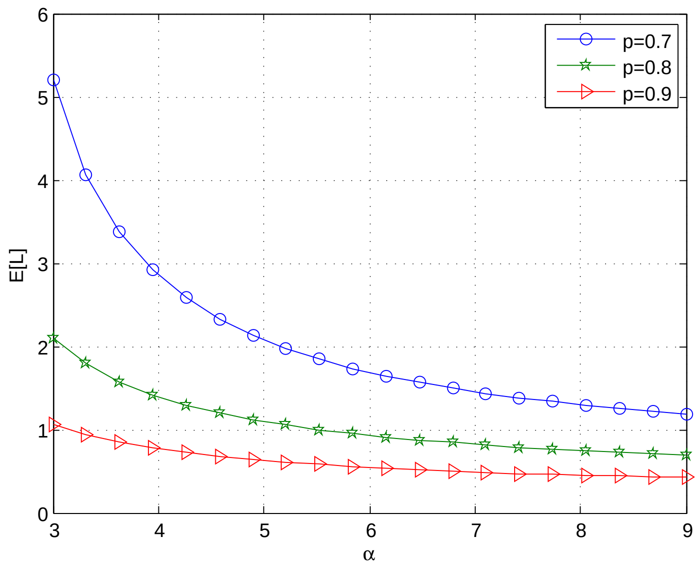

FIGURE 3. The effect of α on $E[L]$ for different values of p

the bigger the probability that the server is busy, which decreases the value of $E[L]$. Since the mean retrial time is $1/\alpha$, when α is large, it can be observed that the effect of α on $E[L]$ is not obvious as α increases. Obviously, the smaller the batch size is, the shorter the average queue length is, i.e., $E[L]$ decreases as p increases.

5. **Conclusion.** In this work, we investigate a single server retrial queue with batch arrival, negative customers, working vacations and vacation interruption, where the server is subject to breakdown and repair. Our model can be considered as a generalized version of many existing queueing models. Using embedded Markov chain and matrix-analytic method, we obtain the condition of stability. Supplementary variable technique is employed to discuss the probability generating function of the number of customers in the orbit. Finally, the effects of various parameters on the mean orbit length are examined numerically. For future study, one can discuss a similar model but without vacation interruption.

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