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FINITE FREQUENCY H_∞ CONTROL FOR NONSTANDARD SINGULARLY PERTURBED MODEL

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Abstract. *This paper is concerned with the finite frequency (FF) bounded realness for nonstandard singularly perturbed model (SPM). By using the generalized Kalman-Yakubovič-Popov (GKYP) lemma, we develop the FF H_∞ controller for nonstandard SPM within the low and high frequency range, respectively. The matrix inequality conditions are derived in terms of bilinear matrix inequalities (BMIs) independent of the small perturbation parameter, which avoid calculation stiffness. These results are also applicable to the standard case of SPM. Finally, some numerical examples are given to demonstrate the validity and advantage of the proposed method.*

Keywords: Finite frequency, Robust control, Nonstandard singularly perturbed model

1. **Introduction.** The slow and fast dynamical phenomena occurring in multiple time scales exist widely in engineering applications. It also brings new challenge to control theory since the direct use of standard control method on multiple time scale systems may lead to calculation stiffness and controller ill-conditioning. To overcome these disadvantages, singularly perturbed theory is introduced to model the multiple time scale system and the controller is developed based on the SPM.

Among the literature, the majority deals with the problem in time domain. Various control strategies have been developed for both standard and nonstandard cases of SPM. For example, optimal control for standard case was developed in [1] and nonstandard case in [2]. The control of SPM has also been applied in various fields, such as miniature unmanned aerial vehicle, hydraulic systems in [3], and optical networks in [4]. It is worth mentioning that all the aforementioned results, regardless of the standard case or nonstandard case, are in the time domain. However, in engineering practice, the frequency information is more intrinsic and used more often. Thus, the frequency analysis of SPM is important for practical meaning.

Compared with the work in time domain, the result of SPM in frequency domain is rarely developed. The foundation of frequency analysis for standard SPM was established in [5]. The model-matching problem for standard two time scale transfer functions was studied in [6] and [7] via minimizing the H_∞ norm of the peak error. [8] provided different frequency domain approaches to compute the upper bound of the small perturbation parameter. To the best of our knowledge, most of the frequency analyses of SPM were dealing with the standard case. Although some papers, such as [9], addressed the frequency analysis of nonstandard SPM by converting nonstandard SPM to standard case, the method had no significant difference and needed additional assumptions for the transformation. In this paper, we directly perform the frequency analysis to the nonstandard case using the GKYP lemma. Moreover, the FF bounded realness, which is less conservative than the full frequency bounded realness, is also obtained by applying the GKYP lemma. Compared with the traditional KYP lemma, FDI in the GKYP lemma need not cover the entire frequency domain but a finite frequency range. So researchers only need to deal with a certain frequency range of the system they concern, which is more precise and practical. Compared with nonstandard SPM, standard SPM needs additional assumption on the state matrix such that fast and slow decomposition can be applied. In our work, the GKYP lemma is introduced into the frequency analysis of nonstandard SPM. The conditions of bounded realness are derived for nonstandard SPM within different frequency ranges. The small perturbation parameter ε is removed from the conditions such that calculation stiffness can be circumvented. Time domain interpretations are presented such that the finite frequency characterization is more intuitive.

The rest of the paper is organized as follows. The problem formulation is presented in Section 2. In Section 3, we derive the conditions of FF bounded realness for SPM and provide its time domain interpretation. The numerical simulation is provided in Section 4 to demonstrate the validity and superiority of the proposed method. Finally, the paper is concluded in Section 5.

2. **Problem Formulation.**

The nonstandard SPM considered in this paper is shown as

$$\begin{aligned} \dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t) + B_{11}w(t) + B_{21}u(t), \\ \varepsilon\dot{x}_2(t) &= A_{21}x_1(t) + A_{22}x_2(t) + B_{12}w(t) + B_{22}u(t), \\ z(t) &= C_{11}x_1(t) + C_{12}x_2(t) + D_{12}u(t), \end{aligned} \tag{1}$$

where $A_{11} \in \mathbf{R}^{n_1\times n_1}$, $A_{12} \in \mathbf{R}^{n_1\times n_2}$, $B_{11} \in \mathbf{R}^{n_1\times l}$, $B_{21} \in \mathbf{R}^{n_1\times m}$, $A_{21} \in \mathbf{R}^{n_2\times n_1}$, $A_{22} \in \mathbf{R}^{n_2\times n_2}$, $B_{12} \in \mathbf{R}^{n_2\times l}$, $B_{22} \in \mathbf{R}^{n_2\times m}$, $C_{11} \in \mathbf{R}^{m_2\times n_1}$, $C_{12} \in \mathbf{R}^{m_2\times n_2}$ and $D_{12} \in \mathbf{R}^{m_2\times m}$ are constant coefficient matrices of the system. A_{22} is the singular matrix. $x_1(t) \in \mathbf{R}^{n_1}$, $x_2(t) \in \mathbf{R}^{n_2}$ are state variables and $z(t) \in \mathbf{R}^{m_2}$ is controlled output. $u(t) \in \mathbf{R}^m$ denotes the control input and $w(t) \in \mathbf{R}^l$ represents the disturbance. ε is the small perturbation parameter which satisfies $0 < \varepsilon \ll 1$. Specifically, the control input $u(t)$ takes the form $u(t) = Kx(t)$ where K is the control gain to be determined. The closed-loop form of (1) is rewritten as

$$E_\varepsilon\dot{x}(t) = (A + B_2K)x(t) + B_1w(t), \tag{2a}$$

$$z(t) = Cx(t), \tag{2b}$$

where

$$E_\varepsilon = \begin{bmatrix} I & 0 \\ 0 & \varepsilon \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad C_1 = \begin{bmatrix} C_{11} & C_{12} \end{bmatrix}, \quad x(t) = \begin{bmatrix} x_1^T(t) & x_2^T(t) \end{bmatrix}^T,$$

$$B_1 = \begin{bmatrix} B_{11}^T & B_{12}^T \end{bmatrix}^T, \quad B_2 = \begin{bmatrix} B_{21}^T & B_{22}^T \end{bmatrix}^T, \quad C = C_1 + D_{12}K.$$

The main objective of this paper is to design controller such that the FF bounded realness is satisfied, that is, the following requirements Q1) and Q2) are met simultaneously.
Q1) System (2) is asymptotically stable with $w(t) = 0$,
Q2) $\|T_{wz}(j\omega)\| < \gamma$, for $\omega \in \Omega$, where symbol γ is the noise attenuation level and Ω denotes the frequency range. $T_{wz}(j\omega)$ represents the transfer function from $w(t)$ to $z(t)$ and

$$T_{wz}(j\omega) = C(j\omega E_\varepsilon - (A + B_2K))^{-1}B_1. \tag{3}$$

3. **Main Results.** This section is devoted to deriving the conditions of FF bounded realness for system (2). Specifically, some sufficient conditions are presented in terms of bilinear matrix inequalities (BMIs) such that FF bounded realness is satisfied in the low and high frequency ranges, respectively. It is worth mentioning that the conditions in this section are all independent of the small perturbation parameter ε, which can cause calculation stiffness.

3.1. **Parameter-independent FF bounded realness.** We first introduce the following theorem such that system (2) satisfies Q1).

Theorem 3.1. *If we have* $P_s = \begin{bmatrix} P_{s11} & 0 \\ P_{s21} & P_{s22} \end{bmatrix}$ *with* $P_{s11} > 0$ *and* $P_{s22} > 0$ *such that*

$$(A + B_2K)^T P_s + P_s^T (A + B_2K) < 0, \tag{4}$$

then there exists an $\varepsilon^* > 0$ *such that system (2a) is asymptotically stable with* $w(t) = 0$ *for* $\varepsilon \in (0, \varepsilon^*]$.

Theorem 3.2. *Considering the closed-loop system* (2)*, the low frequency bounded realness*

$$\|T_{wz}(j\omega)\| < \gamma_1, \qquad \forall|\omega| \le \omega_l \tag{5}$$

holds for $\varepsilon \in (0, \varepsilon^*]$, $\varepsilon^* > 0$*, if there exist matrix variables* $Q_1 > 0$, $P_{l11} \in \mathbf{H}_{n_1}$, $P_{l22} \in \mathbf{H}_{n_2}$, P_{l21} *and* K *such that the following BMIs hold*

$$\Phi = \begin{bmatrix} \Phi(1,1) & \Phi(1,2) & (C_1 + D_{12}K)^T \\ * & -B_1^T Q_1 B_1 - \gamma_1^2 I & 0 \\ * & * & -I \end{bmatrix} < 0, \tag{6}$$

where

$$\begin{aligned} &\Phi(1,1) = P_l^T (A + B_2K) + (A + B_2K)^T P_l + \omega_l^2 H^T Q_1 H, \\ &\Phi(1,2) = -(A + B_2K)^T Q_1 B_1 + P_l^T B_1, \\ &P_l = \begin{bmatrix} P_{l11} & 0 \\ P_{l12}^T & P_{l22} \end{bmatrix}, \qquad H = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Proof: According to [11], low frequency bounded realness (5) can be derived from

$$\begin{bmatrix} A + B_2K & B_1 \\ E_\varepsilon & 0 \end{bmatrix}^T \begin{bmatrix} -Q_1 & P_{l\varepsilon} \\ P_{l\varepsilon} & \omega_l^2 Q_1 \end{bmatrix} \begin{bmatrix} A + B_2K & B_1 \\ E_\varepsilon & 0 \end{bmatrix} + \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & -\gamma_1^2 I \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} < 0, \tag{7}$$

where $Q_1 > 0$. The following inequality is then obtained

$$\Phi_0 = \begin{bmatrix} \Phi_0(1,1) & \Phi_0(1,2) \\ * & -B_1^T Q_1 B_1 - \gamma_1^2 I \end{bmatrix} < 0, \tag{8}$$

with

$$\begin{aligned} \Phi_0(1,1) &= -(A + B_2K)^T Q_1 (A + B_2K) + E_\varepsilon^T P_{l\varepsilon} (A + B_2K) \\ &\quad + (A + B_2K)^T P_{l\varepsilon} E_\varepsilon + \omega_l^2 E_\varepsilon^T Q_1 E_\varepsilon + C^T C, \\ \Phi_0(1,2) &= -(A + B_2K)^T Q_1 B_1 + E_\varepsilon^T P_{l\varepsilon} B_1. \end{aligned}$$

It is worth mentioning that $\Phi_0 < 0$ is the nonlinear matrix inequality and difficult to solve out. In the sequel, we will transform the nonlinear matrix inequality into its special case BMIs which can be solved by calculation software. Since $Q_1 > 0$, we have $-(A + B_2K)^T Q_1 (A + B_2K) < 0$. Thus, the sufficient condition of $\Phi_0 < 0$ is

$$\Phi_1 = \begin{bmatrix} \Phi_1(1,1) & \Phi_1(1,2) \\ * & -B_1^T Q_1 B_1 - \gamma_1^2 I \end{bmatrix} < 0, \tag{9}$$

with

$$\Phi_1(1,1) = E_\varepsilon^T P_{l\varepsilon}(A+B_2K) + (A+B_2K)^T P_{l\varepsilon} E_\varepsilon + \omega_l^2 E_\varepsilon^T Q_1 E_\varepsilon + C^T C,$$
$$\Phi_1(1,2) = -(A+B_2K)^T Q_1 B_1 + E_\varepsilon^T P_{l\varepsilon} B_1.$$

By using the Shur Complement, $\Phi_1 < 0$ can be rewritten as $\Phi_2 < 0$ where

$$\Phi_2 = \begin{bmatrix} \Phi_2(1,1) & \Phi_2(1,2) & (C_1 + D_{12}K)^T \\ * & -B_1^T Q_1 B_1 - \gamma_1^2 I & 0 \\ * & * & -I \end{bmatrix}, \tag{10}$$

with

$$\Phi_2(1,1) = E_\varepsilon^T P_{l\varepsilon}(A+B_2K) + (A+B_2K)^T P_{l\varepsilon} E_\varepsilon + \omega_l^2 E_\varepsilon^T Q_1 E_\varepsilon,$$
$$\Phi_2(1,2) = -(A+B_2K)^T Q_1 B_1 + E_\varepsilon^T P_{l\varepsilon} B_1.$$

Let us denote $P_{l\varepsilon} = \begin{bmatrix} P_{l11} & P_{l12} \\ P_{l12}^T & \frac{1}{\varepsilon} P_{l22} \end{bmatrix}$, and we have

$$P_{l\varepsilon} E_\varepsilon = \begin{bmatrix} P_{l11} & P_{l12} \\ P_{l12}^T & \frac{1}{\varepsilon} P_{l22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \varepsilon \end{bmatrix} = \begin{bmatrix} P_{l11} & 0 \\ P_{l12}^T & P_{l22} \end{bmatrix} + O(\varepsilon), \tag{11}$$

$$E_\varepsilon^T Q_1 E_\varepsilon = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} Q_1 \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + O(\varepsilon). \tag{12}$$

Substituting (11) and (12) into (10), we rewrite Φ_2 as

$$\Phi_2 = \Phi + O(\varepsilon).$$

According to [10], $\Phi_2 < 0$ can be easily obtained from $\Phi < 0$ for all $\varepsilon \in (0, \varepsilon^*]$, $\varepsilon^* > 0$. This completes the proof. □

Theorem 3.3. *Considering the closed-loop system* (2)*, the high frequency bounded realness*

$$\|T_{wz}(j\omega)\| < \gamma_2, \qquad \forall |\omega| \geq \omega_h \tag{13}$$

holds for $\varepsilon \in (0, \varepsilon^*]$*,* $\varepsilon^* > 0$*, if there exist matrix variables* $X_2 > 0$*,* $P_{h11} \in \mathbf{H}_{n_1}$*,* $P_{h22} \in \mathbf{H}_{n_2}$*,* P_{h21}*,* K *and constant* $\alpha > 0$ *such that the following BMIs hold*

$$\Psi = \begin{bmatrix} \Psi(1,1) & P_h^T B_1 & \Psi(1,3) & \Psi(1,4) \\ * & -\gamma_2^2 I & B_1^T & 0 \\ * & * & -X_2 & 0 \\ * & * & * & -I \end{bmatrix} < 0, \tag{14}$$

where

$$\Psi(1,1) = P_h^T(A+B_2K) + (A+B_2K)^T P_h + \omega_h^2 \alpha^2 X_2 - \omega_h^2 \alpha H - \omega_h^2 \alpha H^T,$$
$$\Psi(1,3) = (A+B_2K)^T, \qquad \Psi(1,4) = (C_1 + D_{12}K)^T,$$
$$P_h = \begin{bmatrix} P_{h11} & 0 \\ P_{h12}^T & P_{h22} \end{bmatrix}, \qquad H = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

Proof: According to [11], high frequency bounded realness (13) is derived from

$$\begin{bmatrix} A+B_2K & B_1 \\ E_\varepsilon & 0 \end{bmatrix}^T \begin{bmatrix} Q_2 & P_{h\varepsilon} \\ P_{h\varepsilon} & -\omega_h^2 Q_2 \end{bmatrix} \begin{bmatrix} A+B_2K & B_1 \\ E_\varepsilon & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & -\gamma_2^2 I \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} < 0, \tag{15}$$

where $Q_2 > 0$. The following is then obtained

$$\Psi_1 = \begin{bmatrix} \Psi_1(1,1) & \Psi_1(1,2) \\ * & B_1^T Q_2 B_1 - \gamma_2^2 I \end{bmatrix} < 0, \tag{16}$$

with

$$\begin{aligned}\Psi_1(1,1) &= (A+B_2K)^TQ_2(A+B_2K) + E_\varepsilon^TP_{h\varepsilon}(A+B_2K) + (A+B_2K)^TP_{h\varepsilon}E_\varepsilon, \\ &\quad -\omega_h^2E_\varepsilon^TQ_2E_\varepsilon + C^TC, \\ \Psi_1(1,2) &= (A+B_2K)^TQ_2B_1 + E_\varepsilon^TP_{h\varepsilon}B_1.\end{aligned}$$

Note that $\Psi_1 < 0$ is the nonlinear matrix inequality and difficult to solve out, and we will transform it into its special case BMIs. By using the Shur Complement, $\Psi_1 < 0$ is rewritten as $\Psi_2 < 0$, where

$$\Psi_2 = \begin{bmatrix} \Psi_2(1,1) & E_\varepsilon^TP_{h\varepsilon}B_1 & \Psi_2(1,3) & \Psi_2(1,4) \\ * & -\gamma_2^2I & B_1^T & 0 \\ * & * & -Q_2^{-1} & 0 \\ * & * & * & -I \end{bmatrix}, \tag{17}$$

with

$$\begin{aligned}&\Psi_2(1,1) = E_\varepsilon^TP_{h\varepsilon}(A+B_2K) + (A+B_2K)^TP_{h\varepsilon}E_\varepsilon - \omega_h^2E_\varepsilon^TQ_2E_\varepsilon, \\ &\Psi_2(1,3) = (A+B_2K)^T, \quad \Psi_2(1,4) = (C_1+D_{12}K)^T.\end{aligned}$$

Let us denote $P_{h\varepsilon} = \begin{bmatrix} P_{h11} & P_{h12} \\ P_{h12}^T & \frac{1}{\varepsilon}P_{h22} \end{bmatrix}$, and we have

$$P_{h\varepsilon}E_\varepsilon = \begin{bmatrix} P_{h11} & P_{h12} \\ P_{h12}^T & \frac{1}{\varepsilon}P_{h22} \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & \varepsilon \end{bmatrix} = \begin{bmatrix} P_{h11} & 0 \\ P_{h12}^T & P_{h22} \end{bmatrix} + O(\varepsilon), \tag{18}$$

$$E_\varepsilon^TQ_2E_\varepsilon = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}Q_2\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + O(\varepsilon). \tag{19}$$

Substituting (18) and (19) into (17), we have

$$\Psi_2 = \Psi_3 + O(\varepsilon), \tag{20}$$

where

$$\Psi_3 = \begin{bmatrix} \Psi_3(1,1) & P_h^TB_1 & \Psi_3(1,3) & \Psi_3(1,4) \\ * & -\gamma_2^2I & B_1^T & 0 \\ * & * & -Q_2^{-1} & 0 \\ * & * & * & -I \end{bmatrix}, \tag{21}$$

with

$$\begin{aligned}&\Psi_3(1,1) = P_h^T(A+B_2K) + (A+B_2K)^TP_h - \omega_h^2H^TQ_2H, \\ &\Psi_3(1,3) = (A+B_2K)^T, \quad \Psi_3(1,4) = (C_1+D_{12}K)^T.\end{aligned}$$

According to [10], there exists an $\varepsilon^* > 0$ such that $\Phi_2 < 0$ can be derived from $\Phi_3 < 0$ for all $\varepsilon \in (0, \varepsilon^*)$. The following inequality holds

$$-\omega_h^2H^TQ_2H < \omega_h^2\alpha^2Q_2^{-1} - \omega_h^2\alpha H - \omega_h^2\alpha H^T. \tag{22}$$

Substituting the right hand of (22) into (21), we have

$$\Psi_4 = \begin{bmatrix} \Psi_4(1,1) & P_h^TB_1 & \Psi_4(1,3) & \Psi_4(1,4) \\ * & -\gamma_2^2I & B_1^T & 0 \\ * & * & -X_2 & 0 \\ * & * & * & -I \end{bmatrix} < 0, \tag{23}$$

with

$$\begin{aligned}&\Psi_4(1,1) = P_h^T(A+B_2K) + (A+B_2K)^TP_h + \omega_h^2\alpha^2X_2 - \omega_h^2\alpha H - \omega_h^2\alpha H^T, \\ &\Psi_4(1,3) = (A+B_2K)^T, \quad \Psi_4(1,4) = (C_1+D_{12}K)^T.\end{aligned}$$

Denote Q_2^{-1} as X_2 and the proof is completed. □

Let us denote $\gamma = \gamma_1 = \gamma_2$. Theorems 3.1 to 3.3 actually suggest that, if the following optimization problem

$$\gamma^* = \min_{\gamma, P_s, P_l, P_h, Q_1, X_2, K} \gamma, \text{ subject to } (4), (6), (14) \tag{24}$$

has a solution set $(\gamma, P_s, P_l, P_h, Q_1, X_2, K)$, then the feedback control law is the optimal which ensures the minimization of γ in the given frequency range characterized by ω_l and ω_h.

3.2. **Time domain interpretation.** To characterize the FF bounded realness more intuitively, the following content presents the time domain characterization in terms of input/output signals.

Theorem 3.4. *Considering the transfer function $T_{wz}(j\omega)$ in (3), if $T_{wz}(j\omega)$ is asymptotically stable and satisfies bounded realness in the low frequency range, we have*

$$\int_0^\infty z^T(t)z(t)dt < \gamma_1^2 \int_0^\infty w^T(t)(t)w(t)dt + x_1^T(0)P_{11}x_1(0) + O(\varepsilon) \tag{25}$$

holds for all square integrable inputs $u(t)$ such that

$$\int_0^\infty \dot{x}_1^T(t)\dot{x}_1(t)dt < \omega_l^2 \int_0^\infty x_1^T(t)x_1(t)dt. \tag{26}$$

Proof: Left and right multiplying (7) with $\begin{bmatrix} x^T(t) & w^T(t) \end{bmatrix}$ and its transpose, we have

$$\begin{aligned} &-(E_\varepsilon \dot{x}(t))^T Q_1 E_\varepsilon \dot{x}(t) + \omega_l^2 (E_\varepsilon x(t))^T Q_1 E_\varepsilon x(t) + \frac{d}{dt}(x^T(t)E_\varepsilon^T P_{l\varepsilon} E_\varepsilon x(t)) \\ &+ z^T(t)z(t) - \gamma_1^2 w^T(t)w(t) < 0. \end{aligned} \tag{27}$$

Since the following equations hold

$$E_\varepsilon^T Q_1 E_\varepsilon = \begin{bmatrix} Q_{111} & \varepsilon Q_{112} \\ \varepsilon Q_{112}^T & \varepsilon^2 Q_{122} \end{bmatrix}, \quad E_\varepsilon^T P_{l\varepsilon} E_\varepsilon = \begin{bmatrix} P_{l11} & \varepsilon P_{l12} \\ \varepsilon P_{l12}^T & \varepsilon P_{l22}^T \end{bmatrix},$$

we have

$$\begin{bmatrix} \dot{x}_1^T(t) & \dot{x}_2^T(t) \end{bmatrix} \begin{bmatrix} Q_{111} & \varepsilon Q_{112} \\ \varepsilon Q_{112}^T & \varepsilon^2 Q_{122} \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \dot{x}_1^T(t) Q_{111} \dot{x}_1(t) + O(\varepsilon), \tag{28}$$

$$\begin{bmatrix} x_1^T(t) & x_2^T(t) \end{bmatrix} \begin{bmatrix} Q_{111} & \varepsilon Q_{112} \\ \varepsilon Q_{112}^T & \varepsilon^2 Q_{122} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = x_1^T(t) Q_{111} x_1(t) + O(\varepsilon), \tag{29}$$

$$\begin{bmatrix} x_1^T(t) & x_2^T(t) \end{bmatrix} \begin{bmatrix} P_{l11} & \varepsilon P_{l12} \\ \varepsilon P_{l12}^T & \varepsilon P_{l22}^T \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = x_1^T(t) P_{l11} x_1(t) + O(\varepsilon). \tag{30}$$

Substituting (28)-(30) into (27), we have

$$\begin{aligned} &-\dot{x}_{11}^T(t) Q_{111} \dot{x}_{11}(t) + \omega_l^2 x_{11}^T(t) Q_{111} x_{11}(t) + \frac{d}{dt}\left(x_{11}^T(t) P_{l11} x_{11}(t)\right) \\ &+ z^T(t)z(t) - \gamma_1^2 w^T(t)w(t) + O(\varepsilon) < 0. \end{aligned} \tag{31}$$

Integrating (31) from $t = 0$ to $t = \infty$ and using the stability property

$$\begin{aligned} &\operatorname{tr}\left\{ Q_{111} \int_0^\infty \left[-\dot{x}_{11}^T(t)\dot{x}_{11}(t) + \omega_l^2 x_{11}^T(t) x_{11}(t) \right] dt \right\} \\ &< -\int_0^\infty z^T(t)z(t)dt + \gamma_1^2 \int_0^\infty w^T(t)w(t)dt + x_{11}^T(0) P_{l11} x_{11}(0) + O(\varepsilon). \end{aligned} \tag{32}$$

This completes the proof. □

4. **Numerical Example.** Consider the dynamic model in [2].

$$\begin{aligned} &\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t) + B_{11}w(t) + B_{21}u(t), \\ &\varepsilon\dot{x}_2(t) = A_{21}x_1(t) + A_{22}x_2(t) + B_{12}w(t) + B_{22}u(t), \\ &z(t) = C_{11}x_1(t) + C_{12}x_2(t) + D_{12}u(t), \\ &z^T(t) = \begin{bmatrix} z_1(t) & z_2(t) & z_3(t) & z_4(t) & z_5(t) \end{bmatrix}. \end{aligned} \tag{33}$$

The system model is the nonstandard SPM since the matrix A_{22} is singular. We assign $\omega_{h1} = 5$ and $\omega_{h2} = 5$, $\omega_{l2} = 3$, respectively. The constant α is chosen to be 1. The control results are presented in Table 1.

Table 1. The control results by using Theorems 3.2 and 3.3

Frequency range	γ^*	Feedback gain
$\omega_{h1} = 5$	1.6870	$K_a = 10^3 \times \begin{bmatrix} -0.0663 & -6.0331 & -0.0117 & -0.0035 \end{bmatrix}$
$\omega_{l2} = 3$, $\omega_{h2} = 5$	7.1530	$K_b = 10^3 \times \begin{bmatrix} -2.0642 & -1.0863 & -0.0004 & -0.0001 \end{bmatrix}$

From Table 1, we can see that γ^* is smaller if FF bounded realness covers a more narrow frequency range. Thus, if more prior knowledge of the frequency band of $w(t)$ is available, we can get better noise attenuation level γ^*. Then, we introduce the mean square error (MSE) to describe the combined result of all outputs.

$$\mathrm{MSE}(t) = \frac{\sum_{i=1}^{5} z_i^2(t)}{5} \tag{34}$$

We set the initial values $x(0) = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$ and the perturbation parameter $\varepsilon = 0.005$. The external disturbance is $w(t) = 0.01\sin(6.5t)$. Then, we apply K_a and compare the control result with that of [2]. The resultant curves of MSE(t) are shown in Figure 1. We conclude from Figure 1 that the control method proposed in this paper is superior in the sense of MSE(t) within the given frequency ranges. In order to seek the impact of the small perturbation parameter ϵ, we reassign the initial values $x(0) = \begin{bmatrix} 0 & 0 & 10 & 0 \end{bmatrix}^T$ and apply K_a to system (33). The resultant curves are shown in Figure 2. From Figure 2, we can see that more satisfactory control result can be obtained if the perturbation parameter ε is smaller, which is the character of SPM.

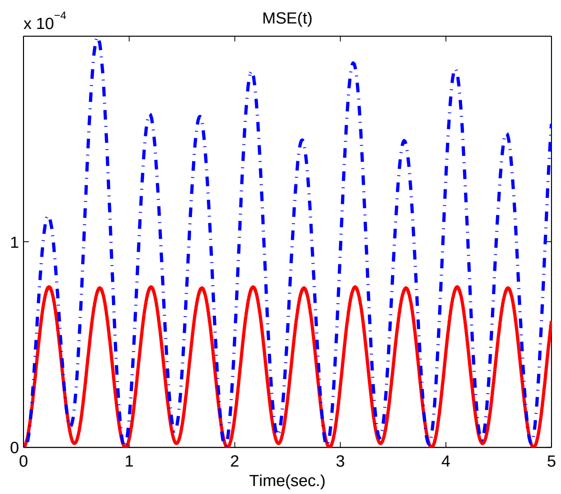

Figure 1. The curves of MSE(t) comparison between FF robust criterion with solid line and robust criterion in full frequency with dash line

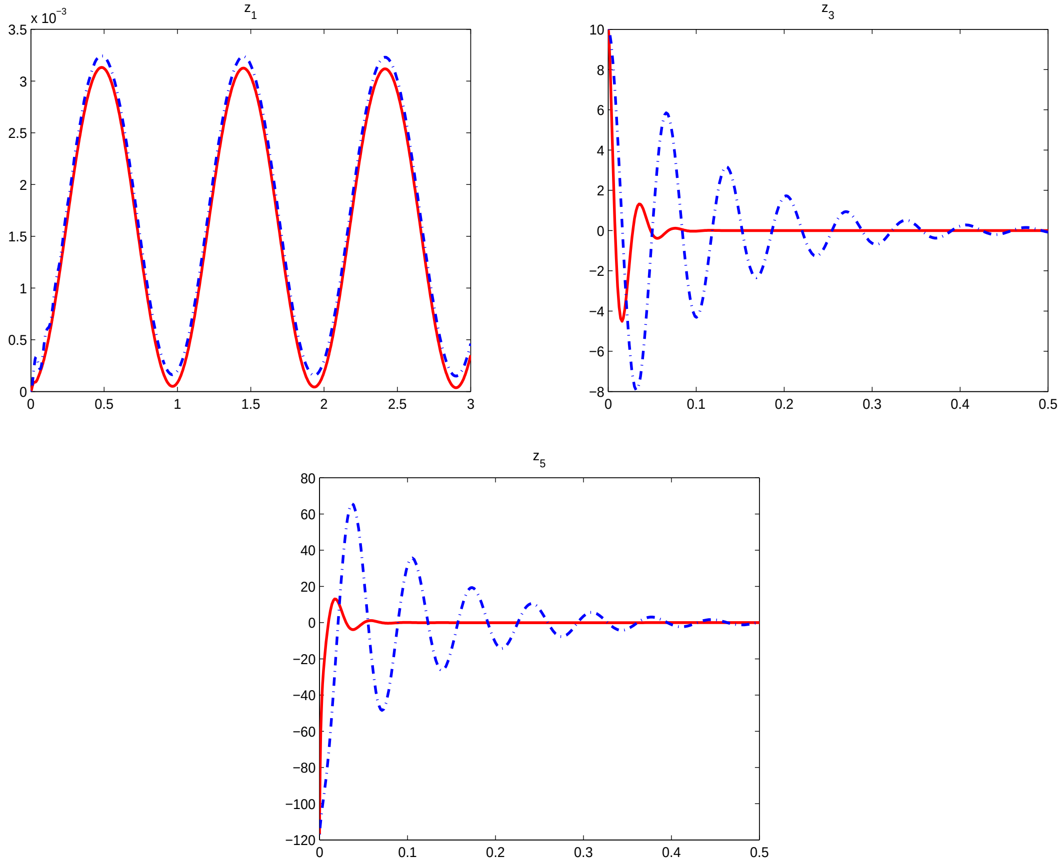

FIGURE 2. Output responses comparison between $\varepsilon = 0.005$ with solid line and $\varepsilon = 0.015$ with dash line

5. **Conclusion.** This paper has investigated the FF bounded realness for nonstandard SPM. The perturbation parameter independent condition for the FF bounded realness has been derived using the GKYP lemma. To characterize the system more intuitively, time domain interpretations are presented. One advantage is that the proposed method, which can be applied to both standard and nonstandard SPM, achieves better H_∞ performance if the frequency range of the noise is known in advance; another advantage is that the BMI conditions are all independent of the small perturbation parameter, avoiding ill-conditioned matrix during calculation. Simulation results have demonstrated the effectiveness and advantage of the proposed method.

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