ICIC Express Letters ICIC International ©2017 ISSN 1881-803X
Volume **11**, Number **1**, January **2017** pp. **169–174**

CONTROLLABILITY OF THE DIRECTED TREE GRAPH CONSISTING OF NON-IDENTICAL NODES UNDER THE BROADCASTING CONTROL SIGNAL

Chuanliang Cheng and Zhijian Ji

College of Automation and Electrical Engineering
Qingdao University
No. 308, Ningxia Road, Qingdao 266071, P. R. China
chengchuanliangok@163.com

Received June 2016; accepted September 2016

Abstract. *Controllability is a key issue in the study of multi-agent systems. The controllability of a multi-agent system with non-identical nodes which is closer to the actual engineering network is paramount. This paper mainly studies the controllability of the directed tree consisting of non-identical nodes under the broadcasting control signal based on the graph theory and the matrix theory. Some necessary and sufficient conditions are summarized for the controllability of the non-identical node dynamics. The controllability of star graph is also studied under the non-identical node dynamics. It is shown that the controllability of the star graph is changed when the non-identical nodes receive the broadcasting control signal. The results in this paper show that the weight and the non-identical agents determine the controllability of the multi-agent systems. Some methods for improving the controllability of the directed tree graph consisting of non-identical nodes are presented.*

Keywords: Multi-agent systems, Broadcasting control signals, A directed tree, Star graph

1. **Introduction.** The technology of multi-agent systems has been a hot topic in recent years. It transmutes the system which is large and complex into small and simple systems through interactive communication. Besides, it has some features such as coordination, autonomy, and self-organization. Therefore, it is widely applied in many fields [1-3], such as formation control of robot, intelligent transportation, and marine technology [4-9].

In 2004, the definition of controllability was first put forward by Tanner for the simple interconnected system model [10]. He raised the classical controllability of a single leader and derived the necessary and sufficient conditions of one-integrator dynamics through controlling the behaviors of the leader. Rahmani et al. [11] proposed the transport control protocol (TCP) and obtained the algebra and graph theory conditions of the controllability of multi-agent systems. Liu et al. [12] developed the controllability of the discrete system with switching topology, and proved that the controllability of multi-agent systems is determined by the information which is transmitted between leaders and followers. With the development of the multi-agent system under leader-follower framework, the study of the system under new framework aroused more and more attention, many people begin to research the controllability of the system under the broadcasting signal framework, and some available results were obtained in this respect [13,14].

Compared with the leader-follower framework, the broadcasting signal framework has the following advantages. 1) It has been widely used in real life, such as television stations. 2) It does not need to provide the communication equipment which is used for leaders and followers. Most networks in practical engineering has non-identical node dynamics. Therefore, the research of the controllability of multi-agent systems under non-identical node dynamics has fundamental significance [15,16].

The contributions of this paper are as follows. The classical concept of controllability from control theory is extended to directed tree topologies with the non-identical node dynamics when all agents receive a broadcasting control signal. Some necessary and sufficient conditions are given in algebraic form. Furthermore, some approaches to improving the controllability are presented.

The paper is organized as follows. Section 2 is a brief review of the graph theory used in the paper. Section 3 follows with an introduction of the non-identical node dynamics system and the necessary and sufficient condition. Our main result is presented in Sections 4 and 5, in which simulation results are included to verify the analytical derivations. We conclude this paper in Section 6.

2. **Problem Statement and Preliminaries.** A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a vertex set $\mathcal{V} = [v_1, v_2, \ldots, v_n]$, and an edge set $\mathcal{E} = [(v_i, v_j) : v_i, v_j \in \mathcal{V}]$, where an edge is an ordered pair of distinct vertices of $\mathcal{V}$. $A = [a_{ij}]$ is the adjacency matrix, if $a_{ij} \in \mathcal{E}$ then $a_{ij} > 0$; otherwise, $a_{ij} = 0$. If $\{j, i\} \in \mathcal{E}$, then we say v_i and v_j are adjacent. The neighborhood set of vertex v_i is denoted by $\mathcal{N}_i = \{j \in v : (j, i) \in \mathcal{E}\}$. The degree of v is $d_v := [\mathcal{N}_v]$. A path in $\mathcal{G}$ of length k is a subgraph of $\mathcal{G}$ consisting of vertices $\{v_0, v_1, \ldots, v_k\} \subset \mathcal{V}$ and edges $\{\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{k-1}, v_k\}\} \subset \mathcal{E}$, where all v_i are distinct. If there is a path between any two vertices of $\mathcal{G}$, we say $\mathcal{G}$ is connected. A graph $\mathcal{G}$ is called undirected graph if $\{j, i\} \in \mathcal{E} \Leftrightarrow \{i, j\} \in \mathcal{E}$ for any vertexes v_i and v_j; otherwise, we say the graph is directed graph. We denote by D the degree matrix of $\mathcal{G}$, the diagonal matrix whose ith diagonal entry is d_i. The Laplacian of $\mathcal{G}$ is defined as

$$L = D - A$$

A digraph is called directed tree if only one vertex is called root which has no any parent, and other vertexes are divided into the mutually disjoint sets.

3. **The Controllability of the Non-Identical Node Dynamics System.** Consider a non-identical node dynamics system:

$$\dot{x}_i = c_i F x_i + \sum_{j=N_i} L_{ij} F (x_i - x_j) + b u_i, \quad i = 1, 2, \ldots, n \tag{1}$$

where $x_i \in \mathbb{R}^m$ is the state vector of agent i, and $u_i \in \mathbb{R}^p$ is the external input or signal. If all agents receive the same control signal, then we say the signal u_i is the broadcasting control signal. $b \in \mathbb{R}^{m\times 1}$ is the control input matrix. F is a constant matrix describing the inner coupling between different agents, the matrix $c_i F x_i$ describing the intrinsic dynamics of node i. Let $x = \left[x_1^T, x_2^T, \ldots, x_n^T\right]^T \in \mathbb{R}^{m\times n}$, and then system (1) can be rewritten in a matrix form as

$$\dot{x} = [(C - L) \otimes F]\, x + (I_n \otimes b)\, u \tag{2}$$

where $C = diag(c_1, c_2, \ldots, c_n)$. As we all know, the Laplacian matrix L and the constant matrix F of directed tree topologies are lower triangular matrixes.

Theorem 3.1. *System (1) containing a directed tree is controllable if and only if the following conditions are satisfied:*

(1) $\begin{bmatrix} F & b \end{bmatrix}$ *is a controllable matrix pair;*

(2) The diagonal entries on $(C - L) \otimes F$ *are all nonzero and distinct.*

Proof: The controllability matrix of system (1) is

$$\begin{bmatrix} (I_n \otimes b) & [(C - L) \otimes F] (I_n \otimes b) & \cdots & [(C - L) \otimes F]^{mn-1} (I_n \otimes b) \end{bmatrix}$$

Because C and L are both lower triangular matrixes, $[(C - L) \otimes F]$ is also a lower triangular matrix, then $[(C - L) \otimes F]$ has the form as $P \Lambda P^T$, where P is a similitude transformation matrix, Λ is a diagonal matrix, and its diagonal entries are equal to the diagonal entries on $[(C-L)\otimes F]$. Then the controllability matrix can be written as

$$\left[\begin{array}{cccc} (I_n\otimes b) & P\Lambda P^T(I_n\otimes b) & \cdots & \left(P\Lambda P^T\right)^{mn-1}(I_n\otimes b) \end{array}\right]$$

Extracting the similitude transformation matrix P, the controllability matrix can be written as

$$P\left[\begin{array}{cccc} P^T(I_n\otimes b) & \Lambda P^T(I_n\otimes b) & \cdots & \Lambda^{mn-1}P^T(I_n\otimes b) \end{array}\right]$$

Because the matrix P is full rank, P does not affect the rank of the controllability matrix. We can just study the following matrix

$$\left[\begin{array}{cccc} P^T(I_n\otimes b) & \Lambda P^T(I_n\otimes b) & \cdots & \Lambda^{mn-1}P^T(I_n\otimes b) \end{array}\right]$$

If the matrix is full rank, then the column vectors of P are not orthogonal to any column vector of $I_n\otimes b$; in other words, the eigenvectors $(C-L)\otimes F$ are not orthogonal to $I_n\otimes b$, if the diagonal entries on $[(C-L)\otimes F]$ are all nonzero, and let v_1 and v_2 be non-zero eigenvectors of $C-L$ and F respectively. So the eigenvectors of $(C-L)\otimes F$ can be written as $v_1\otimes v_2$. Then we say that the eigenvectors of $(C-L)\otimes F$ which are not orthogonal to $I_n\otimes b$ can translate into $\left(v_1^TI_n\right)\otimes\left(v_2^Tb\right)\neq 0^T$.

From above, system (1) is controllable if and only if the following conditions are true:

(1) $v_1^TI_n\neq 0^T$, it is meaning that the eigenvectors of $C-L$ are not orthogonal to any I_n; in other words, the eigenvectors of $C-L$ are non-zero. Since the eigenvectors of $C-L$ are non-zero, we do not need to consider this condition;

(2) $v_2^Tb\neq 0^T$, that is, $\left[\begin{array}{cc} F & b\end{array}\right]$ is a controllable matrix pair.

Λ is a diagonal matrix, when Λ is multiplied by a matrix, the entries on the matrix can only be zoomed at the same rate. When the entries on Λ are the same, the controllability matrix will show rows which are linearly dependent, and the system is uncontrollable. Because Λ is diagonal entries on $(C-L)\otimes F$, the diagonal entries on $(C-L)\otimes F$ must be distinct.

From above, the controllability of the directed tree topology under the broadcasting control signal can translate into the controllability of its subsystem $\dot{x}=Fx+bu$.

4. **The Controllability of the Subsystem.** In this section, we mainly concern with the controllability of the subsystem

$$\dot{x}=Fx+bu \tag{3}$$

Because $F^{m\times m}$ is a constant matrix describing the inner coupling between different agents, F can be decomposed as

$$F=\left[\begin{array}{cc} f_{11} & 0_{1\times(m-1)} \\ F_{(m-1)\times 1} & F_{e_{(m-1)\times 1}} \end{array}\right]\in\mathbb{R}^{m\times m}$$

where $F_{(m-1)\times 1}=[f_{21},f_{31},\ldots,f_{m1}]$, $F_e=\left[\begin{array}{ccc} f_{22} & & 0 \\ \vdots & \ddots & \\ f_{m2} & \cdots & f_{mm}\end{array}\right]$.

Theorem 4.1. *(PBH rank test) System (3) is controllable if and only if it satisfies one of the following conditions:*

(1) $rank\,(sI-F_e,F_1)=m-1$, $\forall s\in\mathbb{C}$;

(2) $rank\,(\lambda_iI-F_e,F_1)=m-1$, $\forall i=1,2,\ldots,m-1$, λ_i *is the eigenvalue of matrix* F_e.

Proof: (1) Consider the system $\dot{x}=Fx+bu$, by using the PBH rank test

$$rank\,(sI-F,b)=rank\left(\begin{array}{ccc} s-f_{11} & 0 & 1 \\ F_{(m-1)\times 1} & sI-F_e & 1_{(m-1)\times 1}\end{array}\right)=m$$

which means that $rank\left(\begin{array}{cc} F_{(m-1)\times 1} & sI-F_e\end{array}\right)=m-1$.

(2) The proof in the same rein as (1) is omitted.

Theorem 4.2. *System (3) is controllable if and only if the following conditions are satisfied:*
(1) The diagonal entries on F are all nonzero and distinct;
(2) The eigenvectors of F are not orthogonal to b.

Proof: The controllability matrix of system (3) is $\begin{bmatrix} b & Fb & \cdots & F^{m-1}b \end{bmatrix}$. Because F is a lower triangular matrix, we have $F = UDU^T$, D is a diagonal matrix, and U is a similitude transformation matrix. The controllability matrix can be written as

$$\begin{bmatrix} b & UDU^Tb & \cdots & \left(UDU^T\right)^{m-1} b \end{bmatrix}$$

Extracting the similitude transformation matrix U, the controllability matrix can be written as

$$U \begin{bmatrix} U^Tb & DU^Tb & \cdots & D^{m-1}U^Tb \end{bmatrix}$$

U is full rank, so U does not affect the rank of the controllability matrix. We can just study the following matrix

$$\begin{bmatrix} U^Tb & DU^Tb & \cdots & D^{m-1}U^Tb \end{bmatrix}$$

If the matrix is full rank, the column vectors of U are not orthogonal to b. Because the column vectors of U are the eigenvectors of F, the eigenvectors of F are not orthogonal to b. Because the entries on D are the entries on the main diagonal of F, the diagonal entries on F must be distinct.

5. **The Controllability of Star Graph.** As we know, the directed star graph is a representative type of the directed tree graph. So we can concern with the controllability of star graph to understand the controllability more comprehensively.

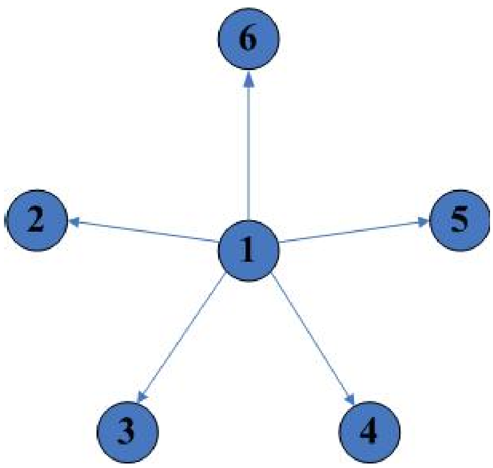

FIGURE 1. Star graph

In Figure 1, suppose that

$$F = \begin{bmatrix} 13 & & & & & \\ 3 & 16 & & & & \\ 1 & & 8 & & & \\ 7 & & & 5 & & \\ 6 & & & & 17 & \\ 3 & & & & & 9 \end{bmatrix}, L = \begin{bmatrix} 0 & & & & & \\ 5 & -5 & & & & \\ 4 & & -4 & & & \\ 3 & & & -3 & & \\ 2 & & & & -2 & \\ 1 & & & & & -1 \end{bmatrix}, C = \begin{bmatrix} -7 & & & & & \\ & -6 & & & & \\ & & -3 & & & \\ & & & -5 & & \\ & & & & -2 & \\ & & & & & -10 \end{bmatrix}$$

The Laplacian matrix of the star graph consisting of non-identical agents is

$$C - L = \begin{bmatrix} -7 & & & & & \\ -5 & -1 & & & & \\ -4 & & 1 & & & \\ -3 & & & -2 & & \\ -2 & & & & 0 & \\ -1 & & & & & -9 \end{bmatrix}$$

The eigenvalues of $C - L$ are -9, -7, -2, -1, 1, so there are more than two eigenvalues of $(C - L) \otimes F$ equal to 0, and the system is uncontrollable.

Let $C = \begin{bmatrix} -7 & & & & & \\ & -6 & & & & \\ & & -3 & & & \\ & & & -5 & & \\ & & & & -5 & \\ & & & & & -10 \end{bmatrix}$, then $C - L = \begin{bmatrix} -7 & & & & & \\ -5 & -1 & & & & \\ -4 & & 1 & & & \\ -3 & & & -2 & & \\ -2 & & & & -3 & \\ -1 & & & & & -9 \end{bmatrix}$. The eigenvalues of $C - L$ are -9, -7, -3, -2, -1, 1. By calculation, there are 36 eigenvalues of $(C - L) \otimes F$ that are different, and the system is controllable.

Following Theorem 3.1, the controllability of the non-identical node dynamics system is determined by $(C - L) \otimes F$ and $\begin{bmatrix} F & b \end{bmatrix}$. F describes the coupling weight between agents. C represents the coupling strength, and L represents the topological structure. The methods for improving the controllability of the directed tree graph consisting of non-identical nodes are as follows:

1) we can make the subsystem $\dot{x} = Fx + bu$ controllable by selecting the appropriate coupling weight matrix;

2) we can change the coupling strength, that is the agent parameter c_i;

3) we can change the coupling strength and the topological structure. For example, we can make the uncontrollable topological structure controllable by adding or removing connection between agents.

6. **Conclusions.**

In this paper, the controllability of the directed tree consisting of non-identical nodes under the broadcasting control signal has been investigated. The controllability of the system is converted to two subproblems. The first subproblem is to analyze the eigenvalues of $(C - L) \otimes F$. The second subproblem is to examine the controllability of the subsystem $\dot{x} = Fx + bu$. The controllability of star graph is studied, and results show that, the controllability of star graph with non-identical node dynamics can be changed. So the non-identical node dynamics system is more complex.

The main difference between our work and the previous works lies in the introduction of directed tree, and the results obtained here are merely a small portion in the study on controllability of the multi-agent systems with non-identical nodes. Future research along the same line might include the cases of directed, uncertain, and time-varying topology.

Acknowledgments. This work was partially supported by the National Natural Science Foundation of China under Grant 61374062, and the Science Foundation of Shandong Province for Distinguished Young Scholars (No. JQ201419). The authors also gratefully acknowledge the helpful comments and suggestions of the reviewers, which have improved the presentation.

REFERENCES

[1] A. Jadbabaie, J. Lin and A. S. Morse, Coordination of groups of mobile autonomous agents using nearest neighbor rule, *IEEE Trans. Automatic Control*, vol.48, no.6, pp.988-1001, 2002.

[2] J. A. Fax, *Optimal and Cooperative Control of Vehicle Formations*, Ph.D. Thesis, Pasadena, California Institute of Technology, 2002.

[3] R. Olfati-Saber and R. M. Murray, Consensus problems in network of agents with switching topology and time-delays, *IEEE Trans. Automatic Control*, vol.49, no.9, pp.1520-1533, 2004.

[4] J. A. Fax and R. M. Murray, Information flow and cooperative control of vehicle formations, *IEEE Trans. Automatic Control*, vol.49, no.9, pp.1465-1476, 2004.

[5] P. Ogren, E. FiorelliI and N. E. Leonard, Cooperative control of mobile sensor networks: Adaptive gradient climbing in a distributed environment, *IEEE Trans. Automatic Control*, vol.49, no.8, pp.1292-1302, 2004.

[6] S. Mu, T. Chu and L. Wang, Coordinated collective motion in a mobile particle group with a leader, *Physical A: Statistical Mechanics and Its Applications*, vol.351, no.24, pp.211-226, 1963.

[7] Y. Hong, J. Hu and L. Gao, Tacking control for multi-agent consensus with an active leader and variable topology, *Automatic*, vol.42, no.7, pp.1177-1182, 2006.

[8] H. Shi, L. Wang and T. Chu, Virtual leader approach to coordinated control of multiple agents with asymmetric interactions, *Physical D: Nonlinear Phenomena*, vol.213, no.1, pp.51-65, 2006.

[9] D. M. Lane and A. G. Mcfadzean, Distributed problem solving and real-time mechanisms in robot architectures, *Engineering Applications of Artificial Intelligence*, pp.105-117, 1994.

[10] H. G. Tanner, On the controllability of nearest neighbor interconnections, *Proc. of the 43rd IEEE Conference on Decision and Control*, Atlantis, Paradise Island, Bahamas, pp.2467-2472, 2004.

[11] A. Rahmani, M. Ji and M. Mesbahi, On the controlled agreement problem, *Proc. of the 2006 American Control Conference*, Minneapolis, USA, vol.26, no.8, pp.1376-1381, 2006.

[12] B. Liu, H. Su, R. Li et al., Switching controllability of discrete-time multi-agent systems with multiple leaders and time-delays, *Applied Mathematics and Computation*, vol.22, no.11, pp.571-588, 2014.

[13] S. Azuma, R. Yoshimura and T. Sugie, Broadcast control of multi-agent systems, *Automatia*, vol.49, no.8, pp.2307-2316, 2013.

[14] M. G. Yoon, P. Rowlinson, D. Cvetkovic et al., Controllability of multiagent dynamical systems with a broadcasting control signal, *Asian Journal of Control*, vol.16, no.4, pp.1-7, 2014.

[15] L. Xiang, J. Zhu, F. Chen et al., *Controllability of Weighted and Directed Networks with Nonidentical Node Dynamics*, http://www.hindawi.com/journals/mpe/2013/405034/.

[16] N. J. Cowan, E. J. Chastain, D. A. Vilhena et al., Node dynamics, not degree distributions, determine the structural controllability of complex networks, *PLOS ONE*, vol.7, no.6, 2012.