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FUZZY SOLUTION OF FUZZY LINEAR SYSTEM

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Abstract. *In this paper, the problem of solution of fuzzy number linear system is studied. The concepts of r-solution and r-interval value solution are given through turning linear systems (equations) with fuzzy number coefficients into equivalent linear systems (equations) with interval number coefficients, and fuzzy number value solution is defined for such fuzzy number linear system. Then we obtain a method of solving trapezoidal fuzzy number solution for linear systems (equations) with trapezoidal fuzzy number coefficients, and give two examples to show the application of the method.*

Keywords: Fuzzy number, Trapezoid fuzzy number, Fuzzy linear equations, Fuzzy solution

1. **Introduction.** The concept of fuzzy set was first put forward by Zadeh [1] in 1965. In 1972, Chang and Zadeh [2] proposed the concept of fuzzy numbers to study the properties of probability functions. With the development of theories and applications of fuzzy numbers, it becomes more and more important.

On the other hand, linear systems and related problems are also an important area of research. In conventional mathematical equation, coefficients of problems are usually determined by the experts as crisp values. However, in reality, in an imprecise and uncertain environment, it is an unrealistic assumption that the knowledge and representation of an expert are so precise. Therefore, fuzzy and stochastic approaches are frequently used to describe and treat imprecise and uncertain elements present in a real decision problem. For example, in 1991, Buckley and Qu [3] discussed fuzzy matrix A is square and always non-singular in fuzzy matrix equation and presented six new solutions and showed that the solutions of fuzzy matrix equation $Ax = b$ is a fuzzy vector; In 1998, Friedman et al. [4] developed mathematical models and numerical procedures that would appropriately treat general fuzzy linear systems and solve them; In 2004, Allahviranloo [5] proposed using the iterative Jacobi and Gauss Sidel methods for solving fuzzy systems of linear equations with convergence theorems; In 2013, Chakraverty and Behera [6] studied the fuzzy system of linear equations which are solved in terms of fuzzy center and get the solution of the original system with width; In 2014, Behera and Chakraverty [7] studied fuzzy complex system of linear equations and handled it in a straight way; Chutia [8] discussed the basic arithmetic operations of the interval-valued trapezoidal fuzzy numbers and the necessary and sufficient condition for existence of solutions of linear equation; Allahviranloo et al. [9] proposed a method to find a trapezoidal solution for a fully fuzzy linear system and consider 1-cut and 0-cut of a fully fuzzy linear system which are two fully interval linear systems. In the above studies, it is found that the coefficient matrix A is a crisp matrix and the unsolved vector x, the right hand b are fuzzy vector in [5, 6, 7]. While the coefficient matrix A is a fuzzy matrix and the unsolved vector x, the right hand b are fuzzy vector in [4, 8, 9]. And most of the authors have considered the fuzzy number as trapezoid fuzzy number in their investigation. Here a new method is proposed and the same is used to solve fuzzy linear system with trapezoid fuzzy number.

As discussed above, the unsolved vector x is considered as fuzzy while it may be a real vector in the equation or in an imprecise and uncertain environment. So we use a new method distinguished from the above method to solve the fuzzy linear system of equations.

Due to the complexity of structure of solutions of linear systems (equations) with fuzzy number coefficients, there are still few good results to solve such a linear system. In this paper, we discuss the solution problem of the fuzzy system of linear equations $Ax = b$, where, the elements of coefficient matrix A and the right hand vector b are all fuzzy numbers. The specific arrangements of this paper are as follows: In Section 2, some basic notions, definitions and results about fuzzy numbers are presented; In Section 3, linear systems (equations) with fuzzy number coefficients are successfully translated into equivalent linear systems (equations) with interval number coefficients. Then, by defining the real solution, horizontal solution and horizontal interval value solution of the fuzzy system of linear equation, the definition of their fuzzy number solution is given; In Section 4, we obtain a method of solving trapezoidal fuzzy number solution for linear systems (equations) with trapezoidal fuzzy number coefficients, and give two examples to show the application of the method. At last, we make a conclusion in Section 5.

2. **Basic Definitions and Notations.** R is the real number field, R^n is the n-dimensional real number column vector space. A fuzzy subset (in short, a fuzzy set) of R is a function $u : R \to [0,1]$. For each such fuzzy set u, we denote by $[u]^r = \{x \in R : u(x) \geq r\}$ for each $r \in [0,1]$, its r-level set.

If u is a normal and fuzzy convex fuzzy set of R, $u(x)$ is upper semicontinuous, and $[u]^0$ is compact, then we call u a 1-dimensional fuzzy number (in short, fuzzy number), and denote the collection of all fuzzy numbers by E.

It is known that for any $u \in E$ and $r \in [0,1]$, $[u]^r$ is a closed interval (denote it as $[\underline{u}(r), \overline{u}(r)]$).

For any $a \in R$, define fuzzy number $\hat{a}$ by

$$\hat{a}(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{if } x \neq a \end{cases}$$

Let $u, v \in E$, the addition, multiplication and scalar multiplication on E are defined by

$$(u+v)(x) = \sup_{y+z=x} \min[u(y), v(z)]$$

$$(uv)(x) = \sup_{yz=x} \min[u(y), v(z)]$$

$$(\lambda u)(x) = \begin{cases} u\left(\lambda^{-1}x\right), & \text{if } \lambda \neq 0 \\ \hat{0}(x), & \text{if } \lambda = 0 \end{cases}$$

Let $u, v \in E$ and $k \in R$, then $u+v, uv, ku \in E$ and $[u+v]^r = [u]^r + [v]^r$, $[uv]^r = [u]^r[v]^r$, $[ku]^r = k[u]^r$, for any $r \in [0,1]$.

We call

$$U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & & \vdots \\ u_{m1} & u_{m2} & \cdots & u_{mn} \end{bmatrix} \quad \text{(in short, denote it as } [u_{ij}]_{m\times n})$$

an $m \times n$ fuzzy number matrix, where, $u_{ij} \in E$, $i = 1, 2, \cdots, m$; $j = 1, 2, \cdots, n$. And we denote the collection of all $m \times n$ fuzzy number matrixes by $F_{m\times n}$. As $n = 1$, we denote $F_m = F_{m\times 1}$ and $u = (u_i)_{i=1}^m = [u_{ij}]_{m\times 1}$ for $u \in F_m$.

The additions and multiplications of interval number matrix and fuzzy number matrix are defined like real number matrix.

For any two $m \times n$ real number matrixes

$$\dot{A} = \begin{bmatrix} \dot{a}_{11} & \dot{a}_{12} & \cdots & \dot{a}_{1n} \\ \dot{a}_{21} & \dot{a}_{22} & \cdots & \dot{a}_{2n} \\ \vdots & \vdots & & \vdots \\ \dot{a}_{m1} & \dot{a}_{m2} & \cdots & \dot{a}_{mn} \end{bmatrix} \quad \text{(in short, denote it as } [\dot{a}_{ij}]_{m\times n}\text{)}$$

and

$$\dot{B} = \begin{bmatrix} \dot{b}_{11} & \dot{b}_{12} & \cdots & \dot{b}_{1n} \\ \dot{b}_{21} & \dot{b}_{22} & \cdots & \dot{b}_{2n} \\ \vdots & \vdots & & \vdots \\ \dot{b}_{m1} & \dot{b}_{m2} & \cdots & \dot{b}_{mn} \end{bmatrix} \quad \left(\text{in short, denote it as } \left[\dot{b}_{ij}\right]_{m\times n}\right)$$

we define $A \leq B$ if and only if $a_{ij} \leq b_{ij}$, where $i = 1, 2, \cdots, m$; $j = 1, 2, \cdots, n$.

For any $u_1, u_2 \in E$, we denote $u_1 \leq u_2$ if and only if $[u_1]^r \leq [u_2]^r$ for any $r \in [0, 1]$, i.e., $\underline{u_1}(r) \leq \underline{u_2}(r)$, $\overline{u_1}(r) \leq \overline{u_2}(r)$ for any $r \in [0, 1]$.

For any $U, V \in F_{m\times n}$, we denote $U \leq V$ if and only if $u_{ij} \leq v_{ij}$, $i = 1, 2, \cdots, m$; $j = 1, 2, \cdots, n$.

For $U, V \in F_{m\times n}$ with $U \leq V$, we say $[U, V] = \{W \in F_{m\times n} : \ U \leq W \leq V\}$ to be an $m \times n$ matrix interval.

For any $U = [u_{ij}]_{m\times n} \in F_{m\times n}$, $r \in [0, 1]$, we denote

$$[U]^r = \begin{bmatrix} [u_{11}]^r & [u_{12}]^r & \cdots & [u_{1n}]^r \\ [u_{21}]^r & [u_{22}]^r & \cdots & [u_{2n}]^r \\ \vdots & \vdots & & \vdots \\ [u_{m1}]^r & [u_{m2}]^r & \cdots & [u_{mn}]^r \end{bmatrix}$$

i.e.,

$$[U]^r = \begin{bmatrix} \left[\underline{u_{11}}(r), \overline{u_{11}}(r)\right] & \left[\underline{u_{12}}(r), \overline{u_{12}}(r)\right] & \cdots & \left[\underline{u_{1n}}(r), \overline{u_{1n}}(r)\right] \\ \left[\underline{u_{21}}(r), \overline{u_{21}}(r)\right] & \left[\underline{u_{22}}(r), \overline{u_{22}}(r)\right] & \cdots & \left[\underline{u_{2n}}(r), \overline{u_{2n}}(r)\right] \\ \vdots & \vdots & & \vdots \\ \left[\underline{u_{m1}}(r), \overline{u_{m1}}(r)\right] & \left[\underline{u_{m2}}(r), \overline{u_{m2}}(r)\right] & \cdots & \left[\underline{u_{mn}}(r), \overline{u_{mn}}(r)\right] \end{bmatrix}$$

and denote

$$\underline{[U]}^r = \begin{bmatrix} \underline{u_{11}}(r) & \underline{u_{12}}(r) & \cdots & \underline{u_{1n}}(r) \\ \underline{u_{21}}(r) & \underline{u_{22}}(r) & \cdots & \underline{u_{2n}}(r) \\ \vdots & \vdots & & \vdots \\ \underline{u_{m1}}(r) & \underline{u_{m2}}(r) & \cdots & \underline{u_{mn}}(r) \end{bmatrix}$$

and

$$\overline{[U]}^r = \begin{bmatrix} \overline{u_{11}}(r) & \overline{u_{12}}(r) & \cdots & \overline{u_{1n}}(r) \\ \overline{u_{21}}(r) & \overline{u_{22}}(r) & \cdots & \overline{u_{2n}}(r) \\ \vdots & \vdots & & \vdots \\ \overline{u_{m1}}(r) & \overline{u_{m2}}(r) & \cdots & \overline{u_{mn}}(r) \end{bmatrix}$$

For $\dot{U} = [\dot{u}_{ij}]_{m\times n} \in R_{m\times n}$, we denote $\dot{U} \in [U]^r$ ($r \in [0, 1]$) if and only if $\dot{u}_{ij} \in \left[\underline{u_{ij}}(r), \overline{u_{ij}}(r)\right]$ for $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$.

Let a_1, a_2, a_3 and $a_4 \in R$ with $a_1 \leq a_2 \leq a_3 \leq a_4$. If fuzzy number $u : R \to [0, 1]$ is defined by

$$u(x) = \begin{cases} 1, & x \in [a_2, a_3] \\ \frac{x-a_1}{a_2-a_1}, & x \in [a_1, a_2) \\ \frac{a_4-x}{a_4-a_3}, & x \in (a_3, a_4] \\ 0, & x \notin [a_1, a_4] \end{cases}$$

then u is called a trapezoid fuzzy number, and denoted as $u = F(a_1, a_2, a_3, a_4)$. Specially, if $a_2 = a_3$, u is called a triangle number, and denoted $u = F(a_1, a_2, a_4)$. And we denote the collection of all trapezoid fuzzy numbers by E_{Tra} and the collection of all triangle fuzzy numbers by E_{Tri}.

2.1. **Fuzzy linear system of equations.** Let $A = [a_{ij}]_{m\times n} \in F_{m\times n}$, $b = (b_i)_{i=1}^m \in F_m$. In this paper, we consider the following linear system of equations with fuzzy number coefficients:

$$Ax = b \tag{1}$$

i.e.,

$$\left\{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}\right. \tag{2}$$

where, $x = (x_j)_{j=1}^n$ is the to be solved n-dimensional column vector.

If the solution of the fuzzy linear system of Equation (1) (i.e., (2)) is regarded as a real number vector, then the following such solution of the fuzzy linear system of equations can be easily thought of.

Definition 2.1. *A real vector $x^* = (x_j^*)_{j=1}^n$ is called a real solution of fuzzy linear system of Equation (1) (i.e., (2)) if $Ax^* = b$.*

However, in fact, to directly solve such solution of (1) is very difficult due to the complexity of the division operation of fuzzy numbers which must be met in solving process. So we should find other methods to solve the problem.

It is known that for $U, V \in E$, $U = V \Longleftrightarrow [U]^r = [V]^r$ for any $r \in [0,1]$. So we have that $Ax = b \Longleftrightarrow [Ax]^r = [b]^r$ for any $r \in [0,1]$. So, if $x \in R^n$, we can turn the fuzzy linear system of Equation (1) into:

$$[Ax]^r = [b]^r \text{ for any } r \in [0,1] \tag{3}$$

Further, if $x = (x_j)_{j=1}^n \in R^n$, from $Ax = \left(\sum_{j=1}^n a_{ij}x_j\right)_{i=1}^m$, we have

$$\begin{aligned} [Ax]^r &= \left[\left(\sum_{j=1}^n a_{ij}x_j\right)_{i=1}^m\right]^r \\ &= \left(\left[\sum_{j=1}^n a_{ij}x_j\right]^r\right)_{i=1}^m \\ &= \left(\sum_{j=1}^n [a_{ij}x_j]^r\right)_{i=1}^m \\ &= \left(\sum_{j=1}^n [a_{ij}]^r x_j\right)_{i=1}^m \\ &= [A]^r x \end{aligned}$$

for any $r \in [0,1]$. Therefore, if $x \in R^n$, we can equivalently turn the fuzzy linear system of Equation (1) into a linear system of equations with interval number coefficients, i.e., we have:

Theorem 2.1. *Let $A = [a_{ij}]_{m\times n} \in F_{m\times n}$, $b = (b_i)_{i=1}^m \in F_m$. Then for $x \in R^n$, x is a solution of fuzzy linear system of Equation (1) (i.e., (2)) if and only if for any $r \in [0,1]$, x is a solution of*

$$[A]^r x = [b]^r \tag{4}$$

i.e.,

$$\left\{\begin{array}{ccccccccc} [a_{11}]^r x_1 & + & [a_{12}]^r x_2 & + & \cdots & + & [a_{1n}]^r x_n & = & [b_1]^r \\ [a_{21}]^r x_1 & + & [a_{22}]^r x_2 & + & \cdots & + & [a_{2n}]^r x_n & = & [b_2]^r \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ [a_{m1}]^r x_1 & + & [a_{m2}]^r x_2 & + & \cdots & + & [a_{mn}]^r x_n & = & [b_m]^r \end{array}\right. \tag{5}$$

Definition 2.2. *Let* $A = [a_{ij}]_{m\times n} \in F_{m\times n}$, $b = (b_i)_{i=1}^m \in F_m$. *For any* $r \in [0,1]$, $\dot{A} = [\dot{a}_{ij}]_{m\times n} \in [A]^r$ *(i.e.,* $\dot{A} = [\dot{a}_{ij}]_{m\times n} \in R^{m\times n}$ *and* $\dot{a}_{ij} \in \left[\underline{a_{ij}}(r), \overline{a_{ij}}(r)\right]$, $i = 1,2,\cdots,m$, $j = 1,2,\cdots,n$*) and* $\dot{b} = \left(\dot{b}_i\right)_{i=1}^m \in [b]^r$ *(i.e.,* $\dot{b} = \left(\dot{b}_i\right)_{i=1}^m \in R^m$ *and* $\dot{b}_i \in \left[\underline{b_i}(r), \overline{b_i}(r)\right]$, $i = 1,2,\cdots,m$*), there exists a real vector* $x^* = \left(x_j^*\right)_{j=1}^n$ *such that*

$$\dot{A}x^* = \dot{b}$$

i.e.,

$$\left\{\begin{array}{ccccccccc} \dot{a}_{11}x_1^* & + & \dot{a}_{12}x_2^* & + & \cdots & + & \dot{a}_{1n}x_n^* & = & \dot{b}_1 \\ \dot{a}_{21}x_1^* & + & \dot{a}_{22}x_2^* & + & \cdots & + & \dot{a}_{2n}x_n^* & = & \dot{b}_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ \dot{a}_{m1}x_1^* & + & \dot{a}_{m2}x_2^* & + & \cdots & + & \dot{a}_{mn}x_n^* & = & \dot{b}_m \end{array}\right.$$

then we call $x^* = \left(x_j^*\right)_{j=1}^n$ *an* r*-solution of fuzzy linear system of Equation (1) (i.e., (2)), and denote it as* $x\left(\dot{A}, \dot{b}, r\right) = \left(x_j\left(\dot{A}, \dot{b}, r\right)\right)_{j=1}^n$.

For $A = [a_{ij}]_{m\times n} \in F_{m\times n}, b = (b_i)_{i=1}^m \in F_m$ and $r \in [0,1]$, we denote $\underline{x}(r) = \min\left\{x\left(\dot{A}, \dot{b}, r\right) :\ \dot{A} \in [A]^r, \dot{b} \in [b]^r\right\}$ and $\overline{x}(r) = \max\left\{x\left(\dot{A}, \dot{b}, r\right) :\ \dot{A} \in [A]^r, \dot{b} \in [b]^r\right\}$.

Definition 2.3. *Let* $A = [a_{ij}]_{m\times n} \in F_{m\times n}$, $b = (b_i)_{i=1}^m \in F_m$. *For any* $r \in [0,1]$, *if* $\underline{x}(r) = \min\left\{x\left(\dot{A}, \dot{b}, r\right) :\ \dot{A} \in [A]^r, \dot{b} \in [b]^r\right\}$ *and* $\overline{x}(r) = \max\left\{x\left(\dot{A}, \dot{b}, r\right) :\ \dot{A} \in [A]^r, \dot{b} \in [b]^r\right\}$ *exist, then we call* $[\underline{x}(r), \overline{x}(r)]$ *an* r*-interval value solution of fuzzy linear system of Equation (1) (i.e., (2)), call the fuzzy set* $\tilde{x} = (\tilde{x}_j)_{j=1}^n$ *which is defined by*

$$\tilde{x}_j(t) = \left\{\begin{array}{ll} \sup\left\{r \in [0,1] :\ t \in \left[\underline{x_j}(r), \overline{x_j}(r)\right]\right\} & t \in \cup_{r\in[0,1]}\left[\underline{x_j}(r), \overline{x_j}(r)\right] \\ 0 & t \in \cup_{r\notin[0,1]}\left[\underline{x_j}(r), \overline{x_j}(r)\right] \end{array}\right., \ j = 1,2,\cdots,n$$

where, $\tilde{x}_j(t)$, $j = 1,2,\cdots,n$ *is a component of* $x(r)$, *a fuzzy set solution of fuzzy linear system of Equation (1) (i.e., (2)). Specially, if* $\tilde{x} = (\tilde{x}_j)_{j=1}^n \in E$, *then we call it a fuzzy number solution of fuzzy linear system of Equation (1) (i.e., (2)).*

3. Solution of Trapezoid Fuzzy Number Linear System.

For linear system of equations with trapezoid fuzzy number coefficients (where $A = [a_{ij}]_{m\times n} \in F_{m\times n}$, $b = (b_i)_{i=1}^m \in F_m$ and $a_{ij}, b_i \in E_{Tra}$, $i = 1,2,\cdots,m$; $j = 1,2,\cdots,n$), we have the following result.

Theorem 3.1. *Let* $A = [a_{ij}]_{m\times n} \in F_{m\times n}, b = (b_i)_{i=1}^m \in F_m$ *be trapezoid fuzzy number matrix and trapezoid fuzzy vector, respectively, i.e.,* $a_{ij}, b_i \in E_{Tra}$ *for any* $i = 1,2,\cdots,m$ *and* $j = 1,2,\cdots,n$. *If* $\left\{x\left(\dot{A}, \dot{b}, 1\right) :\ \dot{A} \in [A]^1, \dot{b} \in [b]^1\right\}$ *and* $\left\{x\left(\dot{A}, \dot{b}, 0\right) :\ \dot{A} \in [A]^0, \dot{b} \in [b]^0\right\}$ *are all nonempty, then* $\underline{x}(1) = \min\left\{x\left(\dot{A}, \dot{b}, 1\right) :\ \dot{A} \in [A]^1, \dot{b} \in [b]^1\right\}$, $\overline{x}(1) = \max\left\{x\left(\dot{A}, \dot{b}, 1\right) :\ \dot{A} \in [A]^1, \dot{b} \in [b]^1\right\}$, $\underline{x}(0) = \min\left\{x\left(\dot{A}, \dot{b}, 0\right) :\ \dot{A} \in [A]^0, \dot{b} \in [b]^0\right\}$ *and* $\overline{x}(0) = \max\left\{x\left(\dot{A}, \dot{b}, 0\right) :\ \dot{A} \in [A]^0, \dot{b} \in [b]^0\right\}$ *satisfy* $\underline{x}(0) \le \underline{x}(1) \le \overline{x}(1) \le \overline{x}(0)$, *and the trapezoid fuzzy number* $F(\underline{x}(0), \underline{x}(1), \overline{x}(1), \overline{x}(0))$ *is the fuzzy number solution of fuzzy linear system of Equation (1) (i.e., (2)).*

Proof: From the definitions of $\underline{x}(1)$ and $\overline{x}(1)$, we see $\underline{x}(1) \le \overline{x}(1)$. So to prove $\underline{x}(0) \le \underline{x}(1) \le \overline{x}(1) \le \overline{x}(0)$, we only need to show $\underline{x}(0) \le \underline{x}(1)$, $\overline{x}(1) \le \overline{x}(0)$. In fact, it can be seen from $[A]^1 \subset [A]^0$, $[B]^1 \subset [B]^0$ and the definitions of $\underline{x}(0)$, $\underline{x}(1)$, $\overline{x}(1)$, $\overline{x}(0)$.

Therefore, trapezoid fuzzy number $F(\underline{x}(0), \underline{x}(1), \overline{x}(1), \overline{x}(0))$ can be indeed defined, and its membership function is

$$F(\underline{x}(0), \underline{x}(1), \overline{x}(1), \overline{x}(0))(x) = \begin{cases} 1, & x \in [\underline{x}(1), \overline{x}(1)] \\ \frac{x-\underline{x}(0)}{\underline{x}(1)-\underline{x}(0)}, & x \in [\underline{x}(0), \underline{x}(1)) \\ \frac{\overline{x}(0)-x}{\overline{x}(0)-\overline{x}(1)}, & x \in (\overline{x}(1), \overline{x}(0)] \\ 0, & x \notin [\underline{x}(0), \overline{x}(0)] \end{cases}$$

From it, we can obtain that

$$\underline{F(\underline{x}(0), \underline{x}(1), \overline{x}(1), \overline{x}(0))}(r) = \underline{x}(0) + (\underline{x}(1) - \underline{x}(0))r$$

and

$$\overline{F(\underline{x}(0), \underline{x}(1), \overline{x}(1), \overline{x}(0))}(r) = \overline{x}(0) - (\overline{x}(0) - \overline{x}(1))r$$

In the following we show that for any $r \in [0, 1]$,

$$\left[\underline{F(\underline{x}(0), \underline{x}(1), \ \overline{x}(1), \overline{x}(0))}(r), \overline{F(\underline{x}(0), \underline{x}(1), \overline{x}(1), \overline{x}(0))}(r)\right],$$

i.e., $[\underline{x}(0) + (\underline{x}(1) - \underline{x}(0))r, \ \ \overline{x}(0) - (\overline{x}(0) - \overline{x}(1))r]$ is an r-interval value solution of fuzzy linear system of Equation (1) (i.e., (2)). Firstly, from $\underline{x}(0) \le \underline{x}(1) \le \overline{x}(1) \le \overline{x}(0)$, we see that for any $r \in [0, 1]$, $(1 - r)\underline{x}(0) + r\underline{x}(1) \le (1 - r)\overline{x}(0) + r\overline{x}(1)$, i.e., $\underline{x}(0) + (\underline{x}(1) - \underline{x}(0))r \le \overline{x}(0) - (\overline{x}(0) - \overline{x}(1))r$, so $[\underline{x}(0) + (\underline{x}(1) - \underline{x}(0))r, \ \overline{x}(0) - (\overline{x}(0) - \overline{x}(1))r]$ is indeed a closed interval. In order to prove that $[\underline{x}(0) + (\underline{x}(1) - \underline{x}(0))r, \ \overline{x}(0) - (\overline{x}(0) - \overline{x}(1))r]$ is an r-interval value solution of fuzzy linear system of Equation (1) (i.e., (2)), we only need to show that for any $r \in [0, 1]$, $\dot{A} = [\dot{a}_{ij}]_{m\times n} \in [A]^r$ and $\dot{b} = \left(\dot{b}_i\right)_{i=1}^{m} \in [b]^r$, there exists $x(\dot{A}, \dot{b}, r) = \left(x_j\left(\dot{A}, \dot{b}, r\right)\right)_{j=1}^{n}$ with $\dot{A}x\left(\dot{A}, \dot{b}, r\right) = \dot{b}$ such that $\underline{x}(0) + (\underline{x}(1) - \underline{x}(0))r = \min\left\{x\left(\dot{A}, \dot{b}, r\right) : \dot{A} \in [A]^r, \dot{b} \in [b]^r\right\}$ and $\overline{x}(0) - (\overline{x}(0) - \overline{x}(1))r = \max\left\{x\left(\dot{A}, \dot{b}, r\right) : \dot{A} \in [A]^r, \dot{b} \in [b]^r\right\}$. In fact, from the definitions of $\underline{x}(0)$, $\underline{x}(1)$, $\overline{x}(0)$ and $\overline{x}(1)$, we see that there exist $\underline{\dot{A}}(0) \in [A]^0$, $\underline{\dot{b}}(0) \in [b]^0$, $\underline{\dot{A}}(1) \in [A]^1$, $\underline{\dot{b}}(1) \in [b]^1$, $\overline{\dot{A}}(0) \in [A]^0$, $\overline{\dot{b}}(0) \in [b]^0$ and $\overline{\dot{A}}(1) \in [A]^1$, $\overline{\dot{b}}(1) \in [b]^1$ such that $\underline{x}(0) = x\left(\underline{\dot{A}}(0), \underline{\dot{b}}(0), 0\right)$, $\underline{x}(1) = x\left(\underline{\dot{A}}(1), \underline{\dot{b}}(1), 1\right)$, $\overline{x}(0) = x\left(\overline{\dot{A}}(0), \overline{\dot{b}}(0), 0\right)$ and $\overline{x}(1) = x\left(\overline{\dot{A}}(1), \overline{\dot{b}}(1), 1\right)$, i.e., $\underline{\dot{A}}(0)\underline{x}(0) = \underline{\dot{b}}(0)$, $\underline{\dot{A}}(1)\underline{x}(1) = \underline{\dot{b}}(1)$, $\overline{\dot{A}}(0)\overline{x}(0) = \overline{\dot{b}}(0)$ and $\overline{\dot{A}}(1)\overline{x}(1) = \overline{\dot{b}}(1)$. $\square$

4. **Numerical Example.** In the following, we give two practical examples to show the application of Theorem 3.1.

Example 4.1. *Consider fuzzy linear system* $(1, 2, 3, 4)x = (6, 7, 8, 9)$.

From the definitions of $\underline{x}(0)$, $\overline{x}(0)$, $\underline{x}(1)$, $\overline{x}(1)$, *we have*

$$\underline{x}(0) = \min\left\{x\left(\dot{A}, \dot{b}, 0\right) : \dot{A} \in [1, 4], \dot{b} \in [6, 9]\right\}$$
$$\overline{x}(0) = \max\left\{x\left(\dot{A}, \dot{b}, 0\right) : \dot{A} \in [1, 4], \dot{b} \in [6, 9]\right\}$$
$$\underline{x}(1) = \min\left\{x\left(\dot{A}, \dot{b}, 1\right) : \dot{A} \in [2, 3], \dot{b} \in [7, 8]\right\}$$
$$\overline{x}(1) = \max\left\{x\left(\dot{A}, \dot{b}, 1\right) : \dot{A} \in [2, 3], \dot{b} \in [7, 8]\right\}$$

We can get

$$\underline{x}(0) = \frac{3}{2}$$
$$\overline{x}(0) = 9$$

and

$$\underline{x}(1) = \frac{7}{3}$$
$$\overline{x}(1) = 4$$

So, by Theorem 3.1 we know trapezoid fuzzy number $\left(\frac{3}{2}, \frac{7}{3}, 4, 9\right)$ *is a fuzzy number solution of the fuzzy linear system (equation)* $(1,2,3,4)x = (6,7,8,9)$.

Example 4.2. *Consider fuzzy linear system*

$$\begin{bmatrix} (2,3,5,6) & (3,4,6,7) \\ (1.5,2,4,5) & (3.5,4,5,6) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (8,15,42,70) \\ (7,12.5,34,59) \end{bmatrix}.$$

From the definitions of $\underline{x}(0)$, $\overline{x}(0)$, $\underline{x}(1)$, $\overline{x}(1)$, *we have*

$$\underline{x}(0) = \min\left\{x\left(\dot{A}, \dot{b}, 0\right) : \dot{A} \in \begin{bmatrix} [2,6] & [3,7] \\ [1.5,5] & [3.5,6] \end{bmatrix}, \dot{b} \in \begin{bmatrix} [8,70] \\ [7,59] \end{bmatrix}\right\}$$
$$\overline{x}(0) = \max\left\{x\left(\dot{A}, \dot{b}, 0\right) : \dot{A} \in \begin{bmatrix} [2,6] & [3,7] \\ [1.5,5] & [3.5,6] \end{bmatrix}, \dot{b} \in \begin{bmatrix} [8,70] \\ [7,59] \end{bmatrix}\right\}$$
$$\underline{x}(1) = \min\left\{x\left(\dot{A}, \dot{b}, 1\right) : \dot{A} \in \begin{bmatrix} [3,5] & [4,6] \\ [2,4] & [4,5] \end{bmatrix}, \dot{b} \in \begin{bmatrix} [15,42] \\ [12.5,34] \end{bmatrix}\right\}$$
$$\overline{x}(1) = \max\left\{x\left(\dot{A}, \dot{b}, 1\right) : \dot{A} \in \begin{bmatrix} [3,5] & [4,6] \\ [2,4] & [4,5] \end{bmatrix}, \dot{b} \in \begin{bmatrix} [15,42] \\ [12.5,34] \end{bmatrix}\right\}$$

so,

$$\begin{cases} 6\underline{x_1}(0) + 7\underline{x_2}(0) = 8 \\ 5\underline{x_1}(0) + 6\underline{x_2}(0) = 7 \end{cases}$$

$$\begin{cases} 2\overline{x_1}(0) + 3\overline{x_2}(0) = 70 \\ 1.5\overline{x_1}(0) + 3.5\overline{x_2}(0) = 59 \end{cases}$$

$$\begin{cases} 5\underline{x_1}(1) + 6\underline{x_2}(1) = 15 \\ 4\underline{x_1}(1) + 5\underline{x_2}(1) = 12.5 \end{cases}$$

$$\begin{cases} 3\overline{x_1}(1) + 4\overline{x_2}(1) = 42 \\ 2\overline{x_1}(1) + 4\overline{x_2}(1) = 34 \end{cases}$$

we can get

$$\begin{cases} \underline{x_1}(0) = -1 \\ \underline{x_2}(0) = 2 \end{cases}$$

$$\begin{cases} \overline{x_1}(0) = 27.2 \\ \overline{x_2}(0) = 5.2 \end{cases}$$

and

$$\begin{cases} \underline{x_1}(1) = 0 \\ \underline{x_2}(1) = 2.5 \end{cases}$$

$$\begin{cases} \overline{x_1}(1) = 8 \\ \overline{x_2}(1) = 4.5 \end{cases}$$

So, by Theorem 3.1 we know trapezoid fuzzy number vector $\begin{pmatrix} (-1,0,8,27.2) \\ (2,2.5,4.5,5.2) \end{pmatrix}$ *is a fuzzy number solution of the fuzzy linear system (equation)* $\begin{bmatrix} (2,3,5,6) & (3,4,6,7) \\ (1.5,2,4,5) & (3.5,4,5,6) \end{bmatrix}$ $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (8,15,42,70) \\ (7,12.5,34,59) \end{pmatrix}$.

From the aboved two examples, if the fuzzy linear system of equations satifies the definations of $\underline{x}(0)$, $\overline{x}(0)$, $\underline{x}(1)$, $\overline{x}(1)$, we can get the trapezoid fuzzy number solution by Theorem 3.1. And the procedures as given may be an easy way to handle the trapezoid fuzzy linear system of equations. The challenge will be how to handle the system of equations when the coefficient matrix and the right hand are general fuzzy numbers.

5. **Conclusion.** In this paper, we gave the concept (Definition 2.1) of real solution of fuzzy linear system of equation with fuzzy number coefficients, and equivalently translated the fuzzy linear systems (equations) into interval number systems (equations) with interval number coefficients by horizontal set (Theorem 2.1). Then we defined horizontal solution of the fuzzy linear system (Definition 2.2), and proposed the concepts of fuzzy set solution and fuzzy number solution of the fuzzy linear system via defining its horizontal interval value solution (Definition 2.3). And then, we obtained a method of solving trapezoidal fuzzy number linear systems (equations) (Theorem 3.1). At last, we gave two examples (Examples 4.1 and 4.2) to show the application of proposed theory. In the future, we can study the structure of the solutions of fuzzy linear systems (equations), and apply the theories related to solutions of fuzzy linear systems (equations) to a wider range of areas.

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