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STABILITY ANALYSIS OF MULTI-AGENT SYSTEMS WITH MULTIPLE LEADERS OF VARIABLE VELOCITIES BASED ON CONSENSUS PROTOCOLS

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Abstract. *Existing works that deal with the problem of the consensus with multi-leaders assume that the velocities of leaders are identical, which amounts to the invariant shape of the target region. However, when the shape of the target region is changed, it is hard to solve. So this paper investigates the collaborative control problem for multi-agent systems with multi-leader architecture of variable velocities. Sufficient and necessary conditions of stability are summarized by utilizing graph theory and control theory, and the efforts of control parameters on stability are analyzed. Furthermore, the steady state of followers is established which has a strong association with the convex hull produced by positions and velocities of leaders. Finally, simulations prove the validity of the theoretical results.*
Keywords: Consensus, Multi-leader, Stability, Stable state

1. **Introduction.** Recently, the problem of collaborative control in networked multi-agent systems [1] has received significant attention due to its important application, such as multiple mobile robots, multi-intelligent vehicles, and multiple unmanned aerial vehicles. At the beginning, it is used to be a single agent to complete the task. However, as the task size and task complexity increased, the traditional agent can hardly meet the need of some complex tasks. Besides, the algorithm of relevant controller will become very complex with the task difficulty increased. So it has practical significance to study the cooperative work of multi-agents, especially the formation control [2-6].

Traditional coordinated formation control strategy mainly contains the master-slave mode [7], the approach based behavior [8] and the virtual structure mode [9]. These three control strategies have their own advantages and disadvantages, so a more general control strategy is needed to be found, which can get rid of their shortcomings and take their advantages. The coordination control strategy based on consensus [10-13] arises at the historic moment. More and more scholars pay attention to consensus with the multi-leader structure in recent years.

The consensus problems of multi-agent systems of multiple leaders in different situations have been studied and the stability of the system plays an important role in the study of the consensus. Considering the case where the graph that captures the underlying networks topology is not connected in some time, Xargay et al. [14] analyzed the stability of multi-leader multi-agent systems with dynamic information flow. Further, considering nonlinear multi-leader systems, two impulsive control algorithms were proposed [15] to make all the agents to track the convex set. Then distributed observe-type containment protocols were applied in the high-order multi-leader system to guarantee the states of the followers to converge to a convex hull [16]. An observer-based distributed controller was proposed to ensure the convergence in finite time [17]. Moreover, distributed finite-time containment control algorithms were designed based on the estimates and the generalized adding a power integrator approach [18]. Mei et al. [19] proved that all followers converge to the convex hull spanned by multiple leaders with zero speed by proposing a distributed adaptive control algorithm. To the best of our knowledge, there is no theoretical result about the stable state of the system of multi-leaders of variable velocities. This is more practical when the shape and position of the task change.

Setting multiple leaders can enhance the anti-interference ability and can also make the follower reach different states. Some people make the velocities of the leaders identical to avoid obstacle by treating the area constituted by leaders as the safe area. However, when encountering an emergency, for example, the position of the obstacle changes, we need to control the leaders to adjust the shape of the area produced by leaders to avoid the obstacle. Thus, giving the leaders a force is necessary. Of course, assuming the force is constant is feasible in the process. So we investigate the multi-agent system of multiple leaders whose accelerations are constant but different among themselves. This paper studies the steady state and stability condition of multi-agent systems with multiple leaders of variable velocities.

The remainder of this paper is organized as follows. Section 2 introduces the graph theory. The main contents of Section 3 are the stable state and the stability conditions of multi-agent systems with multiple leaders of variable velocities. The logistic results are validated with a simulation example in Section 4. Finally, Section 5 concludes the paper.

2. Graph Theory.

In this section, we introduce the basic concepts of the graph theory.

A weighted graph is denoted by $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$, where $\mathcal{V} = \{1, 2, \ldots, n\}$ is the point set, $\mathcal{E} \in \mathcal{V} \times \mathcal{V}$ is the edge set, and $(i, j) \in \mathcal{E}(\mathcal{G})$ means that i can be reached from j, but not necessarily vice versa.

$$\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n},\ a_{ij} = \begin{cases} w_{ij}, & (i,j) \in \mathcal{E}(\mathcal{G}) \\ 0, & (i,j) \notin \mathcal{E}(\mathcal{G}) \end{cases},$$

where $\mathcal{A}$ is the weighted adjacent matrix and w_{ij} is the weight of edge (i, j). A diagonal matrix $\mathcal{D} = diag\{d_1, \ldots, d_n\}$ is a degree matrix of $\mathcal{G}$, whose diagonal elements $d_i = \sum_{j=1}^{n} a_{ij}$ for $i = 1, \ldots, n$. The Laplacian matrix is defined as $\mathcal{L} = \mathcal{D} - \mathcal{A}$, as follows

$$\mathcal{L}(\mathcal{G}) = \begin{cases} \sum\limits_{i\neq j} a_{ij}, & i = j \\ -a_{ij}, & i \neq j \end{cases}.$$

The reachable set is defined as $\mathcal{Q}(j) = \{j\}U\{i|(i, j) \in \mathcal{E}(\mathcal{G})\}$. And $\mathcal{Q}(j)$ is the maximum reachable set only if $\mathcal{Q}(j) \not\subset \mathcal{Q}(i)$, $\forall i \in \mathcal{V}$.

Lemma 2.1. *If the number of maximum reachable sets of $\mathcal{G}$ is q, the number of zero characteristic roots of $\mathcal{L}$ is also q and the other characteristic roots of $\mathcal{L}$ have strictly positive real parts [20].*

3. Steady State and Stability Analysis.

Sometimes, the target region of the system which is composed of the convex hull constituted by leaders may change when the mission changes. So, this paper considers the position and the shape of the convex hull changing at the same time, that is to say, the leaders have variable velocities. We assume that the accelerations of leaders are constant.

3.1. **Controller design.** The system is leader-follower system which includes m leaders and n followers without consideration that leaders accept information from followers. Denote the set of leaders as $\mathcal{R} = \{n+1, n+2, \ldots, n+m\}$ and the set of followers as $\mathcal{F} = \{1, 2, \ldots, n\}$.

Dynamic equation of the kth leader:

$$\begin{cases} \dot{x}_k(t) = v_k(t) \\ \dot{v}_k(t) = u_k(t) \end{cases}, \ k = n+1, n+2, \ldots, n+m,$$

where x is the position, v is the velocity and u is the control input.

Dynamic equation of the ith follower:

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = u_i(t) \end{cases}, \ i = 1, 2, \ldots, n. \tag{1}$$

The consensus controller protocol:

$$\begin{cases} u_i(t) = k_1 \sum\limits_{j \in \mathcal{R} \cup \mathcal{F}} a_{ij}(x_j(t) - x_i(t)) + k_2 \sum\limits_{j \in \mathcal{R} \cup \mathcal{F}} a_{ij}(v_j(t) - v_i(t)), & i \in \mathcal{F} \\ u_i(t) = \psi, & i \in \mathcal{R} \end{cases}, \tag{2}$$

where $\psi \in \mathbb{R}^{m \times 1}$ is a constant matrix composed by accelerations of leaders, k_1, k_2 denote control gains.

According to the definition of leaders and followers, Laplacian matrix of the system can be written as

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_\mathcal{F} & \mathcal{L}_\mathcal{R} \\ 0_{m \times n} & 0_{m \times m} \end{bmatrix}.$$

(2) will be written in the form of a matrix as

$$\begin{aligned} u(t) &= -k_1 \begin{bmatrix} \mathcal{L}_\mathcal{F} & \mathcal{L}_\mathcal{R} \end{bmatrix} \begin{bmatrix} x_\mathcal{F}(t) \\ x_\mathcal{R}(t) \end{bmatrix} - k_2 \begin{bmatrix} \mathcal{L}_\mathcal{F} & \mathcal{L}_\mathcal{R} \end{bmatrix} \begin{bmatrix} v_\mathcal{F}(t) \\ v_\mathcal{R}(t) \end{bmatrix} \\ &= -k_1 \mathcal{L}_\mathcal{F} x_\mathcal{F}(t) - k_2 \mathcal{L}_\mathcal{F} v_\mathcal{F}(t) - k_1 \mathcal{L}_\mathcal{R} x_\mathcal{R}(t) - k_2 \mathcal{L}_\mathcal{R} v_\mathcal{R}(t) \end{aligned} \tag{3}$$

with $u(t) = [u_1(t) \quad u_2(t) \quad \cdots \quad u_n(t)]^T$, $x_\mathcal{F}(t) = [x_1(t) \quad x_2(t) \quad \cdots \quad x_n(t)]^T$, $v_\mathcal{F}(t) = [v_1(t) \quad v_2(t) \quad \cdots \quad v_n(t)]^T$, $x_\mathcal{R}(t) = [x_{n+1}(t) \quad x_{n+2}(t) \quad \cdots \quad x_{n+m}(t)]^T$, $v_\mathcal{R}(t) = \left[v_{n+1}(t) \quad v_{n+2}(t) \quad \cdots \quad v_{n+m}(t)\right]^T$.

Substituting (3) into (1), (4) can be obtained.

$$\begin{bmatrix} \dot{x}_\mathcal{F}(t) \\ \dot{v}_\mathcal{F}(t) \end{bmatrix} = \begin{bmatrix} 0_n & I_n \\ -k_1 \mathcal{L}_\mathcal{F} & -k_2 \mathcal{L}_\mathcal{F} \end{bmatrix} \begin{bmatrix} x_\mathcal{F}(t) \\ v_\mathcal{F}(t) \end{bmatrix} + \begin{bmatrix} 0_{n \times m} & 0_{n \times m} \\ -k_1 \mathcal{L}_\mathcal{R} & -k_2 \mathcal{L}_\mathcal{R} \end{bmatrix} \begin{bmatrix} x_\mathcal{R}(t) \\ v_\mathcal{R}(t) \end{bmatrix}. \tag{4}$$

Further, (4) can be written as

$$\dot{y}_\mathcal{F}(t) = E y_\mathcal{F}(t) + D y_\mathcal{R}(t) \tag{5}$$

with $E = \begin{bmatrix} 0_n & I_n \\ -k_1 \mathcal{L}_\mathcal{F} & -k_2 \mathcal{L}_\mathcal{F} \end{bmatrix}$, $D = \begin{bmatrix} 0_{n \times m} & 0_{n \times m} \\ -k_1 \mathcal{L}_\mathcal{R} & -k_2 \mathcal{L}_\mathcal{R} \end{bmatrix}$, $y_\mathcal{F}(t) = \begin{bmatrix} x_\mathcal{F}(t) \\ v_\mathcal{F}(t) \end{bmatrix}$, $y_\mathcal{R}(t) = \begin{bmatrix} x_\mathcal{R}(t) \\ v_\mathcal{R}(t) \end{bmatrix}$.

3.2. **Stable state.** The relation between the stable state of the system with multiple leaders of variable velocities and the topology will be given in this subsection. It provides a theoretical basis for the design of the topology of the system.

Lemma 3.1. *If each follower can be reached from one leader at least, the row sum of $-\mathcal{L}_\mathcal{F}{}^{-1} \mathcal{L}_\mathcal{R}$ is 1, and all the elements of $-\mathcal{L}_\mathcal{F}{}^{-1} \mathcal{L}_\mathcal{R}$ are not negative [21].*

Theorem 3.1. *Considering the multi-agent system with multiple leaders of variable velocities and in the condition that the system is stable, the stable positions and the stable velocities of followers are as follows*

$$\begin{bmatrix} x^e_{\mathcal{F}}(t) \\ v^e_{\mathcal{F}}(t) \end{bmatrix} = \begin{bmatrix} -\mathcal{L}_{\mathcal{F}}^{-1}\mathcal{L}_{\mathcal{R}}x_{\mathcal{R}}(t) + x_b \\ -\mathcal{L}_{\mathcal{F}}^{-1}\mathcal{L}_{\mathcal{R}}v_{\mathcal{R}}(t) \end{bmatrix} \tag{6}$$

with $x_b = \frac{1}{k_1}(\mathcal{L}_{\mathcal{F}}^{-1})^2\mathcal{L}_{\mathcal{R}}\psi$.

Proof: Assume $t_0 = 0$. Using the knowledge of the state response and by (5), one obtains

$$y_{\mathcal{F}}(t) = e^{Et}y_{\mathcal{F}}(0) + \int_0^t e^{E(t-\tau)}Dy_{\mathcal{R}}(\tau)d\tau. \tag{7}$$

In order to simplify the calculation, assume that the initial positions and initial velocities of leaders are 0. Moreover, when the leaders have initial positions and initial velocities, we can consider the position error and the velocity error. So the following equation also holds. Thus the assumption is reasonable.

The relation between positions, velocities and accelerations of leaders is as follows

$$\begin{bmatrix} x_{\mathcal{R}}(t) \\ v_{\mathcal{R}}(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\psi t^2 \\ \psi t \end{bmatrix}. \tag{8}$$

Substituting (8) into (7), it is easy to say that

$$y_{\mathcal{F}}(t) = e^{Et}y_{\mathcal{F}}(0) + \int_0^t e^{E(t-\tau)}D\begin{bmatrix} \frac{1}{2}\psi t^2 \\ \psi t \end{bmatrix}d\tau. \tag{9}$$

Using integration by parts for (9), one obtains

$$\begin{aligned}
y_{\mathcal{F}}(t) &= e^{Et}y_{\mathcal{F}}(0) + \int_0^t e^{E(t-\tau)}D\begin{bmatrix} \frac{1}{2}\psi\tau^2 \\ \psi\tau \end{bmatrix}d\tau \\
&= e^{Et}y_{\mathcal{F}}(0) - E^{-1}e^{E(t-\tau)}D\begin{bmatrix} \frac{1}{2}\psi\tau^2 \\ \psi\tau \end{bmatrix}_0^t + E^{-1}\int_0^t e^{E(t-\tau)}D\begin{bmatrix} \psi\tau \\ \psi \end{bmatrix}d\tau \\
&= e^{Et}y_{\mathcal{F}}(0) - E^{-1}e^{E(t-\tau)}D\begin{bmatrix} \frac{1}{2}\psi\tau^2 \\ \psi\tau \end{bmatrix}_0^t - (E^{-1})^2e^{E(t-\tau)}D\begin{bmatrix} \psi\tau \\ \psi \end{bmatrix}_0^t \\
&\quad +(E^{-1})^2\int_0^t e^{E(t-\tau)}D\begin{bmatrix} \psi \\ 0 \end{bmatrix}d\tau \\
&= e^{Et}y_{\mathcal{F}}(0) - E^{-1}e^{E(t-\tau)}D\begin{bmatrix} \frac{1}{2}\psi\tau^2 \\ \psi\tau \end{bmatrix}_0^t - (E^{-1})^2e^{E(t-\tau)}D\begin{bmatrix} \psi\tau \\ \psi \end{bmatrix}_0^t \\
&\quad -(E^{-1})^3e^{E(t-\tau)}D\begin{bmatrix} \psi \\ 0 \end{bmatrix}_0^t \\
&= e^{Et}y_{\mathcal{F}}(0) - E^{-1}D\begin{bmatrix} \frac{1}{2}\psi t^2 \\ \psi t \end{bmatrix} - (E^{-1})^2D\begin{bmatrix} \psi t \\ \psi \end{bmatrix} + (E^{-1})^2e^{Et}D\begin{bmatrix} 0 \\ \psi \end{bmatrix} \\
&\quad -(E^{-1})^3D\begin{bmatrix} \psi \\ 0 \end{bmatrix} + (E^{-1})^3e^{Et}D\begin{bmatrix} \psi \\ 0 \end{bmatrix}.
\end{aligned}$$

Assume that the characteristic roots of E have negative real parts in the condition that the system is stable. So there is $\lim_{t\to\infty} e^{Et} = 0$. Therefore, when $t \to \infty$, one has

$$y_{\mathcal{F}}(t) \to -E^{-1}D\begin{bmatrix} \frac{1}{2}\psi t^2 \\ \psi t \end{bmatrix} - (E^{-1})^2D\begin{bmatrix} \psi t \\ \psi \end{bmatrix} - (E^{-1})^3D\begin{bmatrix} \psi \\ 0 \end{bmatrix}. \tag{10}$$

Substituting E and D into (10), the stable positions and the stable velocities of followers are as follows

$$\begin{bmatrix} x^e_{\mathcal{F}}(t) \\ v^e_{\mathcal{F}}(t) \end{bmatrix} = \begin{bmatrix} -L^{-1}_{\mathcal{F}} L_{\mathcal{R}} x_{\mathcal{R}}(t) + x_b \\ -L^{-1}_{\mathcal{F}} L_{\mathcal{R}} v_{\mathcal{R}}(t) \end{bmatrix}$$

with $x_b = \frac{1}{k_1} (\mathcal{L}^{-1}_{\mathcal{F}})^2 \mathcal{L}_{\mathcal{R}} \psi$.

By Lemma 3.1, we can obtain that the stable velocities of followers will converge to the convex hull constituted by velocities of leaders, while the stable positions of followers will be the linear combination of the convex hull constituted by positions of leaders and an offset which is related to the accelerations of leaders.

3.3. **Stability condition.** In this subsection, we will discuss the stability conditions of the system for the controller we designed above and prove the feasibility of the controller.

By (6), the error equation between the state at time t and the stable state of the followers is obtained,

$$\begin{bmatrix} \tilde{x}_{\mathcal{F}}(t) \\ \tilde{v}_{\mathcal{F}}(t) \end{bmatrix} = \begin{bmatrix} x_{\mathcal{F}}(t) \\ v_{\mathcal{F}}(t) \end{bmatrix} - \begin{bmatrix} x^e_{\mathcal{F}}(t) \\ v^e_{\mathcal{F}}(t) \end{bmatrix}. \tag{11}$$

Then the system Equation (12) can be obtained by (4), (6) and (11),

$$\begin{aligned} &\begin{bmatrix} \dot{\tilde{x}}_{\mathcal{F}}(t) \\ \dot{\tilde{v}}_{\mathcal{F}}(t) \end{bmatrix} \\ &= \begin{bmatrix} 0_n & I_n \\ -k_1\mathcal{L}_{\mathcal{F}} & -k_2\mathcal{L}_{\mathcal{F}} \end{bmatrix} \begin{bmatrix} x_{\mathcal{F}}(t) \\ v_{\mathcal{F}}(t) \end{bmatrix} + \begin{bmatrix} 0_{n\times m} & 0_{n\times m} \\ -k_1\mathcal{L}_{\mathcal{R}} & -k_2\mathcal{L}_{\mathcal{R}} \end{bmatrix} \begin{bmatrix} x_{\mathcal{R}}(t) \\ v_{\mathcal{R}}(t) \end{bmatrix} + \begin{bmatrix} \mathcal{L}^{-1}_{\mathcal{F}}\mathcal{L}_{\mathcal{R}} v_{\mathcal{R}}(t) \\ \mathcal{L}^{-1}_{\mathcal{F}}\mathcal{L}_{\mathcal{R}}\psi \end{bmatrix} \\ &= \begin{bmatrix} 0_n & I_n \\ -k_1\mathcal{L}_{\mathcal{F}} & -k_2\mathcal{L}_{\mathcal{F}} \end{bmatrix} \begin{bmatrix} \tilde{x}_{\mathcal{F}}(t) \\ \tilde{v}_{\mathcal{F}}(t) \end{bmatrix} - \begin{bmatrix} 0_n & I_n \\ -k_1\mathcal{L}_{\mathcal{F}} & -k_2\mathcal{L}_{\mathcal{F}} \end{bmatrix} \begin{bmatrix} \mathcal{L}^{-1}_{\mathcal{F}}\mathcal{L}_{\mathcal{R}} x_{\mathcal{R}}(t) - x_b \\ \mathcal{L}^{-1}_{\mathcal{F}}\mathcal{L}_{\mathcal{R}} v_{\mathcal{R}}(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} 0_{n\times m} & 0_{n\times m} \\ -k_1\mathcal{L}_{\mathcal{R}} & -k_2\mathcal{L}_{\mathcal{R}} \end{bmatrix} \begin{bmatrix} x_{\mathcal{R}}(t) \\ v_{\mathcal{R}}(t) \end{bmatrix} + \begin{bmatrix} \mathcal{L}^{-1}_{\mathcal{F}}\mathcal{L}_{\mathcal{R}} v_{\mathcal{R}}(t) \\ \mathcal{L}^{-1}_{\mathcal{F}}\mathcal{L}_{\mathcal{R}}\psi \end{bmatrix} \\ &= \begin{bmatrix} 0_n & I_n \\ -k_1\mathcal{L}_{\mathcal{F}} & -k_2\mathcal{L}_{\mathcal{F}} \end{bmatrix} \begin{bmatrix} \tilde{x}_{\mathcal{F}}(t) \\ \tilde{v}_{\mathcal{F}}(t) \end{bmatrix}. \end{aligned} \tag{12}$$

Further, (12) can be written as

$$\dot{\tilde{y}}_{\mathcal{F}}(t) = E\tilde{y}_{\mathcal{F}}(t) \tag{13}$$

with $\tilde{y}_{\mathcal{F}}(t) = \begin{bmatrix} \tilde{x}_{\mathcal{F}}(t) \\ \tilde{v}_{\mathcal{F}}(t) \end{bmatrix}$.

Lemma 3.2. *Considering continuous linear time-invariant autonomous system*

$$\dot{x}(t) = \Lambda x(t),\ x(0) = x_0,\ t \geq 0,$$

the necessary and sufficient condition of asymptotic stability for the system is that all the eigenvalues of the system matrix have negative real parts [22], such as

$$\mathrm{Re}\,\{\lambda_i(\Lambda)\} < 0,\ i = 1, 2, \ldots, n.$$

By Lemma 3.2 and (13), the necessary and sufficient condition for the stability of the system in this paper is as follows

$$\mathrm{Re}\,\{\lambda_i(E)\} < 0,\ i = 1, 2, \ldots, n.$$

So the assumption before that the characteristic roots of E have negative real parts in the condition that the system is stable is correct.

Theorem 3.2. *If each follower can be reached from one leader at least, all the characteristic roots of $\mathcal{L}_{\mathcal{F}}$ have positive real parts.*

Proof: By the definition of maximum reachable set, we know each leader corresponds to a maximum reachable set and each follower at least belongs to one leader's maximum reachable set. So this topology contains m maximum reachable sets. Therefore, the number of the zero characteristic roots of $\mathcal{L}$ is m and the other characteristic roots of $\mathcal{L}$ have strictly positive real parts by Lemma 2.1. Because of the definition of $\mathcal{L}$, all the characteristic roots of $\mathcal{L}_{\mathcal{F}}$ have positive real parts.

Theorem 3.3. *If each follower can reach from one leader at least, the necessary and sufficient condition of the stability of the system under the control protocol of (2) is as follows*

$$\frac{{k_2}^2}{k_1} > \max\left(\frac{\beta^2}{-\alpha\left[\alpha^2+\beta^2\right]}\right)$$

with $\alpha = \mathrm{Re}(\mu_i)$, $\beta = \mathrm{Im}(\mu_i)$, *where* μ_i *is the eigenvalues of* $-\mathcal{L}_{\mathcal{F}}$ *with* $i = 1, 2, \ldots, n$.

Proof: Define that λ is the eigenvalue of E, then the eigenvector of E is as follows

$$\det(\lambda I_{2n} - E) = \det\begin{bmatrix} \lambda I_n & -I_n \\ k_1\mathcal{L}_{\mathcal{F}} & \lambda I_n + k_2\mathcal{L}_{\mathcal{F}} \end{bmatrix} = \det(\lambda^2 I_n + \lambda k_2\mathcal{L}_{\mathcal{F}} + k_1\mathcal{L}_{\mathcal{F}}).$$

For all μ_i, there is $\det(\lambda I_n - \mathcal{L}_{\mathcal{F}}) = \prod_{i=1}^{n}(\lambda - \mu_i)$ and $\mathrm{Re}(\mu_i) < 0$ by Theorem 3.2. Therefore,

$$\det(\lambda I_{2n} - E) = \prod_{i=1}^{n}(\lambda^2 - \lambda k_2\mu_i - k_1\mu_i).$$

By $\det(\lambda I_{2n} - E) = 0$, each μ_i corresponds to two eigenvalues of E which are λ_{i1} and λ_{i2}.

$$\begin{cases} \lambda_{i1} = \dfrac{k_2\mu_i + \sqrt{{k_2}^2{\mu_i}^2 + 4k_1\mu_i}}{2} \\ \lambda_{i2} = \dfrac{k_2\mu_i - \sqrt{{k_2}^2{\mu_i}^2 + 4k_1\mu_i}}{2} \end{cases}. \tag{14}$$

In order to make the system stable, it is required $\mathrm{Re}(\lambda_{i1}) < 0$, $\mathrm{Re}(\lambda_{i2}) < 0$.

Define $\sqrt{{k_2}^2{\mu_i}^2 + 4k_1\mu_i} = \varphi + \phi i$ with $\varphi, \phi \in \mathbb{R}$. According to (14), we know $\mathrm{Re}(\lambda_i) < 0 \Rightarrow \frac{k_2\alpha \pm \varphi}{2} < 0$. $|k_2\alpha| > |\varphi|$ can be derived by $\alpha < 0$. So

$${k_2}^2\alpha^2 > \varphi^2. \tag{15}$$

By $\sqrt{{k_2}^2{\mu_i}^2 + 4k_1\mu_i} = \varphi + \phi i$, one obtains

$$\begin{array}{l} C = {k_2}^2\alpha^2 - {k_2}^2\beta^2 + 4k_1\alpha = \varphi^2 - \phi^2 \\ B = {k_2}^2\alpha\beta + 2k_1\beta = \varphi\phi \end{array}. \tag{16}$$

By (16), (17) can be calculated.

$$\varphi^4 - C\varphi^2 - B^2 = 0. \tag{17}$$

Solving (17), the result is

$$\varphi^2 = \frac{C \pm \sqrt{C^2 + 4B^2}}{2}. \tag{18}$$

Substituting (15) into (18),

$$\frac{C \pm \sqrt{C^2 + 4B^2}}{2} < {k_2}^2\alpha^2. \tag{19}$$

Substituting B and C into (19),

$$\frac{{k_2}^2}{k_1} > \frac{\beta^2}{-\alpha\left[\alpha^2+\beta^2\right]}. \tag{20}$$

So, the necessary and sufficient condition of the stability of the system which is satisfied with (20) for all μ_i is as follows

$$\frac{{k_2}^2}{k_1} > \max\left(\frac{\beta^2}{-\alpha\left[\alpha^2+\beta^2\right]}\right).$$

4. **Numerical Simulation Results.** In this section, a numerical example is used to demonstrate the validity of the result obtained above.

Consider a topology with 3 leaders and 9 followers as Figure 1 where 10, 11 and 12 are leaders and the weight is as Figure 1.

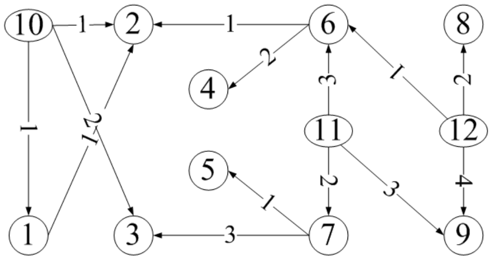


FIGURE 1. Topology structure

Simulation step is taken 0.01s and the initial value of this system can be described by

$$\begin{bmatrix} x_{x0} \\ x_{y0} \\ x_{z0} \\ v_{x0} \\ v_{y0} \\ v_{z0} \\ \psi_{x0} \\ \psi_{y0} \\ \psi_{z0} \end{bmatrix} = \begin{bmatrix} 20 & 10 & 30 & 25 & 15 & 5 & 30 & 0 & 60 & 200 & -200 & -200 \\ 12 & 3 & 22 & 35 & 15 & 9 & 20 & 2 & 10 & -200 & 200 & -200 \\ 4 & 10 & 1 & 6 & 5 & 8 & 20 & 18 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0.4 & 0.3 & 0.1 & 0.5 & 0.9 & 0.7 & 0 & 0 & 0.6 & 0 \\ 0.1 & 0.2 & 0.3 & 0 & 0.4 & 0.9 & 0.8 & 0.6 & 0 & 0.2 & 0 & 0.4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.1 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.1 & 0 & 0 \end{bmatrix},$$

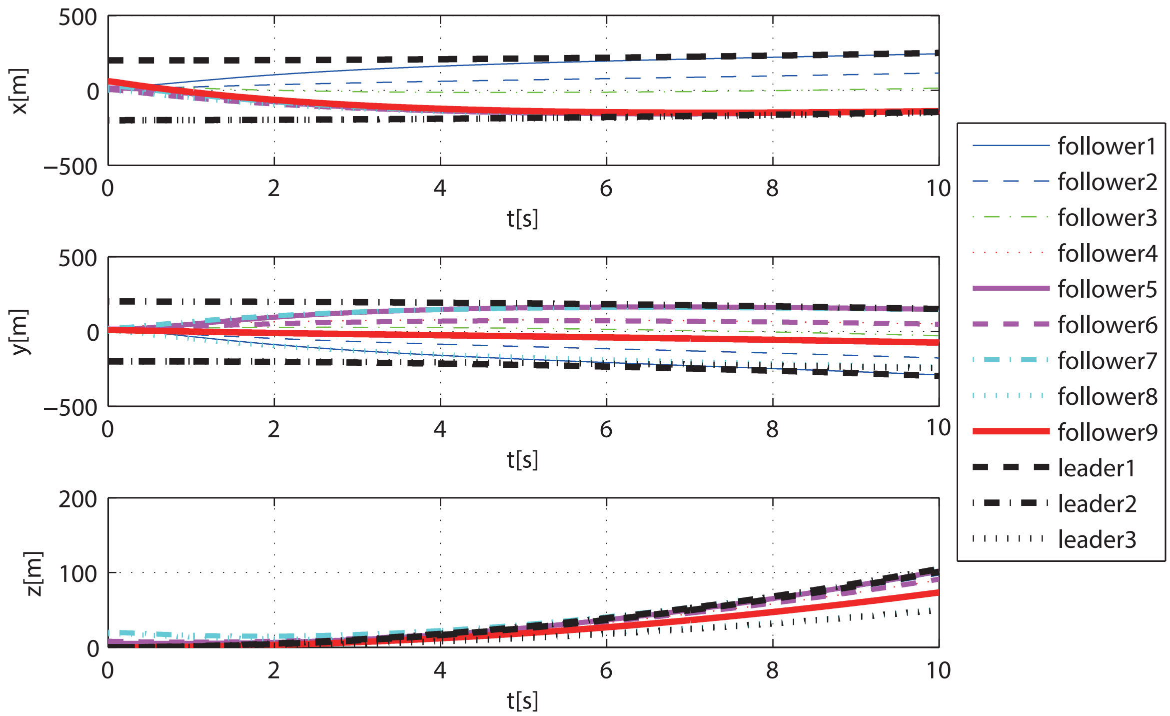
where the dimensions of positions, velocities and accelerations are m, m/s, m/s^2 and x_{x0}, x_{y0}, x_{z0}, v_{x0}, v_{y0}, v_{z0}, ψ_{x0}, ψ_{y0} and ψ_{z0} are the initial positions, initial velocities and initial accelerations in three directions of x, y, z.

From Figure 1, we can obtain $\frac{{k_2}^2}{k_1} > \max\left(\frac{\beta^2}{-\alpha[\alpha^2+\beta^2]}\right) = 0$.

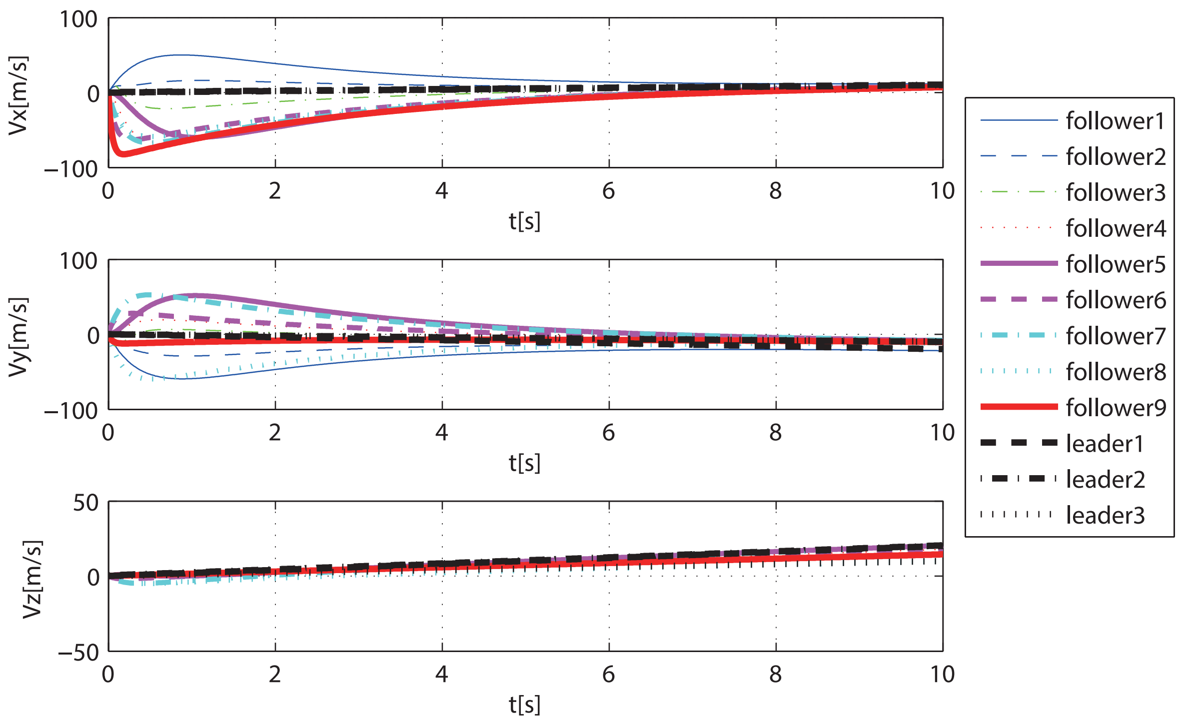
Giving $k_1 = 1$, $k_2 = 3$ and simulation for Figure 1, the positions and velocities of leaders and followers are shown in Figure 2.

Giving $k_1 = -1$, $k_2 = 3$, the positions and velocities of leaders and followers are shown in Figure 3.

In Figure 2, the system is stable, while the system is not stable in Figure 3. Comparing Figure 2 with Figure 3, we obtain that the system is stable only when the condition of Theorem 3.3 is satisfied.

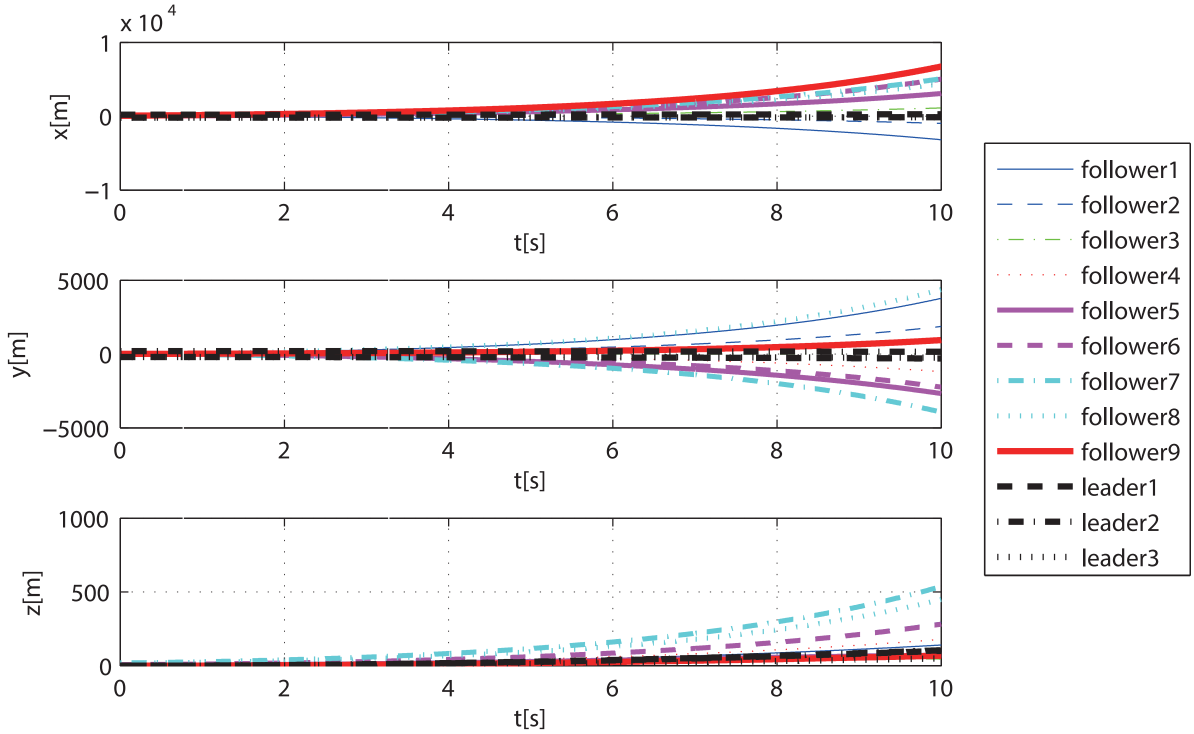


(a) Position

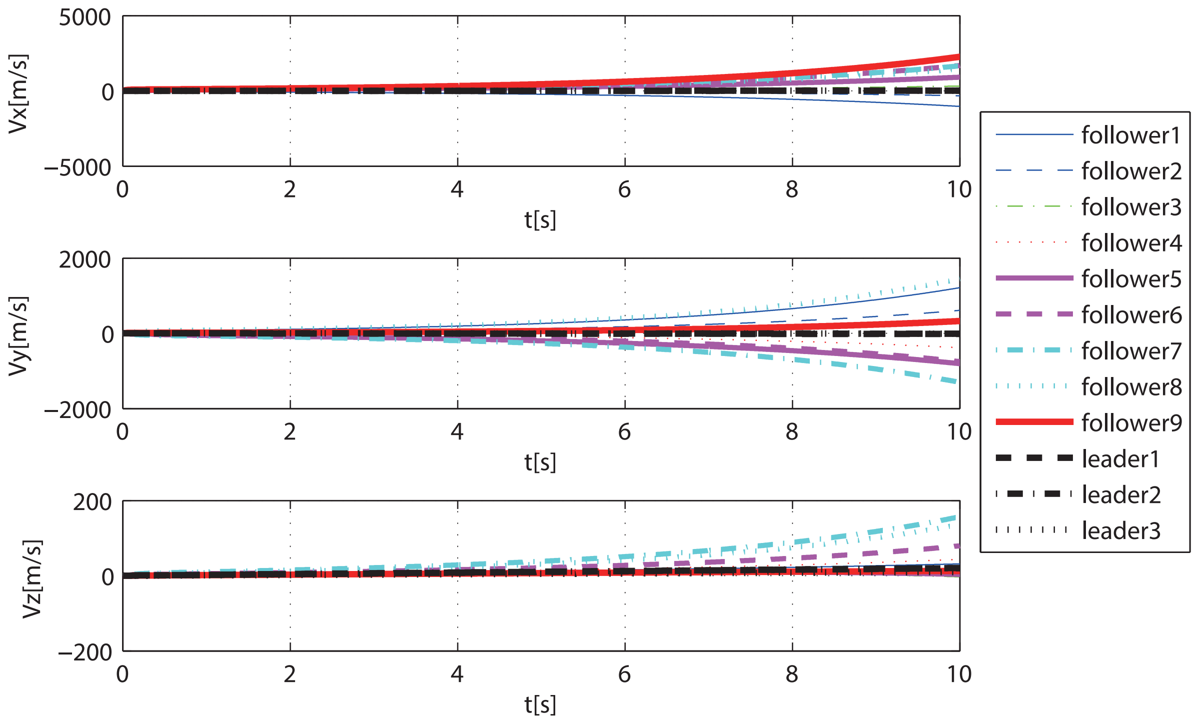


(b) Velocity

FIGURE 2. The positions and velocities of agents



(a) Position



(b) Velocity

Figure 3. The positions and velocities of agents

In Figure 2, considering one direction, the simulation results of $t = 25s$ are

$$\begin{bmatrix} x_{y1} & v_{y1} \\ x_{y2} & v_{y2} \\ x_{y3} & v_{y3} \\ x_{y4} & v_{y4} \\ x_{y5} & v_{y5} \\ x_{y6} & v_{y6} \\ x_{y7} & v_{y7} \\ x_{y8} & v_{y8} \\ x_{y9} & v_{y9} \\ x_{y10} & v_{y10} \\ x_{y11} & v_{y11} \\ x_{y12} & v_{y12} \end{bmatrix} = \begin{bmatrix} -817.234 & -49.767 \\ -614.722 & -41.470 \\ -394.391 & -34.893 \\ -208.864 & -24.877 \\ -110.628 & -24.978 \\ -209.375 & -24.873 \\ -111.653 & -24.970 \\ -501.523 & -24.588 \\ -334.791 & -24.753 \\ -819.252 & -49.760 \\ -112.125 & -24.980 \\ -502.054 & -24.577 \end{bmatrix}, \tag{21}$$

where x_{yi} and v_{yi} $(i = 1, 2, \ldots, 9)$ are the positions and velocities of the followers of the simulation result.

Using the positions and velocities of the leaders of (21), the positions and velocities of the followers can be calculated by (6),

$$\begin{bmatrix} x'_{y1} & v'_{y1} \\ x'_{y2} & v'_{y2} \\ x'_{y3} & v'_{y3} \\ x'_{y4} & v'_{y4} \\ x'_{y5} & v'_{y5} \\ x'_{y6} & v'_{y6} \\ x'_{y7} & v'_{y7} \\ x'_{y8} & v'_{y8} \\ x'_{y9} & v'_{y9} \end{bmatrix} = \begin{bmatrix} -817.252 & -49.760 \\ -614.732 & -41.466 \\ -394.396 & -34.892 \\ -208.857 & -24.879 \\ -110.625 & -24.980 \\ -209.357 & -24.879 \\ -111.625 & -24.980 \\ -501.554 & -24.577 \\ -334.799 & -24.750 \end{bmatrix}, \tag{22}$$

where x'_{yi} and v'_{yi} $(i = 1, 2, \ldots, 9)$ are the positions and velocities of the followers by calculating.

Through the numerical analysis of (21) and (22), it can be seen that the positions and velocities of the leaders and followers are satisfied with (6) in the range of allowable error.

5. Conclusions.

In this paper, we have focused on the stability condition and stable state for multi-agent systems with multiple leaders of variable velocities. Utilizing eigenvalues of Laplacian matrix and control theory, we have proved that the stability of the system depends on the control parameters and the communication topology. An explicit characterization of the stable state has been given: The stable velocities of followers will converge to the convex hull constituted by velocities of leaders, while the stable positions of followers will be the linear combination of the convex hull constituted by positions of leaders and an offset relating to the accelerations of leaders. Numerical simulation has shown the effectiveness of the theoretical results that demonstrates the effect of the control parameters on the stability and the relation between the stable state of followers and the state of leaders.

In particular, an extension of the results presented in this paper is how to solve the problem of the obstacle avoidance where the obstacle is moving randomly by using the result of this paper.

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