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CLOSED-FORM APPROXIMATION OF LARGE SCALE MIMO CHANNEL CAPACITY AT HIGH SNR REGIME

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Abstract. *The channel capacity of a wireless multiple-input multiple-output (MIMO) communication link subject to i.i.d. Rayleigh fading is a critical performance benchmark for the study of many variations of MIMO systems. Several closed-form approximations exist to express the MIMO capacity as a function of the numbers of transmit/receive antennas and the signal-to-noise ratio (SNR). However, these approximations cannot decouple the impacts of antenna numbers and SNR on the MIMO capacity. This paper first proposes a new approximation formula for MIMO channel capacity at the high SNR regime. Moreover, several properties of MIMO channel capacity at the high SNR regime are theoretically proved to validate our approximation. The merit of the proposed approximation is that it provides structural insights into the independent impacts of antenna numbers and SNR on the capacity. This merit is particularly useful for the theoretical study of distributed MIMO and massive MIMO systems at the system level.*

Keywords: Channel capacity, Massive MIMO, Distributed MIMO, Capacity approximation

1. **Introduction.** Massive MIMO technology is considered to be a key enabling technology for the next generation cellular systems and has attracted significant research attention in recent years [1, 2]. A massive MIMO system uses advanced signal processing to exploit the spatial dimension of wireless propagation channels and create multiple parallel channels for high-speed data transmission. An ideal massive MIMO system requires both ends of a communication link to be equipped with large numbers of antennas. However, although it is possible to deploy a large antenna array at base stations (BSs), it is unrealistic to install a large antenna array at mobile terminals due to size and cost constraints. To this end, distribute multiple-input multiple-output (D-MIMO) technology [3, 4] promises to achieve performance gains comparable to conventional MIMO systems by allowing distributed wireless devices to collaborate with each other and form virtual antenna arrays (VAAs). As a result, it is widely expected that massive distributed MIMO transmission with large BS antenna array and random-size user-end antenna array will become a common configuration in future cellular systems.

The performance of an MIMO transmission link is evaluated by its channel capacity. Capacity study reveals the theoretical upper bound of MIMO systems and has been a central theme of MIMO research. There is a wealth of literature regarding MIMO channel capacity, e.g., [5, 6, 7, 8, 9, 10, 11, 12, 13], including studies with respect to different MIMO transmission schemes (e.g., distributed or conventional), different channel models (e.g., Rayleigh or Rician, correlated or uncorrelated, wideband or narrowband), different channel information status (i.e., with or without channel information), different definitions (e.g., ergodic capacity or outage capacity), different asymptotic assumptions (e.g., high/low SNR, very large antenna size), and different levels (e.g., link level and system level). Although various MIMO capacity formulas have been derived for different types of MIMO channel models, to our best knowledge, none of those formulae is able to decouple the impacts of transmit antenna number, receive antenna numbers, and SNR on the capacity, which means the capacity gains originated from increasing the antenna size and increasing SNR cannot be separately quantified.

In this paper, we propose a new approximation formula for ergodic MIMO channel capacity under an i.i.d. Rayleigh channel model in the high SNR regime. Our formula gives structural insights into the separate impacts of transmit/receive antenna numbers and SNR on the MIMO capacity and provides necessary analytical tractability for the system level capacity study of massive distributed MIMO systems. We prove several analytical properties of our approximation. Numerical results show that our approximation is valid for the high SNR regime for different settings of antenna number configurations.

The remainder of the paper is organized as follows. Section 2 proposes an approximation of the MIMO channel capacity and evaluates the accuracy of the approximation. Section 3 proves several critical properties of the approximation. Finally, conclusions and future work are drawn in Section 4.

2. **Approximation of MIMO Channel Capacity.**

We focus on the most fundamental form of MIMO channel, which is the i.i.d. uncorrelated Rayleigh fading MIMO channel model. An accurate closed-form approximation of the MIMO capacity is given by [5]

$$C(t,r,\chi) \approx -\frac{t}{\ln(2)}\left[-(1+\beta)\ln(\sqrt{\chi}) + q_0(\chi)r_0(\chi) + \ln(r_0(\chi)) + \beta\ln\left(q_0(\chi)/\beta\right)\right] \quad (1)$$

where t and r are numbers of transmit and receive antennas, respectively, χ is the SNR, and $\beta = r/t$. In (1), we have

$$q_0(\chi) = \frac{-1-u(\chi)+v(\chi)}{2\sqrt{\chi}} \quad (2)$$

$$r_0(\chi) = \frac{-1+u(\chi)+v(\chi)}{2\sqrt{\chi}} \quad (3)$$

where $u(\chi) = \chi(1-\beta)$ and $v(\chi) = \sqrt{1+2\chi(1+\beta)+\chi^2(1-\beta)^2}$. This accurate approximation has been derived by assuming a large number of t and r. However, its accuracy has been deemed acceptable even for a small number of antennas [5]. Our numerical tests show that the approximation error is lower than 1% given $r \geq 4$.

We notice that in (1), the impacts of antenna numbers and SNR are not decoupled. Motivated by the fact that an accurate and invertible formula exists to calculate the MIMO capacity in the case of symmetrical antennas (i.e., $t = r$) [6], we can rewrite the MIMO capacity as

$$C(t,r,\chi) = \min(t,r)\mathrm{G}(\chi) + \Delta(t,r,\chi) \quad (4)$$

where the function $\mathrm{G}(x)$ is defined as [6]

$$\mathrm{G}(x) = 2\log_2\left[1 + x - \frac{1}{4}\mathrm{L}(x) - \frac{1}{\ln(2)4x}\mathrm{L}(x)\right] \quad (5)$$

with $\mathrm{L}(x) = \left(\sqrt{4x+1}-1\right)^2$.

In the right hand side of (4), the first item is the capacity in the symmetrical antenna case and the second item is the capacity difference. Function $\mathrm{G}(x)$ has a desirable property as being analytically invertible, i.e.,

$$\mathrm{G}^{-1}(x) = -\frac{1}{4} + \frac{1}{4}\left\{1 + \left[\mathrm{W}_0\left(-2^{-\left(\frac{x}{2}+1\right)}e^{-\frac{1}{2}}\right)\right]^{-1}\right\}^2 \quad (6)$$

where $\mathrm{W}_0(x)$ denotes the real branch of the Lambert function. The Lambert W function is the inverse function of $f(w) = we^w$ and satisfies $\mathrm{W}(z)e^{\mathrm{W}(z)} = z$ with complex values of

w and z. Its real branch W_0 maps inputs from interval $\left[-e^{-1}, +\infty\right)$ to interval $[-1, +\infty)$ and is a monotonically increasing function. The aim of this paper is to find a closed-form approximation for the second item given by $\Delta(t, r, \chi) = C(t, r, \chi) - \min(t, r)\mathrm{G}(\chi)$.

It is well known that in the high SNR regime, the MIMO capacity scales linearly with the number of antenna pairs. This suggests that $\Delta(t, r, \chi)$ should approach a constant limit $\Delta^*(t, r)$ when SNR χ tends to infinity. In Figure 1, for a large SNR $\chi = 100$ dB, we compute $\Delta^*(t, r)$ numerically as a function of t with varying r. It is found that $\Delta^*(t, r)$ is a bell-shape curve that exhibits four interesting properties:

- First, there is no capacity difference when $t = 0$ or $t = r$, i.e., $\Delta^*(t = 0, r) = 0$ and $\Delta^*(t = r, r) = 0$;
- Second, $\Delta^*(t = r/2, r) = r$;
- Third, setting $t = r/2$ results in the maximum value of capacity difference;
- Fourth, the capacity difference as a function of t is a symmetric curve centered at $t = r/2$.

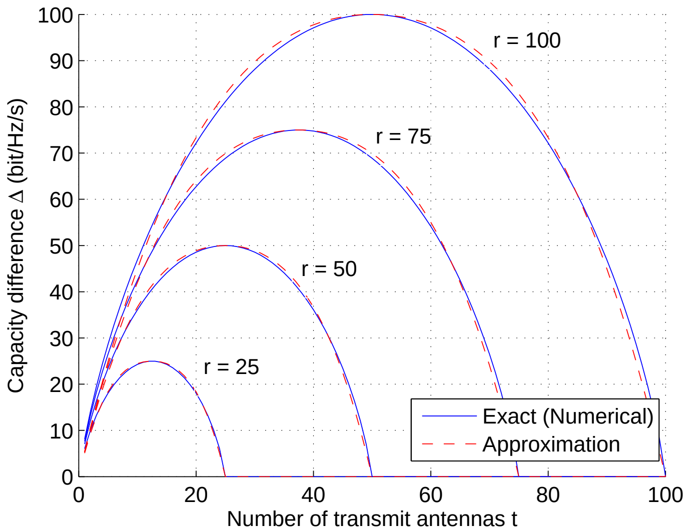

FIGURE 1. MIMO capacity difference $\Delta(t, r, \chi)$ as a function of transmit antenna number t ($\chi = 100$ dB, simplified approximation based on Equation (8))

These four properties inspire us to approximate $\Delta^*(t, r)$ with $\Delta^{apx}(t, r)$, which is given by the following power series

$$\Delta^{apx}(t, r) = r - \sum_{n=2}^{\infty} K_n \frac{2^n}{r^{n-1}} |t - r/2|^n, \quad \sum_{n=1}^{N} K_n = 1 \tag{7}$$

where K_n $(n = 1, 2, \ldots, N)$ are weights that can be optimized numerically. For $N = 4$ and by means of numerical fitting, we are able to get convenient values of weights as $K_2 = 0.5$, $K_3 = 0.25$, $K_4 = 0.25$ for a simple approximation that yields satisfactory accuracy. This simple approximation can be written as

$$\Delta^{apx}(t, r) \approx r - \frac{2}{r}(t - r/2)^2 - \frac{4}{r^2}|t - r/2|^3 - \frac{8}{r^3}(t - r/2)^4 \quad (t \le r) \tag{8}$$

When $t > r$, we have $\Delta^*(t, r) = 0$. The accuracy of the simple approximation is shown in Figure 1. Numerical results show that the maximum normalized root square error is smaller than 1.3% for any combinations of t and r. We note that the approximation accuracy can be improved by increasing N and assigning more accurate values for weighting coefficients. For example, letting $N = 6$ and applying the optimum weights as

$K_2 = 1.1966$, $K_3 = -3.9060$, $K_4 = 11.2102$, $K_5 = -13.1721$, and $K_6 = 5.6713$, we are able to improve the accuracy by one order at $\chi = 100$ dB.

Figure 2 further illustrates the impacts of SNR, receiver antenna number r, and the number of weights N on the approximation accuracy, which is evaluated by the root mean square error (normalized by the range of values) given by

$$\Omega = \frac{\sqrt{\frac{1}{r}\Sigma_{t=1}^{r}\left(\Delta^*(t,r,\chi) - \Delta^{apx}(t,r)\right)^2}}{\max_t \Delta^*(t,r,\chi) - \min_t \Delta^*(t,r,\chi)} \tag{9}$$

where we have $\max_t \Delta^*(t,r,\chi) - \min_t \Delta^*(t,r,\chi) = r$ for large values of χ. It is shown that when the SNR $\chi > 43$ dB, the approximation errors all fall below one percent. Moreover, given r and N, there is an error floor which gets smaller with larger values of N and smaller values of r.

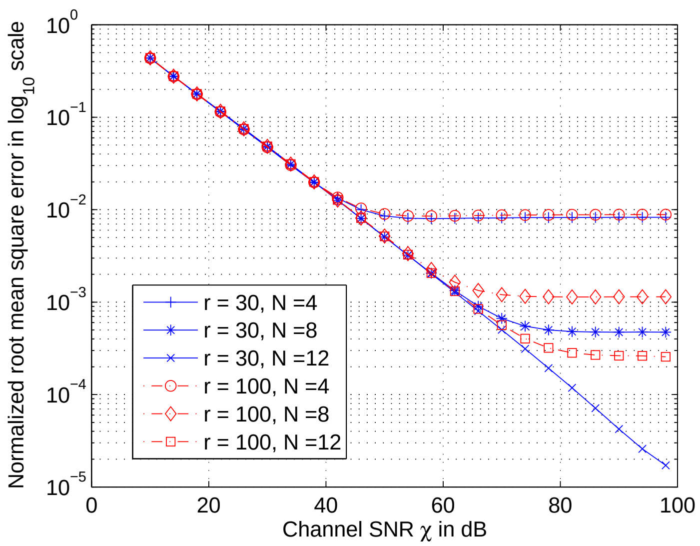

FIGURE 2. Normalized root mean square error as a function of channel SNR χ with varying receive antenna numbers r and number of coefficients N (full approximation based on Equation (7))

3. Proof of Some Properties Used in the Approximation.

In this previous section, we have proposed an approximation that is based on four hypothetical properties observed from numerical results. Property 1 states that $\Delta^*(t = 0, r) = 0$, $\Delta^*(t = r, r) = 0$. Property 2 states that $\Delta^*(t = r/2, r) = r$. Property 3 states that the capacity difference is maximized when $t = r/2$; Property 4 states that the capacity difference is a symmetrical curve centered at $t = r/2$. This section will give a rigid analytical proof of these four properties.

3.1. Proof of Property 1.

When $t = 0$, we have

$$\begin{aligned}\Delta(0,r,\chi) &= C(0,r,\chi) - C\left(0,r/2,\chi\right)\cdot 0 \\ &\cdot\left[-\left(1+\beta\right)\ln\left(\sqrt{\chi}\right) + q_0\left(\chi\right)r_0\left(\chi\right) + \ln\left(r_0\left(\chi\right)\right) + \beta\ln\left(\frac{q_0\left(\chi\right)}{\beta}\right)\right] \\ &- 0\cdot\left[-\left(1+\beta\right)\ln\left(\sqrt{\chi}\right) + q_0\left(\chi\right)r_0\left(\chi\right) + \ln\left(r_0\left(\chi\right)\right) + \beta\ln\left(\frac{q_0\left(\chi\right)}{\beta}\right)\right] = 0.\end{aligned}$$

Moreover, when $t = r$, we have

$$\begin{aligned}\Delta(r,r,\chi) &= C(r,r,\chi) - \min(t,r)G(\chi) \\ &= \min(t,r)G(\chi) - \min(t,r)G(\chi) = 0.\end{aligned}$$

This ends the proof of Property 1.

3.2. **Proof of Property 2.** Under the condition of infinite SNR, i.e., $\chi \to \infty$, we have

$$\begin{aligned}\lim_{\chi\to\infty} \Delta(r/2, r, \chi) &= \lim_{\chi\to\infty} [C(r/2, r/2, \chi) - C(r/2, r, \chi)] \\ &= \lim_{\chi\to\infty} C(r/2, r, \chi) - \lim_{\chi\to\infty} C(r/2, r/2, \chi) \end{aligned} \tag{10}$$

where

$$\begin{aligned}&\lim_{\chi\to\infty} C(r/2, r, \chi) \\ &= \lim_{\chi\to\infty} -\frac{r}{2\ln(2)} \left[-3\ln(\sqrt{\chi}) + q_0(\chi)r_0(\chi) + \ln(r_0(\chi)) + 2\ln\left(\frac{q_0(\chi)}{2}\right)\right]\end{aligned} \tag{11}$$

Substituting the various symbol definitions into (11) we get

$$\begin{aligned}&= \lim_{\chi\to\infty} -\frac{r}{2\ln(2)} \left[-3\ln(\sqrt{\chi}) + \frac{1 - 2v(\chi) + v^2(\chi) - u^2(\chi)}{4\chi} + \ln\left(\frac{r_2(\chi)q_0(\chi)^2}{4}\right)\right] \\ &= \lim_{\chi\to\infty} -\frac{r}{2\ln(2)} \left[\frac{1 - 2v(\chi) + v^2(\chi) - u^2(\chi)}{4\chi} + \ln\left(r_0(\chi)q_0(\chi)\frac{q_0(\chi)}{4\chi^{\frac{3}{2}}}\right)\right] \\ &= \lim_{\chi\to\infty} -\frac{r}{2\ln(2)} \left[\frac{1 - 2\sqrt{1 + 6\chi + \chi^2} + 1 + 6\chi + \chi^2 - \chi^2}{4\chi} + \ln\left(r_0(\chi)q_0(\chi)\frac{q_0(\chi)}{4\chi^{\frac{3}{2}}}\right)\right] \\ &= \lim_{\chi\to\infty} -\frac{r}{2\ln(2)} \left[1 + \ln\left(r_0(\chi)q_0(\chi)\frac{q_0(\chi)}{4\chi^{\frac{3}{2}}}\right)\right]\end{aligned} \tag{12}$$

So far we have obtained the first term of the right-hand side of Equation (10). The second term of the right hand side gives

$$\begin{aligned}\lim_{\chi\to\infty} C(r/2, r/2, \chi) &= \lim_{\chi\to\infty} -\frac{r}{2\ln(2)}\Bigg[- 2\ln(\sqrt{\chi}) + \frac{1 - 2v(\chi) + v^2(\chi) - u^2(\chi)}{4\chi} \\ &\quad + \ln\left(\frac{1 - 2v(\chi) + v^2(\chi) - u^2(\chi)}{4\chi}\right)\Bigg] \\ &= \lim_{\chi\to\infty} -\frac{r}{2\ln(2)}\Bigg[- 2\ln(\sqrt{\chi}) + \frac{1 - 2\sqrt{1 + 4\chi} + 1 + 4\chi}{4\chi} \\ &\quad + \ln\left(\frac{1 - 2\sqrt{1 + 4\chi} + 1 + 4\chi}{4\chi}\right)\Bigg] \\ &= \lim_{\chi\to\infty} -\frac{r}{2\ln(2)} [-2\ln(\sqrt{\chi}) + 1]\end{aligned} \tag{13}$$

Substituting (12) and (13) into (10) we get

$$\begin{aligned}\lim_{\chi\to\infty} \Delta(t, r, \chi) &= \lim_{\chi\to\infty} [C(r/2, r, \chi) - C(r/2, r/2, \chi)] \\ &= \lim_{\chi\to\infty} \left[-\frac{r}{2\ln(2)} \left(\ln\left(r_0(\chi)q_0(\chi)\frac{q_0(\chi)}{4\chi^{\frac{3}{2}}}\right) + \ln(\chi)\right)\right] \\ &= \lim_{\chi\to\infty} \left[-\frac{r}{2\ln(2)} \left(\ln\left(r_0(\chi)q_0(\chi)\frac{q_0(\chi)}{4\chi^{\frac{1}{2}}}\right)\right)\right].\end{aligned}$$

Further considering the fact that $\beta = 2$, we have

$$\begin{aligned}&= \lim_{\chi\to\infty} \left[\ln\left(\frac{1 - 2\sqrt{1 + 6\chi + \chi^2} + 1 + 6\chi}{4\chi}\frac{-1 + \sqrt{1 + 6\chi + \chi^2} - \chi^2}{4\chi^{\frac{1}{2}}}\right)\right] \\ &= -\frac{r}{2\ln(2)}\ln\left(\frac{1}{4}\right) = r\end{aligned} \tag{14}$$

3.3. **Proof of Property 3.** Now we will prove that when $\chi \to \infty$, $\Delta_{\max}(t, r, \chi)$ achieves the maximum value when $t = r/2$. We first express the capacity difference as a function of β as follows

$$\begin{aligned}
&\Delta_{\max}(t, r, \chi) \\
&= C(t, r, \chi) - \min(t, r) G(\chi) \\
&= \lim_{\chi\to\infty} -\frac{t}{\ln(2)} \left[-(1+\beta)\ln(\sqrt{\chi}) + q_0(\chi) r_0(\chi) + \ln(r_0(\chi)) + \beta \ln\left(\frac{q_0(\chi)}{\beta}\right) \right] \\
&\quad - \lim_{\chi\to\infty} C(r, r, \chi) \\
&= \lim_{\chi\to\infty} -\frac{t}{\ln(2)} \left[\ln\left(\frac{1}{\sqrt{\chi}}\right)^{1+\beta} + 1 + \ln(r_0(\chi)) + \beta \ln\left(\frac{q_0(\chi)}{\beta}\right) \right] - \lim_{\chi\to\infty} C(r, r, \chi) \\
&= \lim_{\chi\to\infty} -\frac{t}{\ln(2)} \left[\ln\left(q_0(\chi) r_0(\chi) \left(\frac{q_0(\chi)}{\sqrt{\chi}}\right)^{\beta-1} \frac{1}{\chi} \left(\frac{1}{\beta}\right)^{\beta} \right) + 1 \right] - \lim_{\chi\to\infty} C(r, r, \chi) \\
&= -\frac{r}{\beta\ln(2)} \left[(\beta-1)\ln(\beta-1) + \ln\left(\frac{1}{\chi}\right) - \beta\ln(\beta) + 1 \right] - \left[-\frac{r}{\beta\ln(2)} \left(\ln\left(\frac{1}{\chi}\right) + 1 \right) \right] \\
&= -\frac{r}{\ln(2)} \left[\frac{1}{\beta}(\beta-1)\ln(\beta-1) - \ln(\beta) \right] \qquad (15)
\end{aligned}$$

The maximum value of (15) is a function of β. Taking the first-order derivative of the equation $\frac{1}{\beta}(\beta-1)\ln(\beta-1) - \ln(\beta)$ yields

$$\frac{1}{\beta^2}\ln(\beta-1) \qquad (16)$$

It follows that $\frac{1}{\beta^2}\ln(\beta-1) = 0$ when $\beta = 2$, which means (15) achieves the maximum value when $t = r/2$.

3.4. **Proof of Property 4.** When $\chi \to \infty$ we have

$$\begin{aligned}
\Delta(t, r, \chi) &= -\frac{t}{\ln(2)} \left[(\beta-1)\ln(\beta-1) - \beta\ln(\beta) \right] \\
&= -\frac{t}{\ln(2)} \left[\frac{r-t}{t} \ln\left(\frac{r-t}{t}\right) - \frac{r}{t}\ln\left(\frac{r}{t}\right) \right] \\
&= -\frac{t}{\ln(2)} \left[\ln\left(\frac{t}{r-t}\right) - \frac{r}{t}\ln\left(\frac{r}{r-t}\right) \right] \qquad (17)
\end{aligned}$$

$$\begin{aligned}
\Delta(r-t, r, \chi) &= -\frac{r-t}{\ln(2)} \left[\left(\frac{r}{r-t} - 1\right) \ln\left(\frac{r}{r-t} - 1\right) - \frac{r}{r-t}\ln\left(\frac{r}{r-t}\right) \right] \\
&= -\frac{r-t}{\ln(2)} \left[\frac{t}{r-t}\ln\left(\frac{t}{r-t}\right) - \frac{r}{r-t}\ln\left(\frac{r}{r-t}\right) \right] \\
&= -\frac{1}{\ln(2)} \left[t\ln\left(\frac{t}{r-t}\right) - r\ln\left(\frac{r}{r-t}\right) \right] \\
&= -\frac{t}{\ln(2)} \left[\ln\left(\frac{t}{r-t}\right) - \frac{r}{t}\ln\left(\frac{r}{r-t}\right) \right] \qquad (18)
\end{aligned}$$

It can be observed that (17) and (18) are equal, i.e., $\Delta(t, r, \chi) = \Delta(r-t, r, \chi)$. This proves the symmetric property of $\Delta(t, r, \chi)$.

4. **Conclusions.** In this paper, we have proposed a new closed-form approximation for MIMO channel capacity at the high SNR regime. Our approximation relies on four properties of the MIMO channel capacity at the high SNR regime. These four properties are theoretically proved and the accuracy of our approximation is validated by numerical results. Our approximation can provide structural insights into the separate impacts of transmit antenna number, receive antenna number, and SNR on the capacity, making the approximation useful for the theoretical investigation of the system level performance of large scale distributed massive MIMO systems. Future work can aim to find better approximations in the medium and low SNR regimes, which dominate practical scenarios.

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