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# INTEGRAL OF FUZZY MAPPING ON CURVE SEGMENT AND ITS APPLICATION

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Abstract. *In this paper, the concept of integral of fuzzy number valued mapping on curve segment is defined, and its properties and existence and calculation are investigated. Some results on its property are obtained, and the existence of the integral of continuous fuzzy number valued mapping on piecewise smooth curve segment is proved, and its calculating formulas are set up. In addition, it is proved that such integral for trapezoidal fuzzy number valued mapping is still a trapezoidal fuzzy number, and its specific calculating formula is also obtained. At last, we give an example to show the application of the proposed theory.*
**Keywords:** Curve segment, Fuzzy number, Fuzzy mapping, Integral

1. **Introduction.** In the 1972, Chang and Zadeh introduced the concept of fuzzy number in [1]. The theory of fuzzy numbers is largely well understood. With the development of theory of fuzzy numbers and its application, the concept of fuzzy number becomes more and more important. Many studies have been done on the integral for fuzzy mapping from the real line $R$ into the fuzzy number space $E$. For example, in 1982, Dubois and Prade introduced the concept of integral of fuzzy mapping on a crisp real interval in [2]; in 1986, Goetschel and Voxman presented the definition of integral on an interval $[a, b]$ in a way similar to the definition for real Riemann integration in [3]; in 2001, Wu and Gong investigated the nonabsolute integral of fuzzy mapping from $[a, b]$ into $E$ in [4]. In addition, other kinds of integral for fuzzy number valued mapping have been also studied. For example, in 2004, Wang and Wu dealt with the integral for fuzzy number valued mapping from a directed line segment $[u_0, v_0]$ of $E$ into $E$, and applied the proposed theory to obtaining digital information in an uncertain or imprecise environment in [5]; in 2007, Li and Wu studied the integral of fuzzy mapping on directed line in [6]. In 2012, Li and Wang researched Lebesgue integral of a fuzzy closed set-valued stochastic process with respect to the time $t$ in [7]; in 2015, Musial studied the decomposition theorem for Henstock integral in Banach space in [8]. In the above studies, the authors researched different kinds of fuzzy integral and solved them. While the fuzzy curve integral is studied very few in $E^3$. Here a new method is proposed and a very simple fuzzy integral calculation method in $E^3$ is given.

The purpose of this paper is to deal with the problems of integral of fuzzy number valued mapping on curve segment. In Section 2, we briefly review some basic notions, definitions and results about fuzzy numbers. In Section 3, we define the integral of fuzzy number valued mapping on curve segment, and give some properties about such integral. In Section 4, we prove the existence for continuous fuzzy number valued mapping on piecewise smooth curve segment, and set up its calculating formula. And we also prove that such integral for trapezoidal fuzzy number valued mapping is still a trapezoidal fuzzy number, and obtain its specific calculating formula. In Section 5, we give an example to show the application of the proposed theory. At last, we make a summary and propose some further research contents in Section 6.

2. **Basic Definitions and Notations.** Let $R$ be the real number set. If $u : R \to [0,1]$ satisfies the following properties (1)-(4):

(1) $u$ is normal, i.e., $u(x_0) = 1$ for some $x_0 \in R$;

(2) $u$ is fuzzy convex, i.e., $u(rx + (1-r)y) \geq \min\{u(x), u(y)\}$ for all $x, y \in R$ and $r \in [0,1]$;

(3) $u$ is upper semi-continuous;

(4) $\mathrm{cl}\{x \in R : u(x) > 0\}$ (denote it by $[u]^0$), i.e., the closure of $\{x \in R : u(x) > 0\}$ is a compact set $R$, then we call $u$ a fuzzy number, call the collection of all fuzzy numbers the fuzzy number space, and denote it by $E$.

For any $u \in E$ and $r \in [0,1]$, obviously, $[u]^r$ are all intervals, where $[u]^r = \{x \in R : u(x) \geq r\}$, $r \in (0,1]$ (are called $r$-levels). We denote $[u]^r = [\underline{u}(r), \bar{u}(r)]$ for $r \in [0,1]$.

For any $a \in R$, define a fuzzy number $\hat{a}$ by

$$\hat{a}(x) = \begin{cases} 1, & x = a \\ 0, & x \neq a \end{cases}$$

The operations of addition, scalar product and multiplication in $E$ are defined via the following operations:

$$(u+v)(x) = \sup_{y+z=x} \min[u(y), v(z)]$$

$$(\lambda u)(x) = \begin{cases} u(\lambda^{-1}x), & \lambda \neq 0 \\ \hat{0}(x), & \lambda = 0 \end{cases}$$

$$(uv)(x) = \sup_{yz=x} \min[u(y), v(z)]$$

for $u, v \in E$, $\lambda \in R$.

It is known that for $u, v \in E$, $\lambda \in R$, then $u+v, \lambda u, uv \in E$, and

$$[u+v]^r = [u]^r + [v]^r$$

$$[\lambda u]^r = \lambda [u]^r$$

$$\begin{aligned} [uv]^r = \ & [\min\{\underline{u}(r)\underline{v}(r), \bar{u}(r)\underline{v}(r), \underline{u}(r)\bar{v}(r), \bar{u}(r)\bar{v}(r)\}, \\ & \max\{\underline{u}(r)\underline{v}(r), \bar{u}(r)\underline{v}(r), \underline{u}(r)\bar{v}(r), \bar{u}(r)\bar{v}(r)\}] \end{aligned}$$

for every $r \in [0,1]$.

$A$ and $B$ are two nonempty bounded close subsets of $R$, $d(A,B)$ denotes the Hausdorff meteic of $A$ and $B$, that is $d(A,B) = \inf\{\varepsilon > 0 : A \subset U(B,\varepsilon), B \subset U(A,\varepsilon)\}$, where $U(C,\varepsilon)$ denotes the neighborhood of $C$ ($C \subset R$) with the radius $\varepsilon$. Obviously, for any $u, v \in E$ and $r \in [0,1]$, $d([u]^r, [v]^r) = \max(|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|)$. For $u, v \in E$, define

$$D(u,v) = \sup_{r \in I} d([u]^r, [v]^r) = \sup_{r \in I} \max(|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|)$$

Let $a_1, a_2, a_3$ and $a_4 \in R$ with $a_1 \leq a_2 \leq a_3 \leq a_4$. If fuzzy number $u : R \to [0,1]$ is defined by

$$u(x) = \begin{cases} 1, & x \in [a_2, a_3] \\ \dfrac{x - a_1}{a_2 - a_1}, & x \in [a_1, a_2) \\ \dfrac{a_4 - x}{a_4 - a_3}, & x \in (a_3, a_4] \\ 0, & x \notin [a_1, a_4] \end{cases}$$

then $u$ is called a trapezoid fuzzy number, and denoted as $u = (a_1, a_2, a_3, a_4)$. Specially, if $a_2 = a_3$, $u$ is called a triangle fuzzy number, and denoted $u = (a_1, a_2, a_4)$. And we denote the collection of all trapezoid fuzzy numbers by $E_{Tra}$, and the collection of all triangle fuzzy numbers by $E_{Tri}$.

Let $\alpha, \beta \in R$ with $\alpha < \beta$. $\Gamma = \{(x, y, z) : x = x(t), y = y(t), z = z(t), t \in [\alpha, \beta]\}$ is called a curve segment of $R^3$ (it can be also denoted as $\Gamma : \begin{cases} x = x(t) \\ y = y(t) \quad t \in [\alpha, \beta] \\ z = z(t) \end{cases}$ ).

If $x(t), y(t)$ and $z(t)$ are all derivable, and $[x'(t)]^2+[y'(t)]^2+[z'(t)]^2 \neq 0$ for any $t \in [\alpha, \beta]$, then $\Gamma$ is called smooth. If $\Gamma$ is made up by some smooth curve segments, then $\Gamma$ is said to be piecewise smooth.

In this paper, we call $F : \Gamma \to E$ a fuzzy number valued mapping on curve segment $\Gamma$. If there exist $u_0, v_0 \in E$ such that $u_0 \leq F(x, y, z) \leq v_0$ for any $(x, y, z) \in \Gamma$, then $F$ is said to be bounded on $\Gamma$.

3. **Integral of Fuzzy Number Valued Mapping on Curve Segment.** In the following, we give the definition of integral of $F$ on curve segment $\Gamma$.

**Definition 3.1.** *Suppose $\Gamma$ is a smooth curve segment and $F$ is bounded on $\Gamma$. We arbitrarily take $n-1$ partition nodes $M_1, M_2, \ldots, M_{n-1}$ on $\Gamma$ in turn, and denote $P = \{M_0, M_1, \ldots, M_{n-1}, M_n\}$ ($P$ is call a partition of $\Gamma$), where $M_0 = (x(\alpha), y(\alpha), z(\alpha))$ and $M_n = (x(\beta), y(\beta), z(\beta))$. Then $\Gamma$ is divided into $n$ sections (denoted as $\Gamma_{M_0M_1}, \Gamma_{M_1M_2}, \ldots, \Gamma_{M_{n-1}M_n}$) by $P$. Let $\Delta s_i$ be the length of $\Gamma_{M_{i-1}M_i}$ $(i = 1, 2, \ldots, n)$, and $||P|| = \max\{\Delta s_i : i = 1, 2, \ldots, n\}$. If $\lim_{||P||\to 0} \sum_{i=1}^{n} F(\zeta_i, \eta_i, \tau_i)\Delta s_i$ exists for any $(\zeta_i, \eta_i, \tau_i) \in \Gamma_{M_{i-1}M_i}$ and any partition $P$, i.e., there exists $U_0 \in E$ satisfying that $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $D\left(\sum_{i=1}^{n} F(\zeta_i, \eta_i, \tau_i)\Delta s_i, U_0\right) < \varepsilon$ as $||P|| < \delta$ for any $(\zeta_i, \eta_i, \tau_i) \in \Gamma_{M_{i-1}M_i}$ and any partition $P$, then we say $F$ to be integrable on $\Gamma$, call the limit value the integral of $F$ on $\Gamma$, and denote it as $\int_\Gamma F(x, y, z)ds$, i.e.,*

$$\int_\Gamma F(x, y, z)ds = \lim_{||P||\to 0} \sum_{i=1}^{n} F(\zeta_i, \eta_i, \tau_i)\, \Delta s_i$$

In the following, we investigate the properties of such integrals.

**Property 3.1.** For any integrable fuzzy number valued mappings $F$ and $G$ on smooth curve segment $\Gamma$ and $k \in R$, $F + G$ and $kF$ are all integrable on $\Gamma$, and

(1) $\int_\Gamma (F(x, y, z) + G(x, y, z))ds = \int_\Gamma F(x, y, z)ds + \int_\Gamma G(x, y, z)ds$;

(2) $\int_\Gamma kF(x, y, z)ds = k \int_\Gamma F(x, y, z)ds$.

**Property 3.2.** Let $\Gamma$ be smooth curve segment, fuzzy number valued mappings $F$ and $G$ be integrable on $\Gamma$. If $F(x, y, z) \leq G(x, y, z)$ for any $(x, y, z) \in \Gamma$, then

$$\int_\Gamma F(x, y, z)ds \leq \int_\Gamma G(x, y, z)ds$$

Properties 3.1 and 3.2 can be directly shown by Definition 3.1, and the proofs of Properties 3.1 and 3.2 are all omitted.

For curve segments $\Gamma$, $\Gamma_1$ and $\Gamma_2$, if they satisfy that $\Gamma = \Gamma_1 \cup \Gamma_2$, and $\Gamma_1$ and $\Gamma_2$ have only one common endpoint, we denote $\Gamma = \Gamma_1 \oplus \Gamma_2$.

We can also obtain the following result by Definition 3.1.

**Property 3.3.** If $\Gamma$, $\Gamma_1$ and $\Gamma_2$ are smooth curve segments with $\Gamma = \Gamma_1 \oplus \Gamma_2$, and $F$ is integrable on $\Gamma_1$ and $\Gamma_2$, then $F$ is integrable on $\Gamma$, and

$$\int_\Gamma F(x, y, z)ds = \int_{\Gamma_1} F(x, y, z)ds + \int_{\Gamma_2} F(x, y, z)ds$$

Property 3.3 inspired us to give the following definition:

**Definition 3.2.** *Let $\Gamma$ be piecewise smooth curve segment, and $\Gamma = \Gamma_1 \oplus \Gamma_2 \oplus \ldots \oplus \Gamma_n$, where $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ are all smooth. We say $F$ to be integrable on $\Gamma$ if and only if $F$ is* *all integrable on* $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$*, and define* $\int_\Gamma F(x,y,z)ds$ *as*

$$\int_\Gamma F(x,y,z)ds = \sum_{i=1}^{n} \int_{\Gamma_i} F(x,y,z)ds$$

**Remark 3.1.** *In Properties 3.1-3.3, if "smooth" is replaced by "piecewise smooth", the conclusions still hold.*

**Property 3.4.** Let $\Gamma$ be piecewise smooth curve segment. If fuzzy number valued mapping $F$ is integrable on $\Gamma$, then for any curve segment $\Gamma_1$ with $\Gamma_1 \subset \Gamma$, $F$ is integrable on $\Gamma_1$. Further, if $F(x,y,z) \geq \hat{0}$ for any $(x,y,z) \in \Gamma$, then $\int_{\Gamma_1} F(x,y,z)ds \leq \int_\Gamma F(x,y,z)ds$.

**Proof:** Let $P_1$, $P_1'$ be two arbitrary partitions of $\Gamma_1$, $(\zeta_1, \eta_1, \tau_1)$, $(\zeta_1', \eta_1', \tau_1')$ are corresponding sets of selected points. Let $P_0$, $P_0'$ be two arbitrary partitions of $\Gamma$, $(\zeta_0, \eta_0, \tau_0)$, $(\zeta_0', \eta_0', \tau_0')$ are corresponding sets of selected points, $P_0, P_0'$ limited on $\Gamma_1$ are $P_1, P_1'$, respectively, and $P_0, P_0'$ are the same out of $\Gamma_1$, $(\zeta_0, \eta_0, \tau_0)$, $(\zeta_0', \eta_0', \tau_0')$ limited on $\Gamma_1$ are $(\zeta_1, \eta_1, \tau_1)$, $(\zeta_1', \eta_1', \tau_1')$ and $(\zeta_0, \eta_0, \tau_0)$, $(\zeta_0', \eta_0', \tau_0')$ are the same out of $\Gamma_1$. Denote $P_0 \setminus P_1$, $P_0' \setminus P_1'$ are the partitions of $\Gamma \setminus \Gamma_1$, $(\zeta_0, \eta_0, \tau_0) \setminus (\zeta_1, \eta_1, \tau_1), (\zeta_0', \eta_0', \tau_0') \setminus (\zeta_1', \eta_1', \tau_1')$ are the selected points of $\Gamma \setminus \Gamma_1$. By the integrability of $F(x,y,z)$, when $||P_0|| \to 0$, $||P_0'|| \to 0$,

$$\begin{aligned}
&D(S_\Gamma(P_0, (\zeta_0, \eta_0, \tau_0)), S_\Gamma(P_0', (\zeta_0', \eta_0', \tau_0'))) \\
\leq\, &D(S_\Gamma(P_0, (\zeta_0, \eta_0, \tau_0)), \int_\Gamma F(x,y,z)ds) + D(S_\Gamma(P_0', (\zeta_0', \eta_0', \tau_0')), \int_\Gamma F(x,y,z)ds) \\
\to\, &0
\end{aligned}$$

From

$$\begin{aligned}
&D(S_\Gamma(P_0, (\zeta_0, \eta_0, \tau_0)), S_\Gamma(P_0', (\zeta_0', \eta_0', \tau_0'))) \\
=\, &D(S_{\Gamma_1}(P_1, (\zeta_1, \eta_1, \tau_1)) + S_{\Gamma\setminus\Gamma_1}(P_0 \setminus P_1, (\zeta_0, \eta_0, \tau_0) \setminus (\zeta_1, \eta_1, \tau_1)), \\
&\quad S_{\Gamma_1}(P_1', (\zeta_1', \eta_1', \tau_1')) + S_{\Gamma\setminus\Gamma_1}(P_0' \setminus P_1', (\zeta_0', \eta_0', \tau_0') \setminus (\zeta_1', \eta_1', \tau_1'))) \\
=\, &D(S_{\Gamma_1}(P_1, (\zeta_1, \eta_1, \tau_1)), S_{\Gamma_1}(P_1', (\zeta_1', \eta_1', \tau_1')))
\end{aligned}$$

we see

$$\begin{aligned}
&\lim_{||P_0||,||P_0'||\to 0} D(S_\Gamma(P_0, (\zeta_0, \eta_0, \tau_0)), S_\Gamma(P_0', (\zeta_0', \eta_0', \tau_0'))) \\
=&\lim_{||P_0||,||P_0'||\to 0} D(S_\Gamma(P_1, (\zeta_1, \eta_1, \tau_1)), S_\Gamma(P_1', (\zeta_1', \eta_1', \tau_1'))) \\
=&\lim_{||P_1||,||P_1'||,||P_0\setminus P_1||,||P_0'\setminus P_1'||\to 0} D(S_{\Gamma_1}(P_1, (\zeta_1, \eta_1, \tau_1)), S_{\Gamma_1}(P_1', (\zeta_1', \eta_1', \tau_1'))) \\
=\,&0
\end{aligned}$$

Since $D(S_{\Gamma_1}(P_1, (\zeta_1, \eta_1, \tau_1)), S_{\Gamma_1}(P_1', (\zeta_1', \eta_1', \tau_1')))$ is independent of $P_0 \setminus P_1, P_0' \setminus P_1'$,

$$\lim_{||P_0||,||P_0'||\to 0} D(S_{\Gamma_1}(P_1, (\zeta_1', \eta_1', \tau_1')), S_{\Gamma_1}(P_1', (\zeta_1', \eta_1', \tau_1'))) = 0$$

Thus $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall ||P_1||, ||P_1'|| < \delta$,

$$D\left(S_{\Gamma_1}(P_1, (\zeta_1, \eta_1, \tau_1)), S_{\Gamma_1}(P_1', (\zeta_1', \eta_1', \tau_1'))\right) < \varepsilon$$

So $\forall \varepsilon > 0$, take $P, P' \in \Omega$ with $||P|| < \delta, ||P'|| < \delta$, then $\forall P \subseteq P_1, P' \subseteq P_1'$, we have

$$D(S_{\Gamma_1}(P_1, (\zeta_1, \eta_1, \tau_1)), S_{\Gamma_1}(P_1', (\zeta_1', \eta_1', \tau_1'))) < \varepsilon$$

where $\Omega$ is the set of partitions of $\Gamma_1$. By the above proof, $S_{\Gamma_1}(P, (\zeta, \eta, \tau))_{P\in\Omega}$ is a Cauchy net in $E$. Since $(E, D)$ is a complete space, there is a $u \in E$, such that net $S_{\Gamma_1}(P, (\zeta, \eta, \tau))_{P\in\Omega}$ converges to $u$ in $(E, D)$, which means $F$ is integrable on $\Gamma_1$ and $\int_{\Gamma_1} F(x,y,z)ds = u$.

From $F(x,y,z) \geq \hat{0}$, $\forall(x,y,z) \in \Gamma$, we know for any $(\zeta_i, \eta_i, \tau_i) \in \Gamma_2$, $F(\zeta_i, \eta_i, \tau_i) \geq \hat{0}$, so

$$\int_{\Gamma_2} F(x,y,z)ds = \lim_{||P|| \to 0} \sum_{i=1}^{n} F(\zeta_i, \eta_i, \tau_i)\Delta s_i \geq \hat{0}$$

It implies

$$\int_{\Gamma} F(x,y,z)ds - \int_{\Gamma_1} F(x,y,z)ds = \int_{\Gamma_2} F(x,y,z)ds \geq \hat{0}$$

So

$$\int_{\Gamma_1} F(x,y,z)ds \leq \int_{\Gamma} F(x,y,z)ds$$

□

## 4. **Calculation of Integral.**

**Theorem 4.1.** $\Gamma : \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$ $t \in [\alpha, \beta]$ *is a piecewise smooth curve segment,* $F(x,y,z)$ *is a continuous function on* $\Gamma$*. Then*

$$\int_{\Gamma} F(x,y,z)ds = \int_{\alpha}^{\beta} F(x(t),y(t),z(t))\sqrt{x'^2(t) + y'^2(t) + z'^2(t)}dt$$

**Proof:** As we know, the length of curve segment from $(x(t_{i-1}), y(t_{i-1}), z(t_{i-1}))$ to $(x(t_i), y(t_i), z(t_i))$ is

$$\Delta s_i = \int_{t_{i-1}}^{t_i} \sqrt{x'^2(t) + y'^2(t) + z'^2(t)}dt$$

Since the continuity of $\sqrt{x'^2(t) + y'^2(t) + z'^2(t)}$ by the mean value theorem of integrals, we have

$$\Delta s_i = \sqrt{x'^2(\tau_i') + y'(\tau_i'^2) + z'^2(\tau_i')}\Delta t_i \qquad (t_{i-1} < \tau_i' < t_i)$$

So

$$\sum_{i=1}^{n} F(\zeta, \eta, \tau)\Delta s_i = \sum_{i=1}^{n} F(x(\tau_i''), y(\tau_i''), z(\tau_i''))\sqrt{x'^2(\tau_i') + y'^2(\tau_i') + z'^2(\tau_i')}\Delta t_i$$
$$(t_{i-1} < \tau_i', \tau_i'' < t_i)$$

Suppose

$$\tilde{\sigma} = \sum_{i=1}^{n} F(x(\tau_i''), y(\tau_i''), z(\tau_i''))\Big[\sqrt{x'^2(\tau_i') + y'^2(\tau_i') + z'^2(\tau_i')} - \sqrt{x'^2(\tau_i'') + y'^2(\tau_i'') + z'^2(\tau_i'')}\Big]\Delta t_i$$

Then

$$\sum_{i=1}^{n} F(\zeta, \eta, \tau)\Delta s_i = \sum_{i=1}^{n} F(x(\tau_i''), y(\tau_i''), z(\tau_i''))\sqrt{x'^2(\tau_i'') + y'^2(\tau_i'') + z'^2(\tau_i'')}\Delta t_i + \tilde{\sigma} \quad (1)$$

Let $\Delta t = \max\{\Delta t_1, \Delta t_2, \ldots, \Delta t_n\}$, when $||P|| \to 0$, we have $\Delta t \to 0$. Now we prove $\lim_{\Delta t \to 0} \tilde{\sigma} = \hat{0}$. As compound function $F(x(t), y(t), z(t))$ is continuous about $t$, it is bounded on closed interval $[\alpha, \beta]$. There exists fuzzy number $\tilde{M} > \hat{0}$, for any $t \in [\alpha, \beta]$, we have

$$-\tilde{M} \leq F(x(t), y(t), z(t)) \leq \tilde{M}$$

As $\sqrt{x'^2(t)+y'^2(t)+z'^2(t)}$ is continuous on $[\alpha,\beta]$, it is uniformly continuous on $[\alpha,\beta]$, that is to say that, $\forall\varepsilon>0$, $\exists\delta>0$, when $\Delta t<\delta$, we have

$$\left|\sqrt{x'^2(\tau_i')+y'^2(\tau_i')+z'^2(\tau_i')}-\sqrt{x'^2(\tau_i'')+y'^2(\tau_i'')+z'^2(\tau_i'')}\right|<\varepsilon$$

So

$$-\varepsilon\tilde{M}(b-a)\le\tilde{\sigma}\le\varepsilon\tilde{M}\sum_{i=1}^{n}\Delta t_i=\varepsilon\tilde{M}(b-a)$$

So

$$\lim_{\Delta t\to 0}\tilde{\sigma}=\hat{0}$$

According to the definition of integral of fuzzy number valued mapping on curve segment, we have

$$\lim_{\Delta t\to 0}\sum_{i=1}^{n}F(x(\tau_i''),y(\tau_i''),z(\tau_i''))\sqrt{x'^2(\tau_i'')+y'^2(\tau_i'')+z'^2(\tau_i'')}\Delta t_i$$
$$=\int_\alpha^\beta F(x(t),y(t),z(t))\sqrt{x'^2(t)+y'^2(t)+z'^2(t)}dt$$

So for Equation (1), we take the limitation on both sides of the equation, and we can get the result that we want. □

**Theorem 4.2.** *Suppose* $\Gamma:\begin{cases}x=x(t)\\y=y(t)\\z=z(t)\end{cases}$ $t\in[\alpha,\beta]$ *is a piecewise smooth curve segment,* $F:\Gamma\to E$ *be continuous. Then* $\int_\Gamma F(x,y,z)ds$ *exists, and for any* $r\in[0,1]$*, we have*

$$\left[\int_\Gamma F(x,y,z)ds\right]^r$$
$$=\left[\int_\alpha^\beta\sqrt{x'^2(t)+y'^2(t)+z'^2(t)}\,\underline{F(x(t),y(t),z(t))}(r)dt,\right.$$
$$\left.\int_\alpha^\beta\sqrt{x'^2(t)+y'^2(t)+z'^2(t)}\;\overline{F(x(t),y(t),z(t))}(r)dt\right]$$

**Proof:** By Theorem 4.1, we have

$$\int_\Gamma F(x,y,z)ds=\int_\alpha^\beta F(x(t),y(t),z(t))\sqrt{x'^2(t)+y'^2(t)+z'^2(t)}dt$$

So by Theorem 3.2 in [6], we have

$$\left[\int_\Gamma F(x,y,z)ds\right]^r$$
$$=\left[\int_\alpha^\beta F(x(t),y(t),z(t))\sqrt{x'^2(t)+y'^2(t)+z'^2(t)}dt\right]^r$$
$$=\left[\int_\alpha^\beta\underline{F(x(t),y(t),z(t))\sqrt{x'^2(t)+y'^2(t)+z'^2(t)}}(r)dt,\right.$$
$$\left.\int_\alpha^\beta\overline{F(x(t),y(t),z(t))\sqrt{x'^2(t)+y'^2(t)+z'^2(t)}}(r)dt\right]$$
$$=\left[\int_\alpha^\beta\sqrt{x'^2(t)+y'^2(t)+z'^2(t)}\,\underline{F(x(t),y(t),z(t))}(r)dt,\right.$$
$$\left.\int_\alpha^\beta\sqrt{x'^2(t)+y'^2(t)+z'^2(t)}\;\overline{F(x(t),y(t),z(t))}(r)dt\right]$$

□

**Theorem 4.3.** *Suppose* $\Gamma : \begin{cases} x = x(t) \\ y = y(t) \quad t \in [\alpha, \beta] \\ z = z(t) \end{cases}$ *is a piecewise smooth curve segment,* $\Gamma : F \to E_{Tra}$ *is continuous on* $\Gamma$ *with* $F(x(t), y(t), z(t)) = \left(\underline{f}_0(t), \underline{f}_1(t), \overline{f}_1(t), \overline{f}_0(t)\right)$ $(\forall t \in [\alpha, \beta])$. *Then*

$$\begin{aligned}&\int_\Gamma F(x,y,z)ds \\ &= \Bigg(\int_\alpha^\beta \sqrt{x'^2(t) + y'^2(t) + z'^2(t)}\underline{f}_0(t)dt, \int_\alpha^\beta \sqrt{x'^2(t) + y'^2(t) + z'^2(t)}\underline{f}_1(t)dt, \\ &\quad \int_\alpha^\beta \sqrt{x'^2(t) + y'^2(t) + z'^2(t)}\overline{f}(t)dt, \int_\alpha^\beta \sqrt{x'^2(t) + y'^2(t) + z'^2(t)}\overline{f}_0(t)dt\Bigg)\end{aligned}$$

**Proof:** As $F(x(t), y(t), z(t))$ is a trapezoid fuzzy function, by the closeness of the linear operation of trapezoidal fuzzy number and the closeness of trapezoidal fuzzy number space, we know that $\lim_{||P||\to 0} \sum_{i=1}^n F(x(t_i), y(t_i), z(t_i))\Delta s_i$ is a trapezoid fuzzy number, i.e., $\int_\Gamma F(x, y, z)ds$ is a trapezoid fuzzy number.

On the other hand, from $F(x(t), y(t), z(t)) = \left(\underline{f}_0(t), \underline{f}_1(t), \overline{f}_1(t), \overline{f}_0(t)\right)$ and

$$\begin{aligned}\left[\int_\Gamma F(x,y,z)ds\right]^0 = \Bigg[&\int_\alpha^\beta \sqrt{x'^2(t) + y'^2(t) + z'^2(t)}\underline{F(x(t), y(t), z(t))}(0)dt, \\ &\int_\alpha^\beta \sqrt{x'^2(t) + y'^2(t) + z'^2(t)}\overline{F(x(t), y(t), z(t))}(0)dt\Bigg]\end{aligned}$$

and

$$\begin{aligned}\left[\int_\Gamma F(x,y,z)ds\right]^1 = \Bigg[&\int_\alpha^\beta \sqrt{x'^2(t) + y'^2(t) + z'^2(t)}\underline{F(x(t), y(t), z(t))}(1)dt, \\ &\int_\alpha^\beta \sqrt{x'^2(t) + y'^2(t) + z'^2(t)}\ \overline{F(x(t), y(t), z(t))}(1)dt\Bigg]\end{aligned}$$

we see that

$$\begin{aligned}\left[\int_\Gamma F(x,y,z)ds\right]^0 = \Bigg[&\int_\alpha^\beta \sqrt{x'^2(t) + y'^2(t) + z'^2(t)}\underline{f}_0(t)dt, \\ &\int_\alpha^\beta \sqrt{x'^2(t) + y'^2(t) + z'^2(t)}\overline{f}_0(t)dt\Bigg]\end{aligned}$$

and

$$\begin{aligned}\left[\int_\Gamma F(x,y,z)ds\right]^1 = \Bigg[&\int_\alpha^\beta \sqrt{x'^2(t) + y'^2(t) + z'^2(t)}\underline{f}_1(t)dt, \\ &\int_\alpha^\beta \sqrt{x'^2(t) + y'^2(t) + z'^2(t)}\overline{f}_1(t)dt\Bigg]\end{aligned}$$

So we can see that

$$\begin{aligned}&\int_\Gamma F(x,y,z)ds \\ &= \Bigg(\int_\alpha^\beta \sqrt{x'^2(t) + y'^2(t) + z'^2(t)}\underline{f}_0(t)dt, \int_\alpha^\beta \sqrt{x'^2(t) + y'^2(t) + z'^2(t)}\underline{f}_1(t)dt, \\ &\quad \int_\alpha^\beta \sqrt{x'^2(t) + y'^2(t) + z'^2(t)}\overline{f}_1(t)dt, \int_\alpha^\beta \sqrt{x'^2(t) + y'^2(t) + z'^2(t)}\overline{f}_0(t)dt\Bigg)\end{aligned}$$

□

5. **Application.** We consider the problem: For a metal object of line status with trajectory $\Gamma : \begin{cases} x = t^2 \\ y = t \\ z = t \end{cases}$ $t \in [0, 10]$, we cannot see its accurate line density, only know the linear density and the parameter $t^2$ are approximately proportional, and the measured line density values are about 2 (Unit: kg/M) as $t = 0$ and 7 (Unit: kg/M) as $t = 10$, respectively. What we want to do is to estimate the quality of the metal object.

We can use triangle fuzzy number $(1.5, 2, 2.5)$ to represent the "about 2" and triangle fuzzy number $(5.5, 7, 8.5)$ to represent the "about 7". Since the linear density and the parameter $t^2$ are approximately proportional, we can use $\rho = \rho(x, y, z) = \rho(x(t), y(t), z(t)) = (\alpha t^2+1.5, \beta t^2+2, \gamma t^2+2.5)$ to represent the linear density of the metal object corresponding to parameter $t$. From $\alpha 10^2+1.5 = 5.5$, $\beta 10^2+2 = 7$ and $\gamma t^2+2.5 = 8.5$, we can obtain that $\alpha = 0.04$, $\beta = 0.05$ and $\gamma = 0.06$, so $\rho = \rho(x, y, z) = (0.4t^2 + 1.5, 0.5t^2 + 2, 0.6t^2 + 2.5)$.

By Theorem 4.3, and note the definition of triangle fuzzy number, we have

$$
\begin{aligned}
\int_\Gamma F(x,y,z)ds = \Bigg( & \int_0^{10} (0.4t^2+1.5)\sqrt{(2t)^2+(1)^2+(1)^2}dt, \\
& \int_0^{10} (0.5t^2+2)\sqrt{(2t)^2+(1)^2+(1)^2}dt, \\
& \int_0^{10} (0.6t^2+2.5)\sqrt{(2t)^2+(1)^2+(1)^2}dt \Bigg) \\
= \Bigg( & 0.1 \times \left( \int_0^{10} 4t^2\sqrt{4t^2+2}dt + 15\int_0^{10} \sqrt{4t^2+2}dt \right), \\
& 0.1 \times \left( \int_0^{10} 5t^2\sqrt{4t^2+2}dt + 2\int_0^{10} \sqrt{4t^2+2}dt \right), \\
& 0.1 \times \left( \int_0^{10} 6t^2\sqrt{4t^2+2}dt \right) + 25\int_0^{10} \sqrt{4t^2+2}dt \Bigg) \\
= \Bigg( & 0.1 \times 8^{-1} \times \left(16040\sqrt{402} - 4\ln\left(20+\sqrt{402}\right) - 4\ln\sqrt{2}\right) \\
& + 0.75\left(20\sqrt{402} + 2\ln\left(20+\sqrt{402}\right) - 2\ln\sqrt{2}\right), \\
& 0.5 \times 32^{-1} \times \left(16040\sqrt{402} - 4\ln\left(20+\sqrt{402}\right) - 4\ln 2\right) \\
& + 0.1\left(20\sqrt{402} + 2\ln\left(20+\sqrt{402}\right) - 2\ln\sqrt{2}\right), \\
& 0.3 \times 16^{-1} \times \left(16040\sqrt{402} - 4\ln\left(20+\sqrt{402}\right) - 4\ln 2\right) \\
& + 1.25\left(20\sqrt{402} + 2\ln\left(20+\sqrt{402}\right) - 2\ln\sqrt{2}\right) \Bigg) \\
\approx & (2162.8, 2532.8, 3269.7)
\end{aligned}
$$

So we can denote the approximate quality of the metal object by the triangle fuzzy number $(2162.8, 2532.8, 3269.7)$kg.

6. **Conclusion.** In this paper, we defined a kind of integral of fuzzy number valued mapping on smooth curve segment and piecewise smooth curve segment (Definitions 3.1 and 3.2), and obtained some properties about such integral (Properties 3.1-3.4). And we proved the existence for continuous fuzzy number valued mapping on piecewise smooth curve segment, and set up its calculating formula (Theorems 4.1 and 4.2). And then, we also proved that the such integral for trapezoidal fuzzy number valued mapping is still

a trapezoidal fuzzy number, and obtained its specific calculating formula (Theorem 4.3). At last, we gave an example to show the application of the proposed theory. In the future, we can study such kind of integral for fuzzy number valued mapping $R^n \to E$ on smooth curve segment in $R^n$, and apply this kind of integral to a wider range of areas.

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