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CONTROLLABILITY ANALYSIS OF MULTI-AGENT SYSTEMS WITH SYSTEM MODEL TRANSFORMATION

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Abstract. *With multi-signal inputs, the controllability of graph for networked multi-agent systems is analyzed in this paper. Meanwhile, a new system model is constructed. Under this model, a new model is obtained which is similar to the single signal input system. The controllability of graph for networked multi-agent systems is gained by using the controllability rank criterion and PBH (Popov-Belevitch-Hautus) criterion under the new model. It is more convenient and simpler to judge the controllability of the system by using the new model. In addition, the controllability of graph for networked multi-agent systems is improved on the basis of the existing conclusions.*
Keywords: Controllability, Multi-signal input systems, System model, Rank criterion and PBH criterion

1. **Introduction.** In recent years, academic circles have extensive research and attention on multi-agent systems [1-6], and the multi-agent system has been widely used in many fields, such as UAV formation control, robot formation control, even widely used in the military. As we all know, the core problem of multi-agent systems is the controllability problem, and the controllability is defined by applying an external control input to the leader, so as to make the followers from any initial states to any desired states on the basis of the multi-agent network. Because controllability reflects leaders' ability to control followers from outside, studying the controllability of multi-agent systems has very important significance [7-9].

In multi-agent systems, studies of controllability on the leader-follower topological graph are mostly based on single input system model or a simpler model [6,10]. The establishment of the model has an important influence on the controllability of systems. In the established model, fully understanding the impact of the system on controllability could provide good methods and help to solve the problem of the controllability of the multi-agent systems undoubtedly. So it is a hot spot to study the influence of the controllability of systems under the specific model.

As early as in [11], Tanner first studied the controllability through the link between the nodes in the system. He proposed a necessary and sufficient condition to guarantee that system is controllable with only one leader by using the neighbors' information. And the controllability theorem of undirected graph was obtained. This is of great help for the follow-up study on controllability. In this paper, a more general multi-signal input model is studied. Each node may receive a number of different signal inputs. This model is more accurate in showing the generality of multi-agent systems. The main contribution of this paper is to simplify the complex model of multi-signal input systems, and makes it easier to judge whether the system is controllable, especially for a connected graph with a large number of nodes. The method of this paper can simplify the process of judging whether the graph is controllable.

Some articles have made a lot of research about controllability based on Laplacian matrix in recent years [12,13]. This paper also has investigated the relationship between the controllability of systems and the Laplacian matrix. Especially under the multi-signal input system model, further research about the relationship between the model and the controllability of the systems is a necessity.

The rest of paper is organized as follows. Some preliminaries and the general model of systems are reviewed in Section 2. The difference between the single signal and multi-signal input system and model transformation are considered, respectively, in Sections 3 and 4. Our main results are presented in Section 5. Finally, conclusions are drawn in Section 6.

2. **Preliminaries.** In multi-agent systems, graph of the nodes represents agents, and the edges in the graph represent the communication link between agents. In this paper, we consider a simple graph, that is, there is no weight without a closed ring or multiple edges. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph of order n with vertex set $\mathcal{V} = \{1, 2, \dots, n\}$ and edge set $\mathcal{E} \subseteq [\mathcal{V}]^2 := \{\{i, j\} \,|i, j \in \mathcal{V}\}$. Denote the set of input nodes in the n nodes graph is $\boldsymbol{S} = \{i_1, i_2, \dots, i_q\}$ for $i_1 < i_2 < \cdots < i_q$, the input node is called the leader node, and the remaining follower node set is $\mathcal{V} \backslash \boldsymbol{S}$. An edge exists between two neighboring nodes i and j if $(i, j) \in \mathcal{E}$. The degree d_i of the node i is given by the number of its neighbors, and the degree matrix of $\mathcal{G}$ is the diagonal matrix $\boldsymbol{D} \in R^{n\times n}$ whose ith diagonal entry is d_i. The adjacency matrix of $\mathcal{G}$ is the matrix $\boldsymbol{A} \in R^{n\times n}$ defined as $a_{ij} = 1$ if $(i, j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The Laplacian matrix of $\mathcal{G}$ is given by $\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{A}$, and the Laplacian matrix is positive and symmetric.

Given nodes i and j in the graph $\mathcal{G}$, we define the distance $d_{\mathcal{G}}(i, j)$ between i and j as the length of a shortest path which connects nodes i and j. A graph $\mathcal{G}$ is connected if there is a path between any pair of distinct nodes. We denote by $\langle \boldsymbol{L}; \boldsymbol{B}\rangle$ the smallest $\boldsymbol{L}$-invariant subspace containing $\boldsymbol{B}$. It is well-known that $\langle \boldsymbol{L}; \boldsymbol{B}\rangle = span \left\{\boldsymbol{L}^k\boldsymbol{B} \,|k \in \boldsymbol{N}_0\right\}$, and that if $\dim(\langle \boldsymbol{L}; \boldsymbol{B}\rangle) = k + 1$, then $\{\boldsymbol{B}, \boldsymbol{LB}, \dots, \boldsymbol{L}^k\boldsymbol{B}\}$ is a basis for $\langle \boldsymbol{L}; \boldsymbol{B}\rangle$. The pair $(\boldsymbol{L}, \boldsymbol{B})$ is called controllable if $\dim(\langle \boldsymbol{L}; \boldsymbol{B}\rangle) = n$.

The focus of this paper is the analysis of the problem of controllability, where $x_i(t) \in R$ represents the state of node $i \in \mathcal{V}$ at time t with interaction between nodes. An external control vector $\boldsymbol{U}(t)$ at time t is applied to node i through the state matrix $\boldsymbol{B}_i \in R^q$. The dynamics of an individual node is given by $\dot{x}_i(t) = -\sum\limits_{\{i,j\}\in\mathcal{E}} (x_i - x_j) + \boldsymbol{B}_i^T\boldsymbol{U}(t)$. In addition, the dynamics at time t is observed by a vector $\boldsymbol{y}(t) \in R^p$ through a matrix $\boldsymbol{C} \in R^{p\times n}$. The full system dynamics is

$$\begin{cases} \dot{\boldsymbol{x}} = -\boldsymbol{L}(\mathcal{G})x(t) + \boldsymbol{BU}(t) \\ \dot{\boldsymbol{y}} = \boldsymbol{Cx}(t) \end{cases} \tag{1}$$

where $\boldsymbol{B} = [\boldsymbol{B}_1, \boldsymbol{B}_2, \dots, \boldsymbol{B}_n]^T \in R^{n\times q}$.

3. **The Difference between the Single Signal and Multi-Signal Input System.** The difference between the single signal and multi-signal input system can be seen from the following definition.

Definition 3.1. *The set of input nodes in the n node graph is $\boldsymbol{S} = \{i_1, i_2, \dots, i_q\}$ for $i_1 < i_2 < \cdots < i_q$. Let $U(t) = [u_1(t), u_2(t), \dots, u_q(t)]^T$ be a vector of q controllers. System (1) is a multi-signal input system if $u_1(t), u_2(t), \dots, u_q(t)$ are different, and system (1) is a single signal input system if $u_1(t), u_2(t), \dots, u_q(t)$ are the same, that is $u_1(t) = u_2(t) = \ldots = u_q(t) = u(t)$, where the input matrix is a column vector $\boldsymbol{b}$.*

Compared with the single signal input system, the multi-signal input system has two kinds of complex situations. In the multi-signal input model, the system can be divided into two kinds. One is that each leader only receives a signal input, the received signal is not necessarily the same, and the corresponding matrix of the input is $\boldsymbol{B} = \left[\boldsymbol{e}_{i_1}, \boldsymbol{e}_{i_2}, \ldots, \boldsymbol{e}_{i_q}\right] \in R^{n\times q}$. The other one is that each node may receive multiple signal input (This paper only considers the situation that has q controllers), and the corresponding matrix of the input is

$$\begin{aligned}\boldsymbol{B} = &\left[\mathrm{b}_{11}\boldsymbol{e}_{i_1} + \mathrm{b}_{21}\boldsymbol{e}_{i_2} + \cdots + \mathrm{b}_{q1}\boldsymbol{e}_{i_q}, \mathrm{b}_{12}\boldsymbol{e}_{i_1} + \mathrm{b}_{22}\boldsymbol{e}_{i_2} + \cdots + \mathrm{b}_{q2}\boldsymbol{e}_{i_q}, \ldots, \mathrm{b}_{1q}\boldsymbol{e}_{i_1}\right.\\ &\left.+ \mathrm{b}_{2q}\boldsymbol{e}_{i_2} + \cdots + \mathrm{b}_{qq}\boldsymbol{e}_{i_q}\right],\end{aligned}$$

where $\mathrm{b}_{ij} \in \{0, 1\}$, $1 \le i, j \le q$.

4. **Model Transformation.** For system (1), when each leader only receives a signal input and the received signal is not necessarily the same, let $\boldsymbol{U}(t)$ and $\boldsymbol{M}u(t)$ be the following equation:

$$\begin{aligned}\boldsymbol{U}(t) &= [u_1(t), u_2(t), \ldots, u_q(t)]^{\mathrm{T}} = \left[u_1'(t)u(t), u_2'(t)u(t), \cdots, u_q'(t)u(t)\right]^{\mathrm{T}}\\ &= \left[u_1'(t), u_2'(t), \cdots, u_q'(t)\right]^{\mathrm{T}} u(t),\\ \boldsymbol{M}u(t) &= \boldsymbol{B}\boldsymbol{U}(t) = \left[\boldsymbol{e}_{i_1}u_1'(t)u(t), \boldsymbol{e}_{i_2}u_2'(t)u(t), \cdots, \boldsymbol{e}_{i_q}u_q'(t)u(t)\right]\\ &= \left[\boldsymbol{e}_{i_1}u_1'(t), \boldsymbol{e}_{i_2}u_2'(t), \cdots, \boldsymbol{e}_{i_q}u_q'(t)\right] u(t).\end{aligned}$$

Then system (1) can be converted into the following equation:

$$\left\{\begin{array}{l}\dot{\boldsymbol{x}} = -\boldsymbol{L}(\mathcal{G})\boldsymbol{x}(t) + \boldsymbol{M}u(t)\\ \dot{\boldsymbol{y}} = \boldsymbol{C}\boldsymbol{x}(t)\end{array}\right. \tag{2}$$

where $u_i(t) = u_i'(t)u(t)$ with $u_i'(t)$ being variable and they are not all zero and $u(t)$ being the cardinal number. By observing, each column of $\boldsymbol{M}$ has only a nonzero number, and the locations of the nonzero number in each column are different, the matrix can be converted into the following vector:

$$\boldsymbol{m} = \boldsymbol{e}_{i_1}u_1'(t) + \boldsymbol{e}_{i_2}u_2'(t) + \cdots + \boldsymbol{e}_{i_q}u_q'(t).$$

When each leader receives multiple signal inputs, let $\boldsymbol{U}(t)$ and $\boldsymbol{M}u(t)$ be the following equation:

$$\begin{aligned}\boldsymbol{U}(t) &= [u_1(t), u_2(t), \ldots, u_q(t)]^{\mathrm{T}} = \left[u_1'(t), u_2'(t), \ldots, u_q'(t)\right] [u(t), u(t), \ldots, u(t)]^{\mathrm{T}}\\ &= \left[u_1'(t), u_2'(t), \ldots, u_q'(t)\right] u(t),\end{aligned}$$

$$\begin{aligned}\boldsymbol{M}u(t) =& \boldsymbol{B}\boldsymbol{U}(t)\\ =& \left[\mathrm{b}_{11}\boldsymbol{e}_{i_1} + \mathrm{b}_{21}\boldsymbol{e}_{i_2} + \cdots + \mathrm{b}_{q1}\boldsymbol{e}_{i_q}, \mathrm{b}_{12}\boldsymbol{e}_{i_1} + \mathrm{b}_{22}\boldsymbol{e}_{i_2} + \cdots + \mathrm{b}_{q2}\boldsymbol{e}_{i_q}, \ldots, \mathrm{b}_{1q}\boldsymbol{e}_{i_1}\right.\\ &\left.+\mathrm{b}_{2q}\boldsymbol{e}_{i_2} + \cdots + \mathrm{b}_{qq}\boldsymbol{e}_{i_q}\right] U(t)\\ =& \left[(\mathrm{b}_{11}\boldsymbol{e}_{i_1} + \mathrm{b}_{21}\boldsymbol{e}_{i_2} + \cdots + \mathrm{b}_{q1}\boldsymbol{e}_{i_q})u_1'(t), \ldots, \left(\mathrm{b}_{1q}\boldsymbol{e}_{i_1} + \mathrm{b}_{2q}\boldsymbol{e}_{i_2} + \cdots\right.\right.\\ &\left.\left.+\mathrm{b}_{qq}\boldsymbol{e}_{i_q}\right)u_q'(t)\right] u(t).\end{aligned}$$

Then system (1) can also be converted into system (2). And the matrix $\boldsymbol{M}$ can also be converted into the following vector:

$$\begin{aligned}\boldsymbol{m} =& \left(\mathrm{b}_{11}\boldsymbol{e}_{i_1} + \mathrm{b}_{21}\boldsymbol{e}_{i_2} + \cdots + \mathrm{b}_{q1}\boldsymbol{e}_{i_q}\right) u_1'(t) + \left(\mathrm{b}_{12}\boldsymbol{e}_{i_1} + \mathrm{b}_{22}\boldsymbol{e}_{i_2} + \cdots + \mathrm{b}_{q2}\boldsymbol{e}_{i_q}\right) u_2'(t)\\ &+ \cdots + \left(\mathrm{b}_{1q}\boldsymbol{e}_{i_1} + \mathrm{b}_{2q}\boldsymbol{e}_{i_2} + \cdots + \mathrm{b}_{qq}\boldsymbol{e}_{i_q}\right) u_q'(t)\\ =& \left(\mathrm{b}_{11}u_1'(t) + \mathrm{b}_{12}u_2'(t) + \cdots + \mathrm{b}_{1q}u_q'(t)\right) \boldsymbol{e}_{i_1} + \left(\mathrm{b}_{21}u_1'(t) + \mathrm{b}_{22}u_2'(t) + \cdots + \mathrm{b}_{2q}u_q'(t)\right) \boldsymbol{e}_{i_2}\\ &+ \cdots + \left(\mathrm{b}_{q1}u_1'(t) + \mathrm{b}_{q2}u_2'(t) + \cdots + \mathrm{b}_{qq}u_q'(t)\right) \boldsymbol{e}_{i_q}.\end{aligned}$$

In conclusion, we have a new model of the multi-signal input system:

$$\begin{cases} \dot{\boldsymbol{x}} = -\boldsymbol{L}(\mathcal{G})\boldsymbol{x}(t) + \boldsymbol{m}u(t) \\ \dot{\boldsymbol{y}} = \boldsymbol{C}\boldsymbol{x}(t) \end{cases} \tag{3}$$

5. **Main Results.** With the above model transformation, we give the following theorem.

Theorem 5.1. *Suppose that* $\boldsymbol{L}$ *has distinct eigenvalues, the pair* $(-\boldsymbol{L}(\mathcal{G}), \boldsymbol{B})$ *is controllable if and only if the pair* $(-\boldsymbol{L}(\mathcal{G}), \boldsymbol{m})$ *is controllable.*

Before proving Theorem 5.1, we make the following lemma.

Lemma 5.1. *Suppose that* $\boldsymbol{L}$ *has distinct eigenvalues and let* $\boldsymbol{U}$ *be a matrix whose columns are linear independent eigenvectors of* $\boldsymbol{L}$*. Then, the dimension of* $\langle \boldsymbol{L}; \boldsymbol{B} \rangle$ *is equal to the number of nonzero row vectors of* $\boldsymbol{W} = \boldsymbol{U}^{-1}\boldsymbol{B}$*. In particular,* $(\boldsymbol{L}, \boldsymbol{B})$ *is controllability if and only if no row vector of* $\boldsymbol{W}$ *is zero row vector.*

Proof: Since matrix $\boldsymbol{L}$ is symmetric, it can be expressed as $\boldsymbol{L} = \boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^{-1}$, where the columns of $\boldsymbol{U}$ contain the orthogonal eigenvectors of $\boldsymbol{L}$, and $\boldsymbol{D}$ is the diagonal matrix of the eigenvalues of $\boldsymbol{L}$. Vector $\boldsymbol{u}_i$ is the eigenvector corresponding to eigenvalue λ_i. Denote $\boldsymbol{U} = (\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n)$. Let $\boldsymbol{W} = \boldsymbol{U}^{-1}\boldsymbol{B} = [\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n]^T$, where $\boldsymbol{W} \in R^{n\times q}$, $\boldsymbol{w}_i$ is a column vector, $i \in \{1, 2, \cdots, n\}$. Then the controllability matrix of system (1) is

$$\begin{aligned} [\lambda_i \boldsymbol{I} - \boldsymbol{L}, \boldsymbol{B}] &= \left[\lambda_i \boldsymbol{I} - \boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^{-1}, \boldsymbol{B}\right] = \boldsymbol{U}\left[\lambda_i \boldsymbol{U}^{-1} - \boldsymbol{D}\boldsymbol{U}^{-1}, \boldsymbol{U}^{-1}\boldsymbol{B}\right] \\ &= \boldsymbol{U}\left[(\lambda_i \boldsymbol{I} - \boldsymbol{D})\boldsymbol{U}^{-1}, \boldsymbol{U}^{-1}\boldsymbol{B}\right] \end{aligned}$$

Since $\boldsymbol{U}$ is a nonsingular matrix, it does not affect the rank of the matrix on the right, and the right of the matrix can be expanded as follows

$$\left[\begin{pmatrix} \lambda_i - \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_i - \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_i - \lambda_n \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_1^{\mathrm{T}} \\ \boldsymbol{u}_2^{\mathrm{T}} \\ \vdots \\ \boldsymbol{u}_n^{\mathrm{T}} \end{pmatrix}, \begin{pmatrix} \boldsymbol{w}_1^{\mathrm{T}} \\ \boldsymbol{w}_2^{\mathrm{T}} \\ \vdots \\ \boldsymbol{w}_n^{\mathrm{T}} \end{pmatrix}\right].$$

When $\boldsymbol{L}$ has distinct eigenvalues, the above formula is full rank if and only if $\boldsymbol{w}_i \neq \boldsymbol{0}$, that is, the system is controllable. This completes the proof.

Lemma 5.2. *Under the single signal input system, suppose that* $\boldsymbol{L}$ *has distinct eigenvalues. Let* $\boldsymbol{U}$ *be a matrix whose columns are linearly independent eigenvectors of* $\boldsymbol{L}$*, and then the dimension of* $\langle \boldsymbol{L}; \boldsymbol{b} \rangle$ *is equal to the number of nonzero components of* $\boldsymbol{v} = \boldsymbol{U}^{-1}\boldsymbol{b}$*. In particular,* $(\boldsymbol{L}, \boldsymbol{b})$ *is controllable if and only if no component of* $\boldsymbol{v}$ *is zero.*

Proof: Denote $\boldsymbol{D}$ as the diagonal matrix of the eigenvalues of $\boldsymbol{L}$. Let $\boldsymbol{v} = \boldsymbol{U}^{-1}\boldsymbol{b} = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n]^{\mathrm{T}}$. The controllability matrix of system (1) is

$$\begin{aligned} &\left[\boldsymbol{b}, \boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^{-1}\boldsymbol{b}, \left(\boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^{-1}\right)^2 \boldsymbol{b}, \ldots, \left(\boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^{-1}\right)^{n-1} \boldsymbol{b}\right] \\ &= \left[\boldsymbol{b}, \boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^{-1}\boldsymbol{b}, \boldsymbol{U}\boldsymbol{D}^2\boldsymbol{U}^{-1}\boldsymbol{b}, \ldots, \boldsymbol{U}\boldsymbol{D}^{n-1}\boldsymbol{U}^{-1}\boldsymbol{b}\right] \\ &= \boldsymbol{U}\left[\boldsymbol{U}^T\boldsymbol{b}, \boldsymbol{D}\boldsymbol{U}^{-1}\boldsymbol{b}, \ldots, \boldsymbol{D}^{n-1}\boldsymbol{U}^{-1}\boldsymbol{b}\right] = \boldsymbol{U}\left[\boldsymbol{v}, \boldsymbol{D}\boldsymbol{v}, \ldots, \boldsymbol{D}^{n-1}\boldsymbol{v}\right] \\ &= \boldsymbol{U}\begin{bmatrix} v_1 & \lambda_1 v_1 & \cdots & \lambda_1^{n-1} v_1 \\ v_2 & \lambda_2 v_2 & & \lambda_2^{n-1} v_2 \\ \vdots & \vdots & & \vdots \\ v_n & \lambda_n v_n & \cdots & \lambda_n^{n-1} v_n \end{bmatrix}. \end{aligned}$$

Clearly, the dimension of $\langle \boldsymbol{L}; \boldsymbol{b} \rangle$ is equal to the number of nonzero components of $\boldsymbol{v} = \boldsymbol{U}^{-1}\boldsymbol{b}$. Moreover, the rank of $(\boldsymbol{L}, \boldsymbol{b})$ is n if and only if no component of $\boldsymbol{v}$ is zero. This completes the proof.

We now prove Theorem 5.1.

Proof: Let

$$\boldsymbol{U}^{-1} = \begin{bmatrix} \boldsymbol{u}_1^{\mathrm{T}} \\ \vdots \\ \boldsymbol{u}_n^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & \ddots & & u_{2n} \\ \vdots & & \ddots & \vdots \\ u_{n1} & \cdots & \cdots & u_{nn} \end{bmatrix} \text{ and } \boldsymbol{W} = \boldsymbol{U}^{-1}\boldsymbol{B}.$$

Then we have that

$$\begin{aligned} \boldsymbol{W} &= \boldsymbol{U}^{-1}\boldsymbol{B} \\ &= \begin{bmatrix} b_{11}u_{1i_1} + b_{21}u_{1i_2} + \cdots + b_{q1}u_{1i_q} & b_{12}u_{1i_1} + b_{22}u_{1i_2} + \cdots + b_{q2}u_{1i_q} & \cdots & b_{1n}u_{1i_1} + \cdots + b_{qn}u_{1i_q} \\ b_{11}u_{2i_1} + b_{21}u_{2i_2} + \cdots + b_{q1}u_{2i_q} & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ b_{11}u_{ni_1} + b_{21}u_{ni_2} + \cdots + b_{q1}u_{ni_q} & \cdots & \cdots & b_{1n}u_{ni_1} + \cdots + b_{qn}u_{ni_q} \end{bmatrix}. \end{aligned}$$

Let $\boldsymbol{v} = \boldsymbol{U}^{-1}\boldsymbol{m}$. Then we have that

$$\begin{aligned} \boldsymbol{v} &= \boldsymbol{U}^{-1}\boldsymbol{m} \\ &= \begin{bmatrix} \left(\mathrm{b}_{11}u_1'(t) + \cdots + \mathrm{b}_{1q}u_q(t)\right)u_{1i_1} + \left(\mathrm{b}_{21}u_1'(t) + \cdots + \mathrm{b}_{2q}u_q'(t)\right)u_{1i_2} + \cdots + \left(\mathrm{b}_{q1}u_1'(t) + \cdots + \mathrm{b}_{qq}u_q'(t)\right)u_{1i_q} \\ \vdots \\ \left(\mathrm{b}_{11}u_1'(t) + \cdots + \mathrm{b}_{1q}u_q'(t)\right)u_{ni_1} + \left(\mathrm{b}_{21}u_1'(t) + \cdots + \mathrm{b}_{2q}u_q'(t)\right)u_{ni_2} + \cdots + \left(\mathrm{b}_{q1}u_1'(t) + \cdots + \mathrm{b}_{qq}u_q'(t)\right)u_{ni_q} \end{bmatrix} \\ &= \begin{bmatrix} \left(b_{11}u_{1i_1} + \cdots + b_{q1}u_{1i_q}\right)u_1'(t) + \left(b_{12}u_{1i_1} + \cdots + b_{q2}u_{1i_q}\right)u_2'(t) + \cdots + \left(b_{1n}u_{1i_1} + \cdots + b_{qn}u_{1i_q}\right)u_q'(t) \\ \vdots \\ \left(b_{11}u_{ni_1} + \cdots + b_{q1}u_{ni_q}\right)u_1'(t) + \left(b_{12}u_{ni_1} + \cdots + b_{q2}u_{ni_q}\right)u_2'(t) + \cdots + \left(b_{1n}u_{ni_1} + \cdots + b_{qn}u_{ni_q}\right)u_q'(t) \end{bmatrix}. \end{aligned}$$

Compared with $\boldsymbol{W} = \boldsymbol{U}^{-1}\boldsymbol{B}$ and $\boldsymbol{v} = \boldsymbol{U}^{-1}\boldsymbol{m}$, we can conclude that the combination of the row vector elements in $\boldsymbol{W}$ is the corresponding elements in $\boldsymbol{v}$. When no row vector of $\boldsymbol{W}$ is zero row vector, no component of $\boldsymbol{v}$ is zero. Hence, when $\boldsymbol{L}$ has distinct eigenvalues, the pair $(-\boldsymbol{L}(\mathcal{G}), \boldsymbol{B})$ is controllable if and only if the pair $(-\boldsymbol{L}(\mathcal{G}), \boldsymbol{m})$ is controllable. This completes the proof.

6. **Conclusions.** In this paper, the problem of controllability was researched and analyzed in detail in the multi-signal input system. In addition, a new system model is constructed for a general multi-signal input system so that it is more convenient to judge the controllability of the system in the new system model. Then this paper explained and demonstrated the relationship between the controllability of graph for the multi-signal input system and the Laplacian matrix. Further research is directed on the situation that there are not only q controllers. Under the new model, the research methods and results of controllability are provided, which can provide help and direction for the further study on the controllability of complex graph.

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