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FREQUENCY ESTIMATION WITH ATOMIC NORMS USING COPRIME SAMPLING

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Abstract. *In some applications of frequency estimation, it is challenging to sample at as high as Nyquist rate due to hardware limitations. The compressed sensing theory asserts that one can recover sparse signals using undersampled measurements. Recently the methods based on atomic norm techniques deal with continuous-valued frequencies and completely eliminate basis mismatch of existing compressed sensing. However, compressed sensing usually requires random sampling in data acquisition, which leads to complex hardware. In this letter, we use coprime sampling instead of random sampling to acquire data, and then utilize the method based on atomic norms to estimate the frequency components of a mixture of several complex sinusoids. The simulations show that coprime sampling can be brought into the method based on atomic norms and can replace impractical random sampling. We combine the advantages of atomic norms and coprime sampling to achieve continuous frequency estimation with deterministic sub-Nyquist sampling. The proposed method obtains good results in some applications of frequency estimation.*
Keywords: Frequency estimation, Atomic norms, Coprime sampling, Sub-Nyquist sampling

1. **Introduction.** Frequency estimation of multiple sinusoids has wide applications in communications, audio, medical instrumentation and electric systems. Frequency estimation methods cover classical modified discrete Fourier transform (DFT), subspace techniques such as "multiple signal classification" (MUSIC) [1] and "estimating signal parameter via rotational invariance techniques" (ESPRIT) [2], and other advanced spectral estimation approaches. In general, the sampling rate of the signal is required to be higher than twice the highest frequency (i.e., Nyquist rate). However, it is challenging to build sampling hardware when signal bandwidth is large. When the signal is sampled at sub-Nyquist rate, it often leads to aliasing and attendant problem of frequency ambiguity.

Compressed sensing (CS) studies sparse signal recovery from far fewer measurements and has brought significant impact on signal processing and information theory in the past decade. Since the frequency components are usually assumed to lie on a fixed uniform grid, limitations are present in applications such as array processing, radar and sonar, where the dictionary is typically specified by one or more continuous parameters. A breakthrough came up recently. Candès and Fernandez-Granda deal directly with the continuous frequency recovery problem and therefore completely eliminate "basis mismatch" caused by grid discretization [3]. Inspired by [3], Tang et al. study the problem of continuous frequency recovery from partial observations (i.e., incomplete data) based on the atomic norm minimization [4]. An atomic norm soft thresholding method is presented in [5] in the presence of stochastic noise, a common assumption in the literature. [6] investigates the frequency recovery problem in the presence of multiple measurement vectors (MMVs) which share the same frequency components and extends the single measurement vector (SMV) atomic norm methods and their theoretical guarantees to the MMV case. However, the sampling pattern in these methods is usually required to be random, for example, the indices of the samples are selected uniformly at random from a set $\{1, 2, \dots, N\}$, which becomes an obstacle to practical applications.

Coprime sampling and coprime sensor arrays are introduced and discussed in considerable detail in [7]. Two uniform samplers with sample spacings pT and qT are used where p and q are coprime and T has the dimension of space or time. In [8], coprime sampling is combined with the MUSIC algorithm and a new approach to super resolution line spectrum estimation in both temporal and spatial domain is proposed. However, this combination does not eliminate the effect of basis mismatch because MUSIC still searches the directions according to a fixed grid. In this letter, we combine the coprime sampling with atomic norms to complement each other. In other words, we use coprime sampling to acquire data and then utilize the method based on atomic norms to process these data. It will be seen that the proposed method can estimate the frequency components of a mixture of complex sinusoids accurately from deterministic sub-Nyquist measurements. The proposed method avoids complex random sampling process, while holds high estimation accuracy, which even outperforms conventional ESPRIT with normal sampling.

The remainder of the document is organized as follows. Section 2 will review the basic theory of atomic norms. Section 3 will demonstrate our method. Section 4 will present the simulation results. Finally, Section 5 draws conclusions and concludes the paper.

2. Problem Statement and Preliminaries.

2.1. Problem formulation.

Consider a signal containing K frequency components with unknown constant amplitudes and phases, and that additive noise is assumed to be a zero-mean stationary complex white Gaussian random process. Suppose that we obtain L observations of the signal with the Nyquist rate:

$$y^{\star}_{ml} = \sum_{k=1}^{K} s_{kl} e^{j2\pi f_k m}, \quad (m, l) \in [N] \times [L], \tag{1}$$

which form an $N \times L$ matrix $\boldsymbol{Y}^{\star} = [y^{\star}_{ml}]$, where $[N] = \{1, 2, \dots, N\}$, $[L] = \{1, 2, \dots, L\}$ and N is the number of uniform samples in each observation. Here (m, l) indexes the entries of $\boldsymbol{Y}^{\star}$, $f_k \in [0, 1]$ denotes the k-th normalized frequency, $s_{kl} \in \mathbb{C}$ is the complex amplitude of the k-th frequency component in the l-th observation. If the index set of the measurement vectors $\Omega = [N]$, it implies normal sampling. If $|\Omega| = M < N$, this corresponds to compressive sampling or sub-Nyquist sampling. The problem concerned is to estimate the frequency components given $\boldsymbol{Y}^{\star}_{\Omega}$, where $\boldsymbol{Y}^{\star}_{\Omega}$ takes the rows of $\boldsymbol{Y}^{\star}$ indexed by Ω. This problem is referred to as joint sparse frequency recovery in the sense that the multiple measurement vectors (MMVs) (i.e., the L columns of $\boldsymbol{Y}^{\star}_{\Omega}$) share the same K frequencies. The single measurement vector (SMV) can be regarded as a special case of MMVs, where $L = 1$.

2.2. Atomic norms and semidefinite formulation.

To exploit the joint sparsity in the MMVs, we let $\boldsymbol{s}_k = [s_{k1}, \dots, s_{kL}] \in \mathbb{C}^{1\times L}$. It follows that (1) can be written as

$$\boldsymbol{Y}^{\star} = \sum_{k=1}^{K} \boldsymbol{a}(f_k) \boldsymbol{s}_k = \sum_{k=1}^{K} c_k \boldsymbol{a}(f_k) \boldsymbol{\phi}_k, \tag{2}$$

where $\boldsymbol{a}(f) = \left[e^{j2\pi f}, e^{j2\pi 2f}, \dots, e^{j2\pi N f}\right]^T \in \mathbb{C}^N$, $c_k = \|\boldsymbol{s}_k\|_2 > 0$ and $\boldsymbol{\phi}_k = \boldsymbol{s}_k / c_k$ with $\|\boldsymbol{\phi}_k\|_2 = 1$. Let $\mathbb{S}^{2L-1} = \left\{\boldsymbol{\phi} \in C^{1\times L} : \|\boldsymbol{\phi}\|_2 = 1\right\}$ denote the unit complex $(L-1)$-sphere (or real $(2L-1)$-sphere). Define the set of atoms

$$\mathcal{A} \triangleq \left\{\boldsymbol{a}(f, \boldsymbol{\phi}) = \boldsymbol{a}(f)\, \boldsymbol{\phi} : f \in [0, 1], \boldsymbol{\phi} \in \mathbb{S}^{2L-1}\right\}. \tag{3}$$

It follows from (2) that $\boldsymbol{Y}^\star$ is a linear combination of K atoms in $\mathcal{A}$. The atomic norm is defined as the gauge function of $conv(\mathcal{A})$, the convex hull of $\mathcal{A}$ [9]:

$$\|\boldsymbol{Y}\|_{\mathcal{A}} \triangleq \inf\left\{t > 0 : \boldsymbol{Y} \in tconv\left(\mathcal{A}\right)\right\} = \inf\left\{\sum_k c_k : \boldsymbol{Y} = \sum_k c_k \boldsymbol{a}_k, c_k > 0, \boldsymbol{a}_k \in \mathcal{A}\right\}. \quad (4)$$

Roughly speaking, the atomic norm $\|\bullet\|_{\mathcal{A}}$ can enforce sparsity in $\mathcal{A}$ because low-dimensional faces of $conv(\mathcal{A})$ correspond to signals involving only a few atoms.

We need to estimate the frequency components of a sparse sum of complex exponentials from only a subset of entries (i.e., Ω). The natural algorithm is the atomic norm minimization problem

$$\min_{\boldsymbol{Y}} \|\boldsymbol{Y}\|_{\mathcal{A}}, \text{ subject to } \boldsymbol{Y}_\Omega = \boldsymbol{Y}^\star_\Omega. \quad (5)$$

To practically solve (5), a semidefinite programming formulation of $\|\boldsymbol{Y}\|_{\mathcal{A}}$ is provided:

$$\min_{\boldsymbol{W},\boldsymbol{u}} \frac{1}{2\sqrt{N}}\left[tr\left(\boldsymbol{W}\right) + tr\left(Toep\left(\boldsymbol{u}\right)\right)\right], \text{ subject to } \begin{bmatrix} \boldsymbol{W} & \boldsymbol{Y}^H \\ \boldsymbol{Y} & Toep\left(\boldsymbol{u}\right) \end{bmatrix} \geq \boldsymbol{0}, \quad (6)$$

where $\boldsymbol{u} \in \mathbb{C}^N$ and $Toep\left(\boldsymbol{u}\right) \in \mathbb{C}^{N\times N}$ denotes a (Hermitian) Toeplitz matrix with

$$Toep\left(\boldsymbol{u}\right) = \begin{bmatrix} u_1 & u_2 & \cdots & u_N \\ u_2^H & u_1 & \cdots & u_{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ u_N^H & u_{N-1}^H & \cdots & u_1 \end{bmatrix}, \quad (7)$$

where u_i denotes the i-th entry of $\boldsymbol{u}$. This positive semidefinite Toeplitz matrix $Toep\left(\boldsymbol{u}\right)$ of rank K has an order-K Vandermonde decomposition [10]:

$$Toep\left(\boldsymbol{u}\right) = \sum_{k=1}^{K} p_k \boldsymbol{a}\left(f_k\right)\boldsymbol{a}^H\left(f_k\right) = \boldsymbol{A}\left(f\right)\boldsymbol{\Gamma}\boldsymbol{A}\left(f\right), \quad (8)$$

where $\boldsymbol{A}\left(f\right) = \left[\boldsymbol{a}\left(f_1\right), \ldots, \boldsymbol{a}\left(f_K\right)\right] \in \mathbb{C}^{N\times K}$, $\boldsymbol{\Gamma} = diag\left(\gamma_1, \ldots, \gamma_K\right)$ with $\gamma_k > 0$ and $\{f_k\}$ are distinct points in $[0, 1]$. Moreover, the decomposition is unique if $K < N$. In some sense, $Toep\left(\boldsymbol{u}\right)$ is considered the data covariance matrix. Therefore, the true frequencies can be uniquely obtained from the Vandermonde decomposition of $Toep\left(\boldsymbol{u}\right)$ given $K < N$. The Vandermonde decomposition can be computed efficiently via root finding or by solving a generalized eigenvalue problem [11].

As a result, (5) can be cast as the following semidefinite programming

$$\min_{\boldsymbol{Y},\boldsymbol{W},\boldsymbol{u}} tr\left(\boldsymbol{W}\right) + tr\left(Toep\left(\boldsymbol{u}\right)\right), \text{ subject to } \begin{bmatrix} \boldsymbol{W} & \boldsymbol{Y}^H \\ \boldsymbol{Y} & Toep\left(\boldsymbol{u}\right) \end{bmatrix} \geq \boldsymbol{0} \text{ and } \boldsymbol{Y}_\Omega = \boldsymbol{Y}^\star_\Omega. \quad (9)$$

The optimal solution $\boldsymbol{u}^*$ to (9) can be solved by using an off-the-shelf semidefinite programming solver. Then the frequencies can be computed based on the Vandermonde decomposition of $Toep\left(\boldsymbol{u}^*\right)$. In [6], it has been proven that, if the minimum separation of the frequencies Δf satisfies $\Delta f \geq \frac{1}{\lfloor (N-1)/4 \rfloor}$ and Ω is selected uniformly at random from $[N]$, then $\boldsymbol{Y}^\star$ is the unique optimizer to (9) with high probability. This conclusion provides theoretical guarantee for random sampling. Yet some deterministic sampling also can achieve the same effect. Next we employ coprime sampling to acquire the data instead of random sampling, and it also works well as shown later.

3. **Combination of Atomic Norms and Coprime Sampling.** The coprime sampling allows one to sample at two undersampled rates $1/(pT)$ and $1/(qT)$ ($p, q \in \mathbb{Z}^+$, $p \perp q$), while offers $O(pq)$ degrees of freedom, thus estimating the power spectrum of a signal at a significantly higher resolution. This sampling scheme can be applied in spatial arrays and temporal sampling. One example is that, for uniform spatial sampling with p and q sensors with appropriate interelement spacings, the difference co-array has $O(pq)$ freedoms

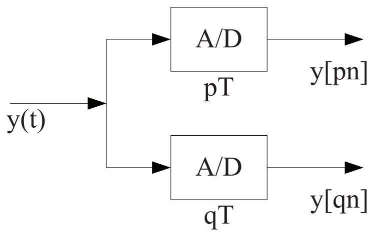

FIGURE 1. The process diagram of coprime sampling in the time domain

which can be exploited in beamforming and in direction of arrival (DOA) estimation. Yet another example is in the identification of sinusoids: from two sparsely sampled sets several number of sinusoidal frequencies can be identified.

Firstly we demonstrate how to estimate the frequencies of time domain sinusoids buried in noise using coprime sampling. The process diagram of coprime sampling is shown in Figure 1, which is much simpler than random sampling. After coprime samping we obtain two sub-Nyquist sequences, and the indices of the samples (or the sampling time points) are

$$\mathcal{I} = \{p, 2p, \cdots\} \cup \{q, 2q, \cdots\}. \tag{10}$$

Let $y[i]$ denote the sample with index i, and the samples are arranged as follows

$$\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{p}_1 & \boldsymbol{p}_2 & \cdots & \boldsymbol{p}_L \\ \boldsymbol{q}_1 & \boldsymbol{q}_2 & \cdots & \boldsymbol{q}_L \end{bmatrix}, \tag{11}$$

where

$$\boldsymbol{p}_l = [y[2pql], y[p(2ql+1)], \cdots, y[p(2ql+q-1)]]^T, \tag{12}$$

$$\boldsymbol{q}_l = [y[(2pl+1)q], y[(2pl+2)q], \cdots, y[(2pl+2p-1)q]]^T, \tag{13}$$

and $l = 1, 2, \cdots, L$. The minimum index and the maximum index in each column of $\boldsymbol{Y}$ are $2pql$ and $(2pl + 2p - 1)q$, respectively. So the equivalent N is $2pq - q + 1$. In some sense, it seems like $M = 2p + q - 1$ samples are selected from $N = 2pq - q + 1$ consecutive samples. In simulations we find that the estimation can attain quite high accuracy even though without restriction on the minimum separation of true frequencies.

In the case of spatial signature, the spatial samples are analogous to the sensor locations, and we will demonstrate to use two uniform sensor arrays with $2p - 1$ and q antennas to estimate DOAs of multiple impinging signals. The essence of uniform linear array DOA is to estimate K arrival angles θ_k from the measurements

$$y_m(t) = \sum_{k=1}^{K} s_k(t) e^{j2\pi dm \cdot \sin\theta_k/\lambda}, \quad m = 0, 1, \ldots, M-1, \tag{14}$$

where $s_k(t)$ is the k-th source signal and λ is the wavelength of the impinging signals. Traditionally, $d = \lambda/2$, and the problem is equivalent to estimating $f_k = \sin\theta_k/2$ as that in (1). In MUSIC and some methods based on CS [12, 13], the arrival angles θ_k are assumed to align with a fixed grid, which does not agree with practical condition. We use the above method based on atomic norms to attain continuous arrival angle estimation. Figure 2 shows a uniform linear array (ULA) with $2p - 1$ sensors having interelement spacing qd, and a ULA with q sensors having spacing pd. The observation matrix is formed as

$$\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{p}(t_1) & \boldsymbol{p}(t_2) & \cdots & \boldsymbol{p}(t_L) \\ \boldsymbol{q}(t_1) & \boldsymbol{q}(t_2) & \cdots & \boldsymbol{q}(t_L) \end{bmatrix}, \tag{15}$$

where $\boldsymbol{p}(t) = \left[y_0(t), y_p(t), \ldots, y_{(q-1)p}(t)\right]^T$ and $\boldsymbol{q}(t) = \left[y_q(t), y_{2q}(t), \ldots, y_{(2p-1)q}(t)\right]^T$. The number of snapshots is corresponding to the measurement times L in MMV. It is worth mentioning that the methods based on CS can estimate the arrival angles even from a

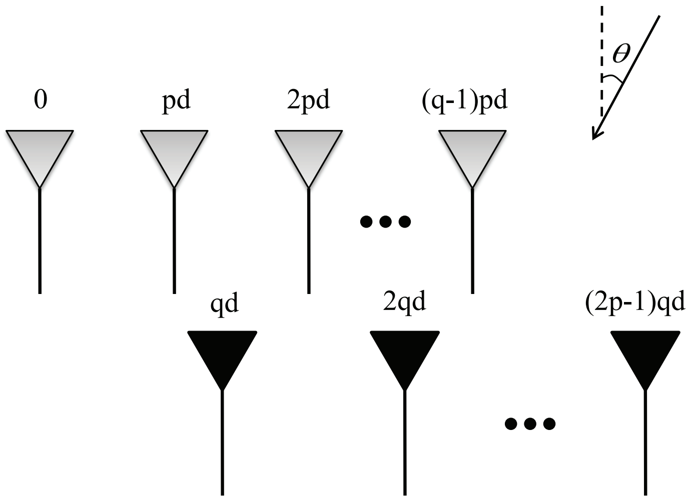

FIGURE 2. Coprime pair of uniform linear arrays with spacings pd and qd

single snapshot (i.e., SMV), which is an advantage compared with subspace-based methods.

4. **Numerical Simulations.** Firstly we simulate frequency estimation of multiple sinusoids buried in noise. The signals contain K frequency components with random amplitudes and random phase angles. The coprime undersampled ratios are set to $p = 3$ and $q = 7$. The normalized frequencies are assumed to distribute uniformly in $(0, 1]$. Complex white Gaussian noise is added to the measurements. $K = 3$ and $L = 20$ are fixed and the signal-to-noise ratio (SNR) varies from 10dB to 30dB. To keep it simple, we assume K is known, so the K frequencies corresponding to the largest K amplitudes are estimated results. The mean square errors (MSE) of estimated frequencies f_{es} different from true frequencies are computed by $MSE = \sqrt{\sum_{k=1}^{K} (f_{es} - f_k)^2} \Big/ K$. The MSE are obtained from 500 experiments for each SNR. For the sake of comparison, we also sample the signals at the normal rate and use conventional ESPRIT to estimate the frequencies. The same number of samples is used for our method and ESPRIT. As shown in Figure 3, the MSE of the proposed method are smaller than ESPRIT. The accuracy of ESPRIT is heavily dependent on the number of measurements. When the number of samples is not large, the method based on CS and atomic norms has a significant advantage over ESPRIT.

Then we apply the proposed method in spatial domain. We provide a simple simulation to demonstrate the performance of the proposed method in gridless DOA estimation. We consider $p = 3$, $q = 7$, $K = 3$ sources with directions of $10.5°$, $20.1°$ and $21.0°$ respectively, and $L = 30$. The source signals of each source are generated with unitary amplitudes and random phases. Complex white Gaussian noise at SNR = 20dB is added to the measurements. The DOA estimation results of the proposed method are presented in Figure 4. The MUSIC algorithm is also considered for comparison. MUSIC uses the same number of snapshots with our method and the grid interval in MUSIC is $0.5°$. It is shown that the three frequency components are correctly identified using the proposed method while MUSIC fails. Obviously, MUSIC requires a sufficiently dense grid to guarantee precision, increasing the amount of computation.

Finally, we compare the errors of DOA estimation with ESPRIT at different SNRs. Meanwhile, in spite of impracticality of random sampling, we construct random arrays in the program to compare with our method. $K = 3$ and $L = 20$ are fixed and the SNR varies from 10dB to 30dB. For convenience, we calculate the MSE of the corresponding frequencies instead of directions. The average values of MSE are obtained from 500 trials

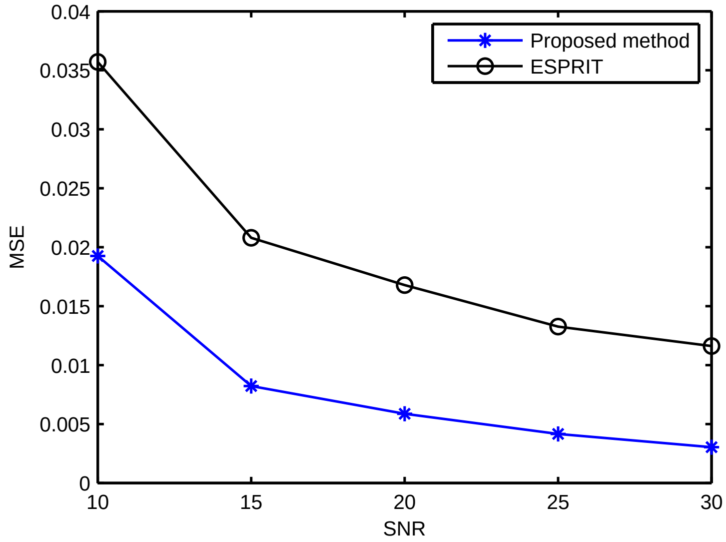

FIGURE 3. The average MSE of frequency estimation for different SNRs

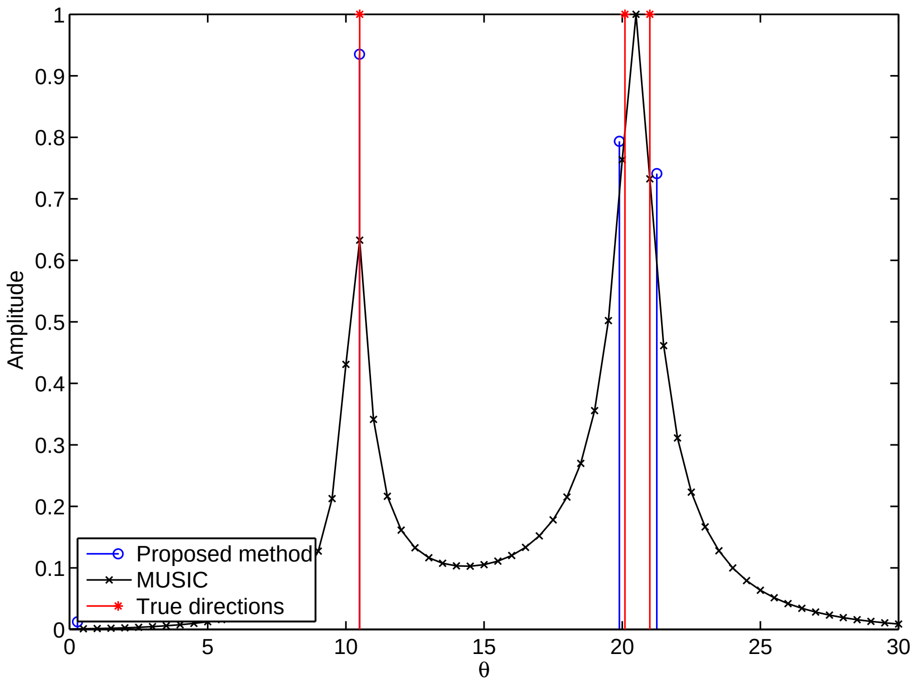

FIGURE 4. DOA estimation using the proposed method and MUSIC (shown only on the direction interval $[0°, 30°]$)

for each SNR. As shown in Figure 5, the proposed method has the same accuracy with that using random arrays, and they both outperform ESPRIT. This simulation illustrates that the deterministic coprime sampling can replace random sampling.

5. **Conclusions.** In this letter, we combine the advantages of atomic norms and coprime sampling to achieve continuous frequency estimation of multiple sinusoids with deterministic sub-Nyquist sampling. The implementation details are illustrated in two typical applications, namely spatial arrays and temporal sampling. The performance of the proposed method is demonstrated through simulations. The method completely eliminates the effect of grid discretization and has higher accuracy than conventional ESPRIT. The coprime sampling even has the same effect with random sampling required in CS. Further research is needed to explain the principle in theory in future studies.

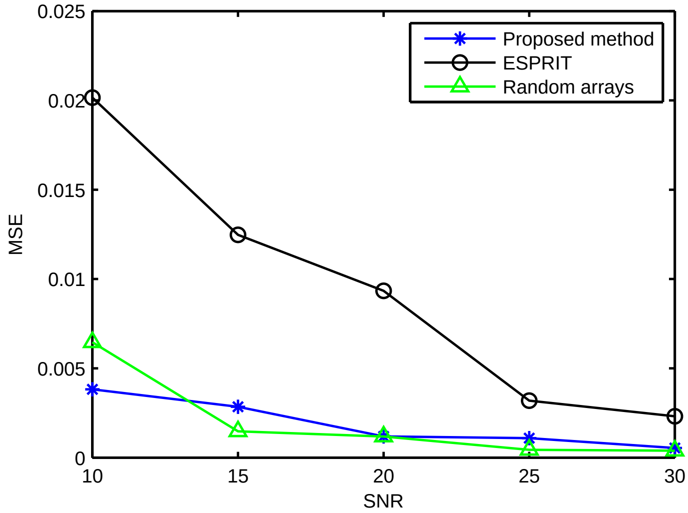

FIGURE 5. The average MSE of DOA estimation for different SNRs

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