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THE (R,Q) CONTROL OF A PERIODIC REVIEW INVENTORY SYSTEM WITH DISCRETE COMPOUND POISSON DEMAND

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Abstract. *The (R,Q) policy has been studied extensively in continuous review inventory system. However, there are few studies concerning the (R,Q) control of a periodic review inventory system. Most of existing periodic review models are controlled by an (R,T) policy. According to Zipkin, an (R,Q) policy is more responsive to demand fluctuations than an (R,T) policy but more complicated. The (R,Q) policy requires information on the inventory position at all times, while the (R,T) policy requires such information only periodically, at each review point. In this paper, we consider the (R,Q) control of a periodic review inventory system with discrete compound Poisson demand in infinite horizon. In our model, through the distribution of inventory position at each review point, we find out the distribution of inventory level at any time. And then, we propose a new way to establish average expected cost per time unit. Finally, an efficient method has been put forward to find the optimal re-order point $R^\star$ and the order quantity $Q^\star$.*
Keywords: Inventory, Periodic review, Discrete compound Poisson demand, (R,Q) policy, Inventory level distribution

1. **Introduction.** As periodic review models have been applied widely in practice, many researchers have studied the periodic review inventory system. Most of them use an (R,T) policy to control the periodic review inventory system. The (R,T) policy is that every time period of length T, it would review the inventory and place an order which will raise the inventory position up to R, which is also called (S,T) policy. The analysis of the (R,T) policy has been studied extensively in periodic review inventory system. The related studies can be seen from [1-3]. Besides, some of researchers adopt an (s,S) policy to control the periodic review inventory system. The (s,S) policy in those models is that every time period of length T, it would review the inventory and if the inventory position is below or at the point s, it would place an order which would raise the inventory position up to S. The related studies can be referred to [4-6].

Compared with an (R,Q) policy (every time period of length T, we review the inventory and if the inventory position is below or at the point R, an order is triggered. The order is for Q units if this is enough to bring inventory position (IP) at the periodic review times above R. If not, the order is for the smallest number of batches needed to make the inventory position larger than R.), adopting the (R,T) policy or the (s,S) policy in the periodic inventory system has some disadvantages. According to Zipkin [7], when considering the control of periodic review inventory system, it is better to use an (R,Q) policy, for that policy is more responsive to demand fluctuations than an (R,T) policy. Furthermore, the (s,S) policy also has a disadvantage of the variation of ordering quantity, which may lead to suppliers more easily to make mistakes. Suppliers prefer to use the (R,Q) policy to control supply chains rather than adopt the (s,S) policy. Therefore, there is a need to extend studies of the (R,Q) control of a periodic review inventory system. Our study aims at complementing the existing studies of the (R,Q) control of a periodic review inventory system and establishing a new model controlled by (R, Q, T) policy to solve the related problems.

As far as our information goes, the (R, Q) policy has been studied extensively in continuous review inventory system and a number of algorithms for finding the optimal (R, Q) policy has been introduced. For example, for models with back orders for shortages, an efficient algorithm is proposed to find optimal (r, q) policy for continuous review stochastic inventory systems in Federgruen and Zheng [8]. Other related work includes Axsäter [9], Zheng [10], Lau et al. [11], Hill [12], Huang et al. [13], Ang et al. [14], and Guan and Zhao [15]. However, there are few studies concerning the (R, Q) control of a periodic review inventory system. Johansen and Hill [16] consider the (r, Q) control of a periodic-review inventory system with continuous demand and lost sales. In their model, they suppose the fixed lead time is an integral number of review periods. They use asymptotic renewal theory to establish expected cost during an order cycle and propose an approximate method to decide the re-order R and the order quantity Q. They do not investigate the distribution of inventory level at any time during a cycle. They regard a cycle as a whole to quantify the holding cost and the lost cost, which do not reveal expected cost at any time during a cycle. In fact, the expected cost at any time varies during a cycle. In this paper, we consider the (R, Q) control of a periodic review inventory system with discrete compound Poisson demand in infinite horizon. We figure out the expected cost at any time during a cycle by finding out the distribution of inventory level at any time, and then establish average expected cost per time unit as a cost oriented function. Axsäter [9] considers a periodic review inventory system and presents a simple formula for the fill rate. The formulas can be applied to the inventory system controlled by (R, Q) policy or (s, S) policy. He just considers the fill rate (service level) of periodic review supply system but does not consider the costs of periodic review supply system, such as holding cost and the shortage cost. Our study extends his model and considers the costs of periodic review supply system. In our study, we construct a cost oriented function depending on the re-order R and the order quantify Q, and propose an effective algorithm to find the optimal re-order $R^\star$, the order quantity $Q^\star$.

In this paper, we consider the (R, Q) control of a periodic review inventory system with discrete compound Poisson demand for the reason that from a modelling perspective, the compound Poisson process can model the demand in the context of fast moving items by considering very low demand time intervals [17]. Furthermore, the compound Poisson demand can be extended to continuous demand and the basic framework of analysis still works. In our model, the length of the periodic review is a given arbitrary constant. Both the periodic review T and the lead time L are exogenous.

The rest of the paper is organized as follows. Section 2 gives a detailed description of the model. Section 3 sets up the general model and establishes per time unit average expected cost function which depends on re-order R and the order quantity Q. In Section 4, we propose an effective algorithm to determine the optimal the re-order $R^\star$ and the order quantity $Q^\star$. Numerical examples will be used to illustrate the effectiveness of the algorithm and to demonstrate the impact of the parameters on the optimal policy in Section 5. Section 6 summarizes our work, where topics for future study are also discussed.

2. **Model Description.** Before concrete description of the model, let us introduce the following notations:

μ average demand per unit of time
μ' average demand of a time interval of length t'
σ' standard deviation of the demand of a time interval of length t'
L lead time
T periodic review
t an arbitrary review time

$IL(t)$	inventory level at time t
$IP(t)$	inventory position at time t
$D(L)$	stochastic demand in a time interval of length L
$P(IL(t) = j)$	probability of the inventory level equal to j at time t in steady state
f_j	probability of a customer's demand size j $(j = 1, 2, \ldots)$
f_j^k	probability that k customers give the total demand j
h	holding cost per unit and time unit
b	shortage cost per unit and time unit
A	ordering cost
R	re-order point
Q	order quantity
C	average expected cost per unit of time in steady state

In this paper, we consider a periodic review inventory system with discrete compound Poisson demand that adopts an (R, Q) policy. And we suppose the system operates for an infinite planning horizon. The length of the fixed periodic review T in our model is an arbitrary given constant. No matter whether it is longer or shorter than the lead time L, it has the same train of thought to construct the model. And the (R, Q) policy we consider is that every time period of length T, we review the inventory and if the inventory position is below or at the point R, an order is triggered. The order is for Q units if this is enough to bring inventory position (IP) at the periodic review times above R. If not, the order is for the smallest number of batches needed to make the inventory position larger than R. In other words, the order can be nQ, $n = 1, 2, \ldots$. Therefore, in steady state we must have $R + 1 \leq IP \leq R + Q$, which is the same to a continuous review model. We assume a common discrete stochastic demand in our model: discrete compound Poisson demand. This demand means that the customers arrive according to a Poisson process with given intensity λ and the size of a customer demand is also a random variable. We further assume that a customer's demand is always a nonnegative integer, which is common for many situation in practice. We shall also assume that not all demands are multiples of some integer larger than one. Obviously, if the demand of a customer is always one, the compound Poisson demand degenerates to pure Poisson demand.

Every time when we place an order, it will incur a fixed cost A. As the same to the continuous review model, when the inventory level is above zero, it leads to inventory holding costs. The holding cost per unit and time unit is denoted h. When the inventory level is below zero, it incurs inventory shortage costs and b is denoted as the shortage cost per unit and time unit. Our model aims to minimize the average expected cost per time unit $C(R, Q)$ by determining the re-order R and the order quantity Q. So we need to propose an effective algorithm to find out the optimal re-order $R^\star$ and order quantity $Q^\star$.

3. **Model Development.** To construct the model of average expected cost per time unit, we need to know the distribution of inventory level at any time. As the same to continuous review inventory system, because orders after time t have not been delivered at time $t+L$, the inventory level IL at time $t+L$ after a possible delivery can be obtained as

$$IL(t + L) = IP(t) - D(L). \tag{1}$$

So we have to find out the distribution of inventory position at time t and get the distribution of the demand over a given lead time L. However, in a periodic review inventory system, finding out the distribution of inventory position at any time is complicated. In our model, we do not try to seek out the distribution of inventory position at any time. We just need to know the distribution of inventory position at all periodic review times and then we can find out the distribution of inventory level at any time during a cycle in infinite horizon.

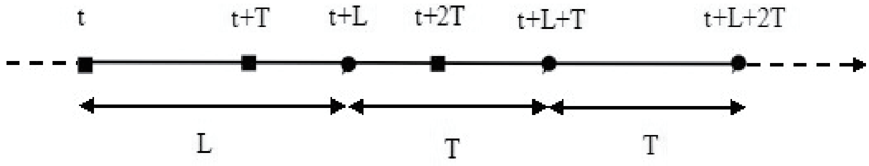

Figure 1. Considered time epochs

Consider Figure 1 and assume t is an arbitrary review time in infinite horizon. This means that the orders can take place at times $\ldots$, t, $t+T$, $\ldots$. If an order takes place at time t, it will be delivered at time $t+L$. It is evidently that the next possible delivery will be at time $t+L+T$. Hence, we just need to consider the distribution of inventory level at any time during the interval $[t+L, t+L+T]$ for another interval is similar to that in steady state. The reason why the distributions of inventory level in all intervals are identical is that all periodic review times have the same distribution that is uniformly distributed on the integers $R+1$, $R+2$,$\ldots$, $R+Q$. We will prove this in **Proposition 3.1**. In other words, the inventory level at time $t+L$ after a possible delivery $IL(t+L)$ is the same to the inventory level at time $t+L+T$ after a possible delivery $IL(t+L+T)$. Apparently, the $IL(t+L+T)$ can be obtained as

$$IL(t+L+T) = IP(t+T) - D(L). \tag{2}$$

Since $IP(t)$ and $IP(t+T)$ have the same distribution, $IL(t+L)$ and $IL(t+L+T)$ after a possible delivery also have the same distribution. Identically, given an arbitrary time z, the distribution of $IL(t+L+z)$ that locates in the interval $[t+L, t+L+T]$ is similar to the distribution of $IL(t+L+T+z)$ that locates in the interval $[t+L+T, t+L+2T]$. Hence, we only need to analyze the distribution of inventory level during an interval in steady state. Without loss of generality, we focus on the interval $[t+L, t+L+T]$. And we regard the interval as a cycle.

From the analysis above, we need to find out the distribution of inventory position at periodic review times firstly. We give the following proposition.

Proposition 3.1. *In steady state the inventory position at periodic review times is uniformly distributed on the integers* $R+1$*,* $R+2$*,* $\ldots$*,* $R+Q$*.*

Proof: In the spirit of Axsäter [9], as the same to the continuous review (R,Q) policy model, we can find out that inventory position at periodic review times ranges $R+1$ from $R+Q$. Recall that we assume that not all demands are multiples of some integer larger than one. So it is evident that IP at periodic review times can reach all the considered inventory positions in the long run. Suppose that the inventory position is $R+i$ at some periodic review time. At the next periodic review time, IP will jump to some other value (or possibly to the same value if the size of the demand of the whole periodic review interval is zero or a multiple of Q). The probability for a jump from $R+i$ to $R+j$ is denoted by $p_{i,j}$. It is apparently that these probability can be confirmed from the distribution of the demand size of the whole periodic review interval. Obviously, we can regard the jumps as a Markov chain. The chain is irreducible for all states can be reached. And the chain is evident not periodic in our model for the uncertain demand of the whole periodic review interval. Therefore, it is an ergodic chain. Since the chain is irreducible and ergodic, it has a unique steady-state distribution. So we just need to prove that the uniform distribution is a steady-state distribution. It means that we should show that

$$\sum_{i=1}^{Q} \frac{1}{Q} p_{i,j} = \frac{1}{Q},\ j = 1, 2, \ldots, Q. \tag{3}$$

It equals to that $\sum_{i=1}^{Q} p_{i,j} = 1$ for all values of j. The chain is said to be doubly stochastic when this equality is satisfied. We assume the state is i and consider a certain demand of the whole periodic review interval is k. If given k, the next state is determined, i.e., $p_{i,j}(k)$ is equal to one for a certain j and zero otherwise. Similarly, if given k and j, the preceding state is known. It means that $p_{i,j}(k)$ is equal to one for a certain i and zero otherwise. Consequently, $\sum_{i=1}^{Q} p_{i,j}(k) = 1$. And the probability $p_{i,j}$ can be expressed as the average of $p_{i,j}(k)$ over k, i.e., $p_{i,j} = E_k p_{i,j}(k)$. Now we have

$$\sum_{i=1}^{Q} p_{i,j} = \sum_{i=1}^{Q} E_k p_{i,j}(k) = E_k \sum_{i=1}^{Q} p_{i,j}(k) = E_k\{1\} = 1. \tag{4}$$

This completes the proof.

Now we discuss the stochastic demand in a time interval of length t'. Since the customers arrive according to a Poisson process with given intensity λ, the number of customers has a Poisson distribution and the probability for k customers is

$$P(k) = \frac{(\lambda t')^k}{k!} \exp^{-\lambda t'},\ k = 0, 1, 2, \ldots. \tag{5}$$

Both the average and the variance of the number of customers are equal to $\lambda t'$. Without loss of generality, we assume that each customer demands an integral number of units and there are no demands of size zero. Evidently, if $f_1 = 1$, the demand process degenerates to pure Poisson demand. Then the demand is equal to the number of customers and the demand distribution is given by Equation (5). We figure out the distribution of $D(t')$ as follows. Firstly, we can know that $f_0^0 = 1$, $f_j^0 = 0$, and $f_j^1 = f_j$. If we know f_j^1, we can get the k-fold convolution of f_j, f_j^k, recursively as

$$f_j^k = \sum_{i=k-1}^{j-1} f_i^{k-1} f_{j-i},\ k = 2, 3, 4, \ldots. \tag{6}$$

Using Equation (5) we then have

$$P(D(t') = j) = \sum_{k=0}^{j} \frac{(\lambda t')^k}{k!} \exp^{-\lambda t'} f_j^k,\ j = 0, 1, 2, \ldots. \tag{7}$$

Now we determine the mean and the standard deviation of the demand during one unit of time. According to Axsäter [9], we have

$$\mu = \lambda \sum_{j=1}^{\infty} j f_j \tag{8}$$

$$\sigma = \lambda \sum_{j=1}^{\infty} j^2 f_j \tag{9}$$

and then we can have $\mu' = \mu t'$ and $(\sigma')^2 = \sigma^2 t'$. From Equations (8) and (9), we can note that $\sigma^2/\mu \geq 1$ with equality merely for pure Poisson demand. So as compound Poisson demand, it is not possible to model demand processes with $\sigma^2/\mu < 1$.

After having determined the distribution of inventory position at all periodic review times and stochastic demand during a time interval of length T (if we let $t' = T$, we can get $D(T)$ by Equation (7)), we can determine the distribution of inventory level at any time of the interval $[t + L, t + L + T]$. As mentioned above, we suppose t is an arbitrary periodic review point in infinite horizon. Then we know that the inventory level IL at time $t + L$ after a possible delivery can be obtained as $IL(t + L) = IP(t) - D(L)$ from

Equation (1). Suppose $t+L+z$ is an arbitrary time in the interval $[t+L, t+L+T]$. Then we have

$$IL(t+L+z) = IP(t) - D(L+z). \tag{10}$$

Particularly, the inventory level at time $t+L+T$ before a possible delivery can be determined in exactly the same way. From Equation (10), we have

$$P(IL(t+L+z)=j) = \frac{1}{Q}\sum_{k=\max\{R+1,j\}}^{R+Q} P(D(L+z)=k-j),\ j \le R+Q, \tag{11}$$

where the inventory level at time $t+L+z$ can never exceed the inventory position at time t, i.e., $k \ge j$. We can see from Equation (11) that the distribution of inventory level during the interval $[t+L, t+L+T]$ is a function that depends on a time variable ranging from L to $L+T$, which is different from the (R,Q) control model in continuous review.

Now we try to establish average expected cost per unit of time in steady state. Since the distribution of inventory level varies during the interval $[t+L, t+L+T]$, firstly we should find out the expected cost at any time in the interval $[t+L, t+L+T]$. As the same to the (R,Q) policy of continuous model [7,9], the expected cost at time $t+L+z$ can be expressed as

$$C(R,Q)[t+L+z] = \frac{A\mu}{Q} + \frac{1}{Q}\sum_{k=R+1}^{R+Q} g(k)[t+L+z], \tag{12}$$

where

$$g(k)[t+L+z] = -b(k-\mu(L+z)) + (h+b)\sum_{j=1}^{k} jP(IL(t+L+z)=j). \tag{13}$$

Note that $g(k)[t+L+z]$ is a convex function of the inventory position k. Furthermore, $g(k)[t+L+z] \to \infty$ as $|k| \to \infty$. It has been proved by Axsäter [9] or Zipkin [7]. Here we just cite the conclusion. Let us go back to the (R,Q) policy of periodic review. The average expected cost per unit of time can be expressed as

$$\begin{aligned} C(R,Q) &= \frac{\dfrac{AT\mu}{Q} + \dfrac{1}{Q}\displaystyle\int_0^T \sum_{k=R+1}^{R+Q} g(k)[t+L+z]dz}{T} \\ &= \frac{A\mu}{Q} + \frac{1}{QT}\sum_{k=R+1}^{R+Q}\int_0^T g(k)[t+L+z]dz. \end{aligned} \tag{14}$$

Since the analysis about the other intervals is the same to the interval $[t+L, t+L+T]$, Equation (14) can be represented per time unit expected cost in steady-state.

4. **Optimal Solution.** In this section, our objective is to optimize $C(R,Q)$ with respect to R and Q. We let $G(k) = \int_0^T g(k)[t+L+z]dz$, and it has a good approximation as

$$G(k) = \int_0^T g(k)[t+L+z]dz \approx \frac{T}{n+1}\sum_{l=0}^{n} g(k)\left[t+L+l\frac{T}{n}\right]. \tag{15}$$

Furthermore, as $n \to \infty$, then

$$\int_0^T g(k)[t+L+z]dz = \frac{T}{n+1}\sum_{l=0}^{n} g(k)\left[t+L+l\frac{T}{n}\right].$$

Therefore, the objective function $C(R,Q)$ reacts approximately as

$$C(R,Q) \approx \frac{\dfrac{AT\mu}{Q} + \dfrac{1}{Q}\dfrac{T}{n+1}\displaystyle\sum_{k=R+1}^{R+Q}\sum_{l=0}^{n} g(k)\left[t+L+l\frac{T}{n}\right]}{T}. \tag{16}$$

For the sake of convenience, we let

$$\bar{G}(k) = \sum_{l=0}^{n} g(k)\left[t+L+l\frac{T}{n}\right], \tag{17}$$

and then we have the following proposition.

Proposition 4.1. *$\bar{G}(k)$ is a convex function of the inventory position k. Furthermore, $\bar{G}(k) \to \infty$ as $|k| \to \infty$.*

Proof: From the above section, we have known that

$$\begin{aligned} &g(k)\left[t+L+l\frac{T}{n}\right] \\ &= -b\left(k-\mu\left(L+l\frac{T}{n}\right)\right) + (h+b)\sum_{j=1}^{k} jP\left(IL\left(t+L+l\frac{T}{n}\right)=j\right),\ l \le n \end{aligned} \tag{18}$$

is a convex function of the inventory position k and $g(k)\left[t+L+l\frac{T}{n}\right] \to \infty$ as $|k| \to \infty$. Then

$$\bar{G}(k) = \sum_{l=1}^{n} g(k)\left[t+L+l\frac{T}{n}\right]$$

is also a convex function of the inventory position k for a theorem that the sum of finite convex functions is also a convex function [18]. Obviously, $\bar{G}(k) \to \infty$ as $|k| \to \infty$. This completes the proof.

From Equations (16) and (17), we note that

$$C(R,Q) \approx \bar{C}(R,Q) = \frac{\dfrac{AT\mu}{Q} + \dfrac{1}{Q}\dfrac{T}{n+1}\displaystyle\sum_{k=R+1}^{R+Q}\bar{G}(k)}{T} \tag{19}$$

where

$$\bar{G}(k) = \sum_{l=0}^{n} g(k)\left[t+L+l\frac{T}{n}\right].$$

Consequently, we can consider the optimization problem as follows.

$$\min_{R,Q}\bar{C}(R,Q) = \min_{R,Q}\frac{\dfrac{AT\mu}{Q} + \dfrac{1}{Q}\dfrac{T}{n+1}\displaystyle\sum_{k=R+1}^{R+Q}\bar{G}(k)}{T} \tag{20}$$

where

$$\bar{G}(k) = \sum_{l=0}^{n} g(k)\left[t+L+l\frac{T}{n}\right],$$

and n should be large enough.

Since $\bar{G}(k)$ is a convex function of the inventory position k, given Q, we can find out the optimal solution $R^\star(Q)$ of the objection function $\bar{C}(Q) = \min_R \bar{C}(R,Q)$. With spirit of Axsäter [9] and Zheng [10], more generally we have

$$R^\star(Q+1) = \begin{cases} R^\star(Q)-1, & \text{if } \bar{G}(R^\star(Q)) \le \bar{G}(R^\star(Q)+Q+1), \\ R^\star(Q), & \text{otherwise,} \end{cases} \tag{21}$$

and

$$\bar{C}(Q+1) = \bar{C}(Q)\frac{Q}{Q+1} + \left[\min\left\{\bar{G}(R^\star(Q)), \bar{G}(R^\star(Q)+Q+1)\right\}\right]\frac{1}{Q+1}. \tag{22}$$

It is apparent from Equation (22) that $\bar{C}(Q+1) \geq \bar{C}(Q)$ if and only if $\min\left\{\bar{G}(R^\star(Q)), \bar{G}(R^\star(Q)+Q+1)\right\} \geq \bar{C}(Q)$. Furthermore, it is evident that $\min\left\{\bar{G}(R^\star(Q)), \bar{G}(R^\star(Q)+Q+1)\right\}$ is increasing with Q. Let $Q^\star$ be the smallest Q such that $\bar{C}(Q+1) \geq \bar{C}(Q)$. It follows from Equation (21) that $\bar{C}(Q) \geq \bar{C}(Q^\star)$ for any $Q \geq Q^\star$. So $Q^\star$ and $R^\star(Q^\star)$ provide the optimal solution.

To conclude, it is easy to determine the optimal solution by using Equations (21) and (22) until the costs increase.

Here, we give an algorithm to find out the optimal solution.

Step 1: Let $Q = 1$, and set n large enough. Since $\bar{G}(k)$ is a convex function of the inventory position k, we can find out an optimal $k^\star$ that minimizes $\bar{G}(k)$. A corresponding optimal re-order point for $Q = 1$ is $R^\star(1) = k^\star - 1$. And then figure out $\bar{C}(R^\star(Q), Q)$.

Step 2: Set $Q' = Q + 1$, we should compare $\bar{G}(R^\star(Q))$ with $\bar{G}(R^\star(Q) + Q + 1)$. If $\bar{G}(R^\star(Q)) \leq \bar{G}(R^\star(Q) + Q + 1)$, $R^\star(Q') = R^\star(Q) - 1$. Otherwise, $R^\star(Q') = R^\star(Q)$. And then figure out $\bar{C}(R^\star(Q'), Q')$.

Step 3: Compare $\bar{C}(R^\star(Q), Q)$ with $\bar{C}(R^\star(Q'), Q')$. If $\bar{C}(R^\star(Q'), Q') \geq \bar{C}(R^\star(Q), Q)$, stop the process. And the optimal solution $Q^\star = Q$, $R^\star(Q^\star) = R^\star(Q)$. Else set $Q = Q'$, and then continue to execute Step 2.

5. **Numerical Examples.** To provide a better understanding of the model and the algorithm of finding the optimal solution described above, we consider two special and common cases.

5.1. **The (R, Q) control of a periodic review inventory system with pure Poisson demand.** We consider the demand of a customer is always one. It means that $f_1 = 1$. So Equation (7) can be simplified as the following:

$$P(D(t') = j) = \frac{(\lambda t')^j}{j!}\exp^{-\lambda t'},\ j = 0, 1, 2, \ldots.$$

And Equation (11) can be simplified as the following:

$$P(IL(t+L+z) = j) = \frac{1}{Q}\sum_{k=\max\{R+1,j\}}^{R+Q}\frac{(\lambda(L+z))^{k-j}}{(k-j)!}\cdot\exp^{-\lambda(L+z)},\ j \leq R+Q.$$

We consider $b = 20$, $h = 5$, $\lambda = 4$, $T = 6$, $L = 4$, $A = 100$. According to Equations (19), (20) and using the algorithm, we can get the optimal $k^\star$ that minimizes $\bar{G}(k)$ is 36, the optimal re-order $R^\star = 27$ and the order quantity $Q^\star = 19$. The algorithm is implemented in Matlab 2012b and we set $n = 1000$ there, which is supposed to be large enough. Consequently, we can figure out the average expected cost per unit of time $\bar{C}(R^\star, Q^\star) = 95.0883$. Since $n = 1000$ is supposed to be large enough, $\bar{C}(R^\star, Q^\star)$ has a good approximation of $C(R^\star, Q^\star)$. Hence, $R^\star = 27$ and $Q^\star = 19$ are the optimal solution of $C(R, Q)$. Figure 2 shows that $\bar{G}(k)$ is a convex function of the inventory position k.

5.2. **The (R, Q) control of a periodic review inventory system with logarithmic compounding distribution.** We assume that the demand size of a customer has a logarithmic distribution. This means that

$$f_j = -\frac{\alpha^j}{\ln(1-\alpha)j},\ j = 1, 2, 3, \ldots.$$

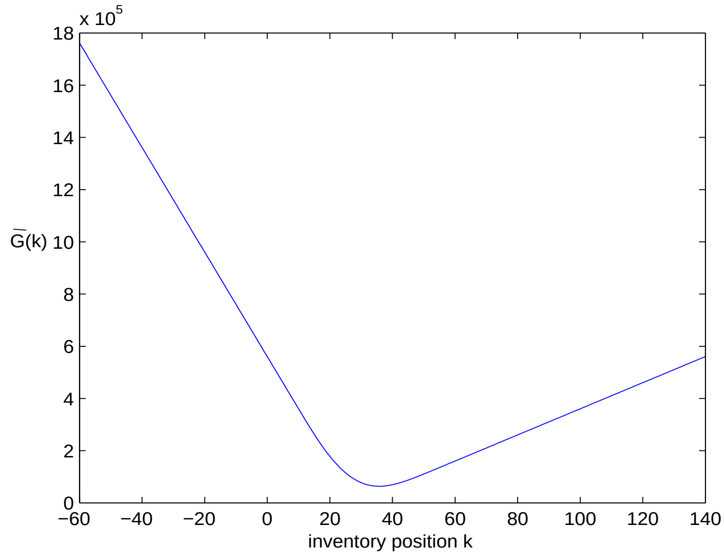

FIGURE 2. The figure of $\bar{G}(k)$ of the model with pure Poisson demand

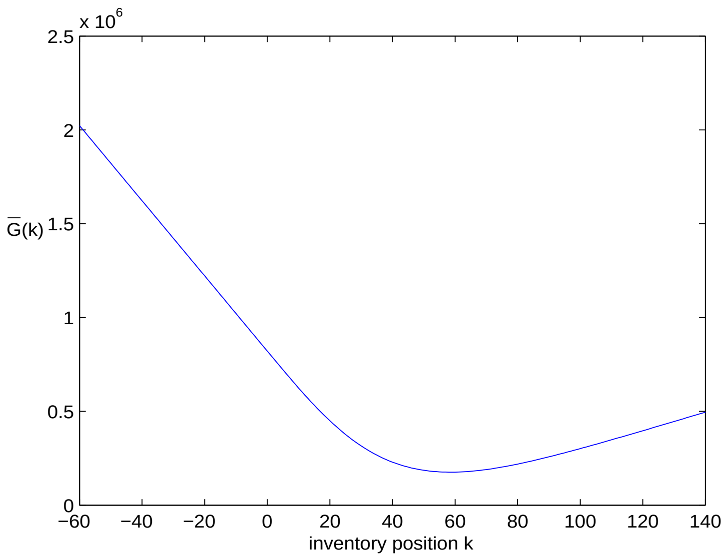

FIGURE 3. The figure of $\bar{G}(k)$ in the model with logarithmic compounding distribution

According to Axsäter [9], Equation (7) can be simplified as the following:

$$\begin{aligned} &P(D(t')=0)=(1-p)^r,\\ &P(D(t')=k)=\frac{r(r+1)\cdots(r+k-1)}{k!}(1-p)^r p^k,\ k=1,2,\ldots \end{aligned}$$

where

$$p=1-\frac{\mu'}{(\sigma')^2}=\alpha,\ r=\mu'\frac{1-p}{p},\ \mu'=-\frac{\lambda t'\alpha}{(1-\alpha)\ln(1-\alpha)},\ (\sigma')^2=-\frac{\lambda t'\alpha}{(1-\alpha)^2\ln(1-\alpha)}.$$

And using Equation (11), we can determine $P(IL(t+L+z)=j)$ at any time during the interval $[t+L, t+L+T]$.

We consider $b=20$, $h=5$, $A=100$, $T=6$, $L=4$, $\alpha=0.9$, $\lambda=1.5$. According to Equations (19), (20) and using the algorithm, we can get the optimal $k^\star$ that minimizes $\bar{G}(k)$ is 59, the optimal re-order $R^\star=44$, and the order quantity $Q^\star=31$. The algorithm is implemented in Matlab 2012b and we set $n=1000$ there, which is supposed to be large enough. Consequently, we can figure out the average expected cost per unit of time

$\bar{C}(R^\star, Q^\star) = 204.2931$. Since $n = 1000$ is supposed to be large enough, $\bar{C}(R^\star, Q^\star)$ has a good approximation of $C(R^\star, Q^\star)$. Hence, $R^\star = 44$ and $Q^\star = 31$ are the optimal solution of $C(R, Q)$. Figure 3 shows that $\bar{G}(k)$ is a convex function of the inventory position k.

6. **Conclusion and Future Research.** In our paper, we consider the (R, Q) control of a periodic review inventory system with discrete compound Poisson demand. We do not regard a cycle as a unit to analyze the related properties and establish expected cost during a cycle, of which some researchers have studied [16]. Instead, we use a new way to establish per time unit average expected cost by finding out the distribution of inventory level at any time, which is the most important contribution in our study. Through our analysis, we have discovered that in steady state, the distribution of inventory level is a function with respect to a time variable ranging from L to $L + T$. Correspondingly, the construction of average expected cost per unit of time is different from the preceding one. Furthermore, we propose an effective method to find out the optimal re-order $R^\star$, and the quantity $Q^\star$. In the numerical examples section, we consider the pure Poisson distribution and the logarithmic compounding distribution, respectively. In fact, other compound Poisson distribution follows the analogous analysis. Besides, we investigate the sensitivity of the holding cost h, the shortage cost b, the ordering cost A, and the length of periodic review T.

The analysis of determination of the distribution of inventory level at any time can be applied into other models. So our research can be further extended along the following lines.

(1) To consider that the inventory system is controlled by (s, S) policy, and establish the average expected cost per unit of time by determining the distribution of inventory level at any time. However, the computation of the inventory controlled by (s, S) is more complex and the order quantity of the inventory controlled by (s, S) is varying.

(2) To relax the assumption of compound Poisson demand and allow for more demand distribution, such as normally distributed demand. For items with higher demand, the normally distributed demand is usually more convenient and efficient to model the demand over a time period by a continuous distribution.

(3) To consider the length of periodic review T as an endogenous variable and introduce a cost function of periodic review T. Correspondingly, our aim is to determine the optimal re-order $R^\star$, the quantity $Q^\star$ and the length of periodic review $T^\star$.

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