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# FUZZY ADAPTIVE PRESCRIBED PERFORMANCE CONTROL FOR A CLASS OF UNCERTAIN CHAOTIC ECONOMICAL SYSTEM

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Abstract. *This paper presents a fuzzy adaptive control method for a class of uncertain chaotic economical system, capable of guaranteeing prescribed performance bounds. By prescribed performance bounds we mean that tracking error should converge to a predefined arbitrarily small set, with convergence rate no more than a prescribed value. The main idea is to transform the original constrained system into an equivalent one via an appropriately defined output error transformation. Furthermore, for updating the parameters of the fuzzy logic systems, a proportional-integral adaptation law is introduced. Finally, an illustrative example is given to demonstrate the effectiveness of the proposed results.*

**Keywords:** Chaotic economical system, Prescribed performance control, Fuzzy adaptive control

1. **Introduction.** In the last decades researchers from almost all fields of natural sciences have studied phenomena that involved nonlinear systems exhibiting chaotic behavior. This is due to the fact that nonlinear systems demonstrate rich dynamics and have sensitivity on initial conditions. In the 1980s, economist Stutzer revealed the chaotic phenomena in economic system for the first time [1], which aroused the human's reflection on the traditional economics theory and after that the issue on nonlinear economics, chaotic economics has become a hot topic [2-4]. The modern research has shown that economic system can exhibit not only stable, unstable and periodic behavior but also chaotic phenomenon. In fact, financial crisis is just a chaotic phenomenon of the economic system [5]. Moreover, economists have noticed the fact that uncertainties in the economic development, such as the impact of non-economic factors, the sudden change of economy in frequency are increasing [6]. Financial risks come from uncertainties, and therefore it has an important theoretical and practical significance by introducing uncertainties into economic system [7]. Taking into consideration the chaotic behaviors and uncertainties in the economic system, it is essential to investigate the chaos control strategies for economic and financial systems in order to solve financial crisis and the related problems. The aim of chaos control is to suppress or eliminate the chaotic behavior of the nonlinear system.

On the other hand, one of the most important issues associated with the adaptive control of nonlinear systems is tracking error performances. However, no systematic procedure exists to accurately compute the required upper bounds of tracking errors. The problem is relaxed for feedback linearizable systems in [8,9]. Performance issues on transient behavior (i.e., overshoot, undershoot convergence rate) are hard to be established analytically, even if the nonlinearities are completely known. In [10], $L_2$ norm of the tracking error which is derived to be a function of initial estimation errors and design parameters is studied. Contributions in guaranteeing prescribed transient and steady state output error bounds can be found in [11-15]. In [11,12], the tracking error can converge to a neighborhood of prescribed radius $\lambda > 0$, while, in [13], funnel control is established in the light of which the achieved transient behavior is governed by a dynamic gain involving the required transient response characteristics. In [14], robust adaptive control schemes for SISO strict feedback nonlinear systems, capable of guaranteeing prescribed performance bounds are considered. In [15], a universal prescribed performance controller is obtained for cascade systems involving dynamic uncertainty, unknown nonlinearities and exogenous disturbances. The synchronization for two different fractional-order chaotic systems, capable of guaranteeing synchronization error with prescribed performance, is investigated in [16]. To ensure desired transient and steady-state behaviours of the tracking error under actuator faults, the dynamic effect caused by the actuator failures on the error dynamics of a transformed model is analysed in [17].

To the best of our knowledge, there are few studies dealing with the prescribed performance control problem for economic system. Inspired by the work in [14-17], we investigate the tracking control with guaranteed prescribed performance for uncertain economic system. The unknown nonlinear functions are approximated by fuzzy logic systems. Compared with the related work, there are two main contributions that are worth to be emphasized. (1) Compared with the existing results, the system we considered consists of not only external disturbances but also unknown model uncertainty. (2) An adaptation PI law based on $e$-modification is proposed to update the fuzzy parameters.

This paper is organized as follows. Problem formulation and preliminaries are given in Section 2, and description of the fuzzy logic system is given in Sections 3. Adaptive fuzzy control design with prescribed performance is proposed in Section 4. Section 5 provides a simulation example to illustrate the effectiveness of our results. Finally, Section 6 gives some concluding remarks.

## 2. Problem Formulation and Preliminaries.

Consider the following uncertain nonlinear economical system:

$$\begin{aligned}
\dot{x}_1 &= x_3 + (x_2 - a)x_1 + \Delta f_1(x_1, x_2, x_3, t) + d_1(t) + u_1(t) \\
\dot{x}_2 &= 1 - bx_2 - x_1^2 + \Delta f_2(x_1, x_2, x_3, t) + d_2(t) + u_2(t) \\
\dot{x}_3 &= -x_1 - cx_3 + \Delta f_3(x_1, x_2, x_3, t) + d_3(t) + u_3(t)
\end{aligned} \tag{1}$$

where the three state variables $x_1$, $x_2$, $x_3$ stand for the interest rate, the investment demand, and the price index, respectively. Constant $a$ is the saving amount, constant $b$ is the cost per investment, and constant $c$ is the elasticity of demand of the commercial markets. $\Delta f_i(x_1, x_2, x_3, t)$, $i = 1, 2, 3$ and $d_i(t)$, $i = 1, 2, 3$ represent unknown model uncertainty and external disturbances of the system, respectively, and $u_i(t)$, $i = 1, 2, 3$ is the control input. Denote

$$\begin{gathered}
x = [x_1, x_2, x_3]^T \\
f(x) = \left[x_3 + (x_2 - a)x_1, 1 - bx_2 - x_1^2, -x_1 - cx_3\right]^T \\
d(t) = [d_1(t), d_2(t), d_3(t)]^T \\
\Delta f(x, t) = [\Delta f_1(x_1, x_2, x_3, t), \Delta f_2(x_1, x_2, x_3, t), \Delta f_3(x_1, x_2, x_3, t)]^T \\
u(t) = [u_1(t), u_2(t), u_3(t)]^T
\end{gathered}$$

Then, (1) can be rewritten as

$$\dot{x} = f(x) + \Delta f(x, t) + u(t) + d(t) \tag{2}$$

The objective of this paper is to construct an adaptive fuzzy controller such that:

P1. The system state $x$ tracks the reference signal $x_d \in R^n$ and all the signals in the closed-loop system remain bounded.

P2. Achieve prescribed transient and steady state behavioral bounds on the tracking error $e_i(t) = x_i(t) - x_{di}(t), i = 1, 2, \cdots, n$.

To meet the objective, we make the following assumption.

**Assumption 2.1.** *The desired trajectory $x_d(t)$ is a known bounded function of time, with bounded derivatives.*

3. **Description of the Fuzzy Logic System.** The fuzzy logic system that employs singleton fuzzification, sum-product inference and center-of-sets defuzzification is modeled by

$$\alpha(x) = \frac{\sum_{j=1}^{N} \theta_j \prod_{i=1}^{n} \mu_{F_i^j}(x_i)}{\sum_{j=1}^{N} \left[\prod_{i=1}^{n} \mu_{F_i^j}(x_i)\right]} \tag{3}$$

where $\alpha(x)$ is the output of the fuzzy system, $x$ is the input vector, $\mu_{F_i^j}(x_i)$ is $x_i$'s membership of the $j$th rule and $\theta_j$ is the centroid of the $j$th consequent set. (3) can be rewritten as following equation:

$$\alpha(x) = \theta^T \psi(x) \tag{4}$$

with $\theta = [\theta_1, \cdots, \theta_N]^T$, $\psi(x) = [p_1(x), p_2(x), \cdots, p_N(x)]^T$, and the fuzzy basis function can be written as

$$p_j(x) = \frac{\prod_{i=1}^{n} \mu_{F_i^j}(x_i)}{\sum_{j=1}^{N} \left[\prod_{i=1}^{n} \mu_{F_i^j}(x_i)\right]}$$

Suppose there are $N$ rules of the fuzzy system used to approximate the unknown function $\alpha(x)$:
Rule $i$: if $x_1$ is $F_1^i$ and $\cdots$ and $x_n$ is $F_n^i$ then $\alpha(x)$ is $B^i$, $i = 1, 2, \cdots, N$.

4. **Adaptive Fuzzy Control Design with Prescribed Performance.** P2 is introduced in the analysis with the help of the performance function which translates the prescribed performance characteristics into tracking error constraints.

**Definition 4.1.** *A smooth function $y : R_+ \to R_+ - \{0\}$ is called a performance function if $y(t)$ is decreasing and $\lim_{t\to\infty} y(t) = y_\infty > 0$.*

Hence, we can guarantee P2 by satisfying:

$$-y(t) \le e(t) \le y(t) \tag{5}$$

for all $t \ge 0$ and $y(t)$ is a performance function associated with the tracking error $e(t)$. The constant $y_\infty$ represents the maximum allowable size of the tracking error $e(t)$ at the steady state, and the decreasing rate of the performance function $y(t)$ represents a lower bound on the required speed of convergence of $e(t)$. The aforementioned statements are shown in Figure 1.

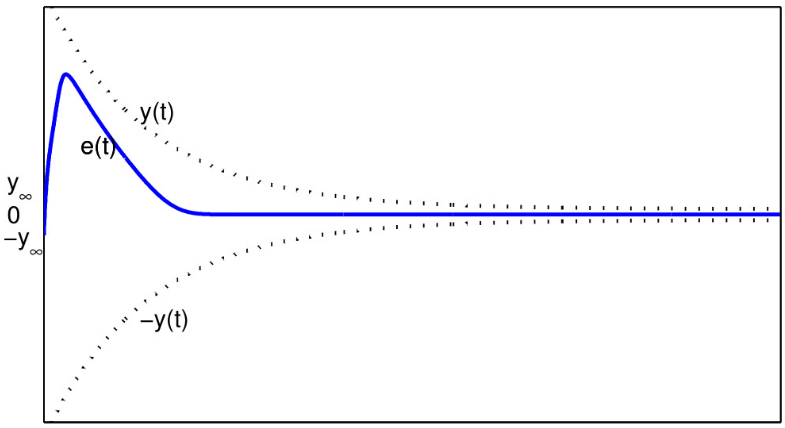


FIGURE 1. Tracking error prescribed performance

4.1. **Error transformation.** In order to meet the control objective, we introduce an error transformation which can transform the original nonlinear system, in the sense of (5), into an equivalent unconstrained system. Let us define:

$$e_i(t) = y_i(t)s_i(z_i), \quad i = 1, 2, \cdots, n \tag{6}$$

where $e_i(t)$ are performance functions, and $z_i$ are the transformed errors. $s_i(\cdot)$ are smooth, strictly increasing and thus invertible functions and satisfy:

$$\begin{aligned} &-1 \le s_i(z_i) \le 1, \\ &\lim_{z_i \to -\infty} s_i(z_i) = -1 \\ &\lim_{z_i \to +\infty} s_i(z_i) = 1 \end{aligned} \tag{7}$$

For example, a candidate function could be $s_i(z_i) = \tanh(z_i)$. If $z_i$ remain bounded, we can obtain $-1 < s_i(z_i) < 1$; furthermore we have $-y(t) < e(t) < y(t)$. Owing to the properties of $s_i(z_i)$ and $y(t) \ge y_\infty > 0$, we have the inverse transformation:

$$z_i = s_i^{-1}\left(\frac{e_i(t)}{y_i(t)}\right), \quad i = 1, 2, \cdots, n \tag{8}$$

are well defined. Then if we can keep $z_i(t)$ bounded, we can guarantee (5). Differentiating (8) with respect to time gives:

$$\dot{z}_i = \frac{\partial s_i^{-1}}{\partial\left(\frac{e_i(t)}{y_i(t)}\right)} \frac{1}{y_i(t)} \left[f_i(x) + \Delta f_i(x,t) + d_i(t) + u_i - \dot{x}_{di} - \frac{e_i(t)\dot{y}_i(t)}{y_i(t)}\right] \tag{9}$$

Define

$$r_i = \frac{\partial s_i^{-1}}{\partial\left(\frac{e_i(t)}{y_i(t)}\right)} \frac{1}{y_i(t)} > 0, \quad v_i = -\dot{x}_{di} - \frac{e_i(t)\dot{y}_i(t)}{y_i(t)}$$

Then (9) can be rewritten as

$$\dot{z}_i = r_i\Big[f_i(x) + \Delta f_i(x,t) + d_i(t) + u_i + v_i\Big], \quad i = 1, 2, \cdots, n \tag{10}$$

Let $z = [z_1, \cdots, z_n]^T$, $v = [v_1, \cdots, v_n]^T$, $\Gamma = \mathrm{diag}[r_1, \cdots, r_n]$, $\gamma(x) = f(x) + \Delta f(x,t) + d(t)$. Then (10) can be written into the following compact form:

$$\dot{z} = \Gamma\big[\gamma(x) + u + v\big] \tag{11}$$

4.2. **Adaptive fuzzy control design.** Since the nonlinear function $\gamma(x)$ is unknown, we employ fuzzy systems to approximate $\gamma_i(x)$. Then, the nonlinear function $\gamma_i(x)$ can be approximated, by the fuzzy logic systems (4) as

$$\hat{\gamma}_i = \theta_i^T \psi(x), \quad i = 1, 2, \cdots, n \tag{12}$$

Let us define the ideal parameters of $\theta_i$ as:

$$\theta_i^* = \arg\min_{\theta_i}\left[\sup|\gamma_i(x) - \hat{\gamma}_i(x)|\right] \tag{13}$$

Define the parameter estimation errors and the fuzzy approximation errors as follows:

$$\tilde{\theta}_i = \theta_i - \theta_i^* \tag{14}$$

$$\varepsilon_i(x) = \gamma_i(x) - \hat{\gamma}_i(x, \theta_i^*) \tag{15}$$

with $\hat{\gamma}_i(x, \theta_i^*) = \theta_i^* \psi(x)$. As in [14], we can assume that the fuzzy approximation error is bounded for all $x$, i.e., $|\varepsilon_i(x)| < \bar{\varepsilon}_i$, $\bar{\varepsilon}_i$ is unknown constant. Let $\varepsilon = [\varepsilon_1(x), \cdots, \varepsilon_n(x)]^T$, $\bar{\varepsilon} = [\bar{\varepsilon}_1, \cdots, \bar{\varepsilon}_n]^T$. Then we can get $|\varepsilon(x)| \le \bar{\varepsilon}$. From above analysis, we have

$$\hat{\gamma}(x,\theta) - \gamma(x) = \hat{\gamma}(x,\theta) - \hat{\gamma}(x,\theta^*) + \hat{\gamma}(x,\theta^*) - \gamma(x) = \tilde{\theta}^T\psi(x) - \varepsilon(x) \tag{16}$$

where $\hat{\gamma}(x,\theta) = [\hat{\gamma}_1(x,\theta_1), \cdots, \hat{\gamma}_n(x,\theta_n)]^T$. Then the controller can be constructed as

$$u = -\theta^T\psi(x) - v - Kz + u_r \tag{17}$$

with

$$u_{ri} = -\text{sign}(z_i)r_i\hat{\bar{\varepsilon}}_i, \quad i = 1, 2, \cdots, n \tag{18}$$

where $K = \text{diag}[k_1, \cdots, k_n]$ with $k_i > 0$, $i = 1, 2, \cdots, n$ are free positive constants of the design. $\hat{\bar{\varepsilon}}_i$ are desgin parameters which will be defined later.

Multiplying $s^T$ to (11) and using (16)-(18) we obtain

$$z^T\dot{z} = -\sum_{i=1}^{n} r_ik_iz_i^2 - \sum_{i=1}^{n} r_iz_i\tilde{\theta}_i^T\psi(x) + \sum_{i=1}^{n} r_iz_i\varepsilon_i(x) - \sum_{i=1}^{n} z_iu_{ri} \tag{19}$$

In order to meet the control objective, $\hat{\bar{\varepsilon}}_i$ and the fuzzy parameters $\theta_i$ are updated by

$$\theta_i = \int_0^t \left[-\sigma_i\gamma_{0i}|z_i|\theta_i + \gamma_{0i}z_i\psi(x)\right] d\tau - \gamma_{1i}\delta_i \tag{20}$$

with $\delta_i = \sigma_i|s_i|\theta_i - s_i\psi(x)$, and $\gamma_{0i}, \gamma_{1i}, \sigma_i > 0$ are design constants.

**Remark 4.1.** *In (20), the term $-\sigma_i\gamma_{0i}|z_i|\theta_i$ keeps all the parameter bounded, and $\delta_i$ makes the fuzzy parameters a fast convergence.*

Now we are ready to give the following results.

**Theorem 4.1.** *Consider the system (2) on the assumption that the desired trajectory $x_d(t)$ is a known bounded function of time, with bounded derivatives. Then the proposed controller defined by (17) with the adaption law (20) guarantees the following properties: (a) all signals in the closed-loop system are bounded; (b) the prescribed performance of the closed-loop system is achieved.*

**Proof:** Let us consider the following Lyapunov function candidate

$$V = \frac{1}{2}z^Tz + \frac{1}{2}\sum_{i=1}^{n} \frac{r_i}{\gamma_{0i}} \left(\tilde{\theta}_i + \gamma_{1i}\delta_i\right)^T \left(\tilde{\theta}_i + \gamma_{1i}\delta_i\right) \tag{21}$$

The time derivative of $V$ is given by

$$\dot{V} = z^T\dot{z} + \sum_{i=1}^{n} \frac{r_i}{\gamma_{0i}} \left(\tilde{\theta}_i + \gamma_{1i}\delta_i\right)^T \left(\dot{\tilde{\theta}}_i + \gamma_{1i}\dot{\delta}_i\right) = \dot{V}_1 + \dot{V}_2 \tag{22}$$

From (19) and (20), we have

$$\dot{V}_1 = z^T\dot{z} \le -\sum_{i=1}^{n} r_ik_iz_i^2 - \sum_{i=1}^{n} r_iz_i\tilde{\theta}_i^T\psi(x) + \sum_{i=1}^{n} r_i|z_i|\bar{\varepsilon}_i - \sum_{i=1}^{n} z_iu_{ri} \tag{23}$$

Note that $-2\tilde{\theta}_i^T\theta_i \le -\sum_{i=1}^{n} \|\tilde{\theta}_i\|^2 + \sum_{i=1}^{n} \|\theta_i^*\|^2$. Then we have

$$\dot{V} \le -\sum_{i=1}^{n} r_ik_iz_i^2 + \sum_{i=1}^{n} |z_i|r_i\bar{\varepsilon}_i + \frac{1}{2}\sum_{i=1}^{n} r_i\sigma_i|z_i|\|\theta_i^*\|^2 - \sum_{i=1}^{n} |z_i|r_i\hat{\bar{\varepsilon}}_i \tag{24}$$

If we choose $\hat{\bar{\varepsilon}}_i \ge \bar{\varepsilon}_i + 0.5\sigma_i\|\theta_i^*\|^2$, it yields

$$\dot{V} \le -\sum_{i=1}^{n} r_ik_iz_i^2 \tag{25}$$

Then $V$ is always negative, which implies that $z_i$ and $\tilde{\theta}_i + \gamma_{1i} \in L_\infty$. Since $z_i$ keep bounded, we have $-y(t) < e(t) < y(t)$. This completes the proof.

**Remark 4.2.** *In order to avoid the algebraic loop problem in (20), the adaption law can be rewritten as*

$$\theta_i = \frac{1}{1+\gamma_{1i}\sigma_i|z_i|}\left(\int_0^t \left[-\sigma_i\gamma_{0i}|z_i|\theta_i + \gamma_{0i}z_i\psi(x)\right]d\tau + \gamma_{1i}z_i\psi(x)\right) \tag{26}$$

5. **Simulation Results.** In this section, an illustrative example is presented to illustrate the effectiveness and applicability of the proposed adaptive fuzzy control approach and to confirm the theoretical results. Consider the following fractional-order economic system with model uncertainties and external disturbances [18].

$$\begin{aligned}\dot{x} &= z + (y-3)x + \Delta f_1(x,y,z,t) + d_1(t)\\ \dot{y} &= 1 - 0.1y - x^2 + \Delta f_2(x,y,z,t) + d_2(t)\\ \dot{z} &= -x - z + \Delta f_3(x,y,z,t) + d_3(t)\end{aligned} \tag{27}$$

In the simulation, the uncertainty term and external noise of the system are selected as follows

$$\begin{aligned}\Delta f_1(x,y,z,t) + d_1(t) &= 2 + \sin(t)\\ \Delta f_2(x,y,z,t) + d_2(t) &= 4 - \cos(t)\\ \Delta f_3(x,y,z,t) + d_3(t) &= 4 - 2\sin(t)z + 3\cos(t)\end{aligned} \tag{28}$$

The desired trajectory is $x_d = [\sin(t), \sin(t), \sin(t)]^T$. The initial values of the system are selected as $x = [-1, 1, 3]^T$. The transient and steady state error are prescribed through the performance functions $y_i(t) = 2e^{-0.7t} + 0.05$, $i = 1, 2, 3$, and the transformation functions are $s_i = \frac{2}{\pi}\arctan(z_i)$. The design parameters are chosen as follows: $\gamma_{0i} = 500$, $\gamma_{1i} = 500$, $\sigma_i = 0.05$, $k_i = 1$, $\hat{\bar{\varepsilon}}_i = 10$, $i = 1, 2, 3$. The discontinuous function $\mathrm{sign}(z_i(t))$ has been replaced by smooth function $\arctan(20z_i(t))$. The simulation results are shown in Figure 2 and Figure 3. The simulation results show that output tracking with prescribed performance is achieved.

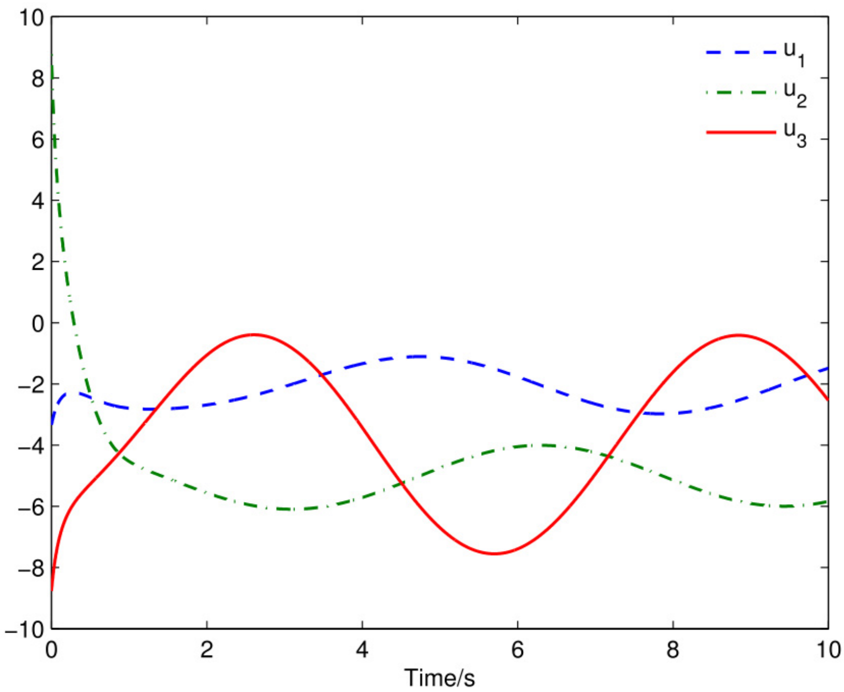


FIGURE 2. The control inputs

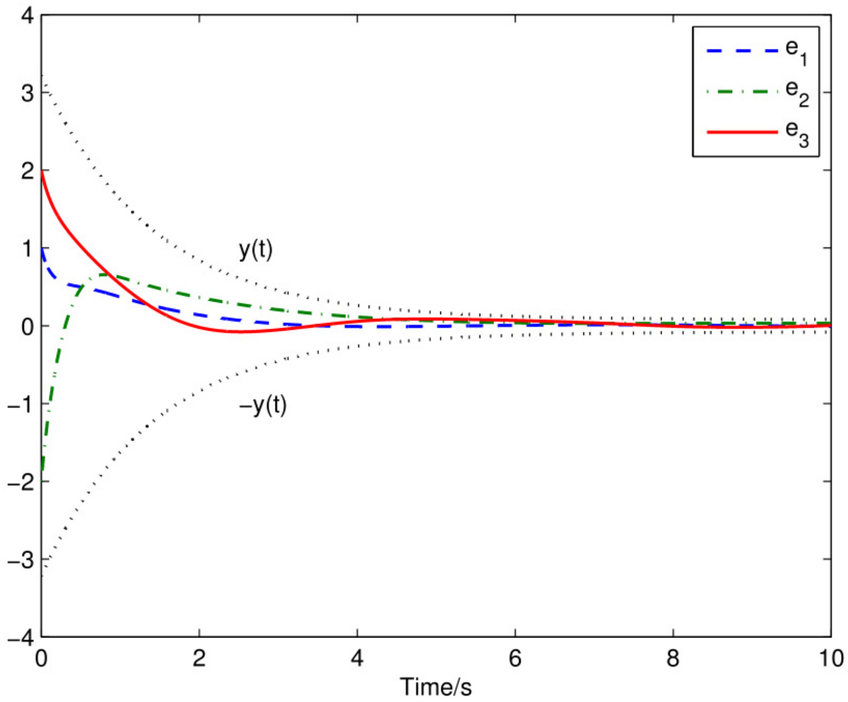


FIGURE 3. Tracking errors response

6. **Conclusion.** This paper proposes a robust adaptive fuzzy control method for uncertain economical system with unknown disturbances, capable of guaranteeing a prescribed performance. By using prescribed performance functions, we transform the system into an equivalent one, and it is sufficient to guarantee ultimate boundedness property of the transformed output error and a uniform boundedness of other signals in the closed-loop system. Simulation results have shown the effectiveness of the proposed scheme. The prescribed performance control of fractional-order economical system is our further investigation direction.

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