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UNCERTAIN MINIMUM EDGE COVERING PROBLEM WITH VERTEX WEIGHT CONSTRAINT

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Abstract. *Minimum weight edge covering problem has been investigated in many scientific and engineering applications. The traditional minimum weight edge covering problem assumes that the edge weight is a crisp value. As an empirical or subjective estimation, however, edge weight is more suitable to be regarded as an uncertain variable. In this paper, the minimum weight edge covering problem with vertex weight constraint under uncertain environment is considered. First, with different criteria, two types of uncertain programming models are constructed for the problem. Furthermore, the proposed models can be transformed into their corresponding deterministic forms by taking advantage of some properties of the uncertainty theory. At the end, two numerical examples are presented to illustrate the applicability of the proposed models.*
Keywords: Edge covering problem, Graph theory, Uncertain programming, Uncertainty theory, Vertex weight constraint

1. **Introduction.** Minimum weight edge covering problem (MWECP) is a classic problem in graph theory. Given an undirect graph with an edge weight, MWECP is to find an edge cover with the minimum weight, where an edge cover is a set of edges such that every vertex in the graph is an endpoint of at least one edge in the set [14, 15]. Recently, MWECP has been employed to model many issues in various real-life applications, such as computer science, combinational mathematics, and management engineering.

Norman and Rabin [16] first studied a particular class of edge covering problem where the weight on every edge is the same number, which is equivalent to minimizing the cardinality of the edge cover. In the following years, the research work on MWECP mostly focused on the deterministic cases. However, due to the lack of history data or emergency events, the edge weights are nondeterministic in many situations. In these cases, it is unsuitable to employ classical models to study the MWECP. Therefore, several researches have been presented within the framework of probability theory. Kardoš et al. [9] studied the maximum edge-cuts in random cubic graphs with large girth. In the meantime, Ni [14] considered the randomness in MWECP and introduced random variables to describe stochastic weights.

It is undeniable that probability theory is a useful tool to deal with indeterminacy factor. However, we often lack in observed data to estimate the probability distribution of the edge weights via statistics. In such case, we have no choice but to invite some domain experts to give the belief degree about each weight of an edge. According to Kahneman and Tversky [8], human beings usually overweight unlikely events, and thus the estimated probability distribution based on experts' estimations may be far from the cumulative frequency. In this situation, if we insist on dealing with the belief degree by using probability theory, some counterintuitive phenomena may happen (Liu [13]).

In the year of 2007, Liu [10] founded an uncertainty theory to deal with experts' belief degree mathematically, and Liu [12] perfected it based on normality, duality, subadditivity, and product measure axioms. In practical aspect, the uncertain programming was proposed by Liu [11] as a spectrum of mathematical programming involving uncertain variables. The concerns of using the uncertain programming to deal with the optimization problem have been elaborated in a number of papers. For example, Chen et al. [2] investigated the minimum weight vertex covering problem under uncertain environment. Yang et al. [17] studied a new class of uncertain furniture production planning problem, where the customer demand and production costs were characterized by mutually independent uncertain variables. Recently, Chen et al. [3] proposed a semivariance method for diversified portfolio selection in which future security returns were characterized by uncertain variables. By using uncertain goal programming, Chen et al. [5] investigated the bicriteria solid transportation problem under uncertain environment.

Motivated by the above mentioned research, this paper concerns about minimum weight edge covering problem with vertex weight constraint under an uncertain environment, in which the edge weights are assumed to be uncertain variables. According to different decision criteria, expected value model and chance-constrained programming model are proposed. Within the framework of uncertainty theory, the proposed models can be transformed into their corresponding deterministic forms, which can be solved by LINGO conveniently. We also conduct a case study to illustrate the applications of the models. The result shows that the proposed models could solve the minimum weight edge covering problem with vertex weight constraint under an uncertain environment effectively.

The remainder of this paper is organized as follows. In Section 2, we present the problem considered in this paper. In Section 3, two types of uncertain programming models for MWECP are presented, which include expected value model and chance-constrained programming model. For the sake of illustrating the modeling idea of the paper, two numerical examples are given in Section 4. A brief summary is presented in the last section. Preliminaries on uncertainty theory are relegated to the appendix for clarity of presentation.

2. Problem Description.

Let $G = (V, E)$ be an undirected and simple graph with the set of vertices $V = \{v_1, v_2, \dots, v_n\}$ and the set of edges $E = \{(v_i, v_j) \mid v_i \in V, v_j \in V, i < j\}$. A subset of edges $X \subset E$ is called an edge cover if each vertex in V is an endpoint of at least one edge in X. All the weights are presented by a vector $\mathbf{w} = \{w_{ij} \mid (v_i, v_j) \in E\}$. We employ a binary decision variable x_{ij} to indicate whether the edge (v_i, v_j) is in the set X or not

$$x_{ij} = \begin{cases} 1, & \text{if } (v_i, v_j) \in X \\ 0, & \text{otherwise.} \end{cases}$$

Then decision variable set $\mathbf{x} = \{x_{ij} \mid (v_i, v_j) \in E\}$ corresponding to X is an edge cover if and only if

$$\sum_{j:j<i} x_{ji} + \sum_{j:j>i} x_{ij} \geq 1, \qquad \forall\ v_i \in V.$$

Thus, the weight of an edge cover X is denoted by

$$W(\mathbf{x}, \mathbf{w}) = \sum_{(v_i,v_j)\in E} w_{ij} x_{ij}.$$

In this paper, we consider the MWECP with vertex weight constraint. Mathematically, the new type of MWECP can be formulated as the following programming model

$$\begin{cases} \min\limits_{\mathbf{x}} \sum\limits_{(v_i,v_j)\in E} w_{ij}x_{ij} \\ \text{subject to:} \\ \qquad \sum\limits_{j:j<i} x_{ji} + \sum\limits_{j:j>i} x_{ij} \geq 1, \quad \forall\ v_i \in V \\ \qquad \sum\limits_{j:j<i} \eta_{ji}x_{ji} + \sum\limits_{j:j>i} \eta_{ij}x_{ij} \geq \eta_j, \quad \forall\ v_i \in V \\ \qquad x_{ij} \in \{0,1\}, \quad \forall\ (v_i,v_j) \in E. \end{cases} \tag{1}$$

In classic deterministic MWECP, each edge (v_i, v_j) is assigned a crisp value w_{ij} that indicates its weight. In practice, however, the weight of edge is usually nondeterministic. If there is enough historical data of each edge weight, we can regard the weight as a random variable, and random MWECP may be considered (see Ni [14]). Unfortunately, sometimes we cannot get enough historical data, or historical data is invalid because of conditions' change. As a result, it is inappropriate to regard subjective estimation weight data as random variables. So, how can we deal with this kind of nondeterministic factors? As mentioned before, uncertainty theory provides a new tool to deal with uncertain information, especially subjective or empirical data. Hence, in this paper, we assume that the edge weights are all independent uncertain variables. Such independent assumption is widely used in recent work under uncertain environment (Chen et al. [2], Chen et al. [4], Zhang et al. [18], to name a few). Then, each edge weight w_{ij} is replaced by an uncertain variable ξ_{ij}, and all the weights are presented by $\xi = \{\xi_{ij} \mid (v_i, v_j) \in E\}$. Note that there exist many uncertain variables in model (1) and they cannot be ranked directly. In order to optimize the objective, it is inevitable to rank uncertain variables according to some criteria.

3. **Uncertain Minimum Weight Edge Covering Models.** In this section, in order to solve minimum weight edge covering problem in an uncertain environment, we introduce two different decision criteria in decision theory. According to the decision criteria, we propose two concepts of minimum weight edge cover and then present two uncertain programming models for the problem.

3.1. **Expected value model.** In order to find the optimal cover, we need to rank all covers according to their weights. However, the weight of any edge is an uncertain variable, which cannot be ranked directly. A natural decision criterion is based on taking the expected values as the indices for ranking uncertain variables. The main idea of expected value model is to optimize the expected objective function.

Definition 3.1. *Let $G = (V, E)$ be an undirected and simple graph with edge weights. An edge cover $\boldsymbol{x}^*$ is called the expected minimum weight edge cover if*

$$E\left[W\left(\boldsymbol{x}^*, \xi\right)\right] \leq E\left[W\left(\boldsymbol{x}, \xi\right)\right],$$

for any edge cover $\boldsymbol{x}$ of G.

When $w_{ij} = \xi_{ij}$ are uncertain variables, model (1) is only a conceptual model rather than a mathematical model because an uncertain objective function cannot be minimized directly. As a first choice, we present the expected value programming model for MWECP to seek an optimal cover as follows:

$$
\begin{cases}
\min\limits_{\mathbf{x}} \quad E\left[\sum\limits_{(v_i,v_j)\in E} \xi_{ij}x_{ij}\right] \\
\text{subject to:} \\
\qquad \sum\limits_{j:j<i} x_{ji} + \sum\limits_{j:j>i} x_{ij} \geq 1, \quad \forall\ v_i \in V \\
\qquad \sum\limits_{j:j<i} \eta_{ji}x_{ji} + \sum\limits_{j:j>i} \eta_{ij}x_{ij} \geq \eta_j, \quad \forall\ v_i \in V \\
\qquad x_{ij} \in \{0,1\}, \quad \forall\ (v_i,v_j) \in E.
\end{cases}
\tag{2}
$$

Theorem 3.1. *Let $G=(V,E)$ be an undirected and simple graph with edge weights, and ξ_{ij} be independent uncertain variables with uncertainty distributions Φ_{ij}, $i,j=1,2,\ldots,n$, respectively. Then model (2) is equivalent to the following one*

$$
\begin{cases}
\min\limits_{\mathbf{x}} \quad \sum\limits_{(v_i,v_j)\in E} x_{ij}\int_0^1 \Phi_{ij}^{-1}(\alpha)\mathrm{d}\alpha \\
\text{subject to:} \\
\qquad \sum\limits_{j:j<i} x_{ji} + \sum\limits_{j:j>i} x_{ij} \geq 1, \quad \forall\ v_i \in V \\
\qquad \sum\limits_{j:j<i} \eta_{ji}x_{ji} + \sum\limits_{j:j>i} \eta_{ij}x_{ij} \geq \eta_j, \quad \forall\ v_i \in V \\
\qquad x_{ij} \in \{0,1\}, \quad \forall\ (v_i,v_j) \in E.
\end{cases}
\tag{3}
$$

Proof: It follows from Theorem A.1 in the appendix that

$$
E\left[\sum_{(v_i,v_j)\in E} \xi_{ij}x_{ij}\right] = \sum_{(v_i,v_j)\in E} E\left[\xi_{ij}\right]x_{ij} = \sum_{(v_i,v_j)\in E} x_{ij}\int_0^1 \Phi_{ij}^{-1}(\alpha)\mathrm{d}\alpha.
$$

Therefore, the theorem is proven.

3.2. **Chance-constrained programming model.** Although the expected value model is often adopted in real-life applications, it cannot be fully trusted in some cases. Charnes and Cooper [1] initialized the chance-constrained programming which is a powerful tool to deal with an indeterminacy system. The essential idea of chance-constrained programming is to optimize some critical values with a given confidence level subject to some chance constraints.

Given an $\alpha \in (0,1)$, the decision maker hopes to get the smallest value $\overline{W}$ such that uncertain variable $W(\mathbf{x},\xi)$ is less than $\overline{W}$ with confidence level α, which causes appearance of the following criterion.

Definition 3.2. *Let $G=(V,E)$ be an undirected and simple graph with edge weights. An edge cover $\boldsymbol{x}^*$ is called the α-minimum weight edge cover if*

$$
\min\left\{\overline{W} \mid M\left\{W\left(\boldsymbol{x}^*,\xi\right) \leq \overline{W}\right\} \geq \alpha\right\} \leq \min\left\{\overline{W} \mid M\left\{W\left(\boldsymbol{x},\xi\right) \leq \overline{W}\right\} \geq \alpha\right\},
$$

for any edge cover $\boldsymbol{x}$ of G, where α is a predetermined confidence level.

If the decision maker prefers treating the MWECP under the chance-constraints, the model can be constructed as follows:

$$\begin{cases} \min\limits_{\mathbf{x}} \quad \overline{W} \\ \text{subject to:} \\ \qquad M\left\{\sum\limits_{(v_i,v_j)\in E} \xi_{ij}x_{ij} \le \overline{W}\right\} \ge \alpha \\ \qquad \sum\limits_{j:j<i} x_{ji} + \sum\limits_{j:j>i} x_{ij} \ge 1, \quad \forall\ v_i \in V \\ \qquad \sum\limits_{j:j<i} \eta_{ji}x_{ji} + \sum\limits_{j:j>i} \eta_{ij}x_{ij} \ge \eta_j, \quad \forall\ v_i \in V \\ \qquad x_{ij} \in \{0,1\}, \quad \forall\ (v_i,v_j) \in E. \end{cases} \tag{4}$$

Theorem 3.2. *Let $G = (V,E)$ be an undirected and simple graph with edge weights, and ξ_{ij} be independent uncertain variables with regular uncertainty distributions Φ_{ij}, $i,j = 1,2,\ldots,n$, respectively. Then model (4) is equivalent to the following one*

$$\begin{cases} \min\limits_{\mathbf{x}} \quad \sum\limits_{(v_i,v_j)\in E} x_{ij}\Phi_{ij}^{-1}(\alpha) \\ \text{subject to:} \\ \qquad \sum\limits_{j:j<i} x_{ji} + \sum\limits_{j:j>i} x_{ij} \ge 1, \quad \forall\ v_i \in V \\ \qquad \sum\limits_{j:j<i} \eta_{ji}x_{ji} + \sum\limits_{j:j>i} \eta_{ij}x_{ij} \ge \eta_j, \quad \forall\ v_i \in V \\ \qquad x_{ij} \in \{0,1\}, \quad \forall\ (v_i,v_j) \in E, \end{cases} \tag{5}$$

where Φ_{ij}^{-1} is the inverse uncertainty distributions of ξ_{ij}.

Proof: Since the total weight W is strictly increasing with respect to each edge weight, and it is assumed that ξ_{ij} have regular uncertainty distributions Φ_{ij}, $i,j = 1,2,\ldots,n$, respectively, then, using the inverse uncertainty distribution, we can transform the constraint

$$M\left\{\sum_{(v_i,v_j)\in E} \xi_{ij}x_{ij} \le \overline{W}\right\} \ge \alpha$$

into a deterministic constraint

$$\sum_{(v_i,v_j)\in E} x_{ij}\Phi_{ij}^{-1}(\alpha) \le \overline{W}.$$

Therefore, model (4) can be equivalently transformed into the following deterministic model:

$$\begin{cases} \min\limits_{\mathbf{x}} \quad \overline{W} \\ \text{subject to:} \\ \qquad \sum\limits_{(v_i,v_j)\in E} x_{ij}\Phi_{ij}^{-1}(\alpha) \le \overline{W} \\ \qquad \sum\limits_{j:j<i} x_{ji} + \sum\limits_{j:j>i} x_{ij} \ge 1, \quad \forall\ v_i \in V \\ \qquad \sum\limits_{j:j<i} \eta_{ji}x_{ji} + \sum\limits_{j:j>i} \eta_{ij}x_{ij} \ge \eta_j, \quad \forall\ v_i \in V \\ \qquad x_{ij} \in \{0,1\}, \quad \forall\ (v_i,v_j) \in E. \end{cases} \tag{6}$$

Clearly, model (6) is equivalent to model (5). The theorem is verified.

It is necessary to point out that models (3) and (5) are both 0-1 programming models. With the aid of some optimization software packages, for example, LINGO, we can solve the above optimization models effectively. Similar to the mainstream literature in the area of uncertain optimization problem (Chen et al. [2], Chen et al. [3], and Yang et al. [17]), we solve the proposed models by using LINGO software.

4. **Numerical Examples.** In this section, we give two numerical examples to show the applications of the models. We usually encounter the uncertain factors in the real-life application. In this case, we have no choice but to invite the domain experts to give the subjective estimations. In the previous literature, the researchers assume the relevant parameters as fuzzy variables, such as Ni [15]. However, the fuzzy set theory is unsuitable to model the imprecise edge weight in the minimum weight edge covering problem with vertex weight constraint when the relevant parameters are given by some domain experts. Therefore, the edge weights are more suitable to be regarded as uncertain variables. Consider the graph shown in Figure 1. Assume that all edge weights are linear uncertain variables ξ_{ij}. The uncertainty distributions of ξ_{ij} are listed in Table 1.

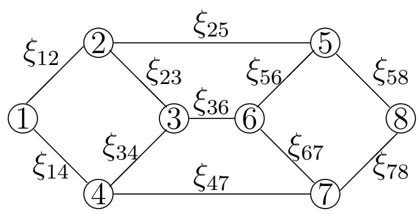


FIGURE 1. Uncertain weighted graph G

TABLE 1. The distributions of weights ξ_{ij} in Figure 1

ξ_{ij}	Φ_{ij}	ξ_{ij}	Φ_{ij}
ξ_{12}	$L(2,3)$	ξ_{47}	$L(5,7)$
ξ_{14}	$L(2,3)$	ξ_{56}	$L(3,7)$
ξ_{23}	$L(2,6)$	ξ_{58}	$L(2,5)$
ξ_{25}	$L(5,8)$	ξ_{67}	$L(4,6)$
ξ_{34}	$L(2,9)$	ξ_{78}	$L(2,6)$
ξ_{36}	$L(5,7)$		

The expected value model (2) for the uncertain minimum weight edge covering problem is formulated as follows:

$$\begin{cases} \min\limits_{\mathbf{x}} \quad E\left[\sum_{i=1}^{8}\sum_{j=1}^{8} x_{ij}\xi_{ij}\right] \\ \text{subject to:} \\ \qquad \sum_{j:j<i} x_{ji} + \sum_{j:j>i} x_{ij} \geq 1, \quad i,j = 1,2,\ldots,8 \\ \qquad \sum_{j:j<i} \eta_{ji}x_{ji} + \sum_{j:j>i} \eta_{ij}x_{ij} \geq \eta_j, \quad i,j = 1,2,\ldots,8 \\ \qquad x_{ij} \in \{0,1\}, \quad i,j = 1,2,\ldots,8. \end{cases} \tag{7}$$

It follows from Theorem 3.1 that model (7) is equivalent to the following model:

$$\begin{cases} \min_{\mathbf{x}} \quad \sum_{i=1}^{8}\sum_{j=1}^{8} x_{ij}\int_0^1 \Phi_{ij}^{-1}(\alpha)\mathrm{d}\alpha \\ \text{subject to:} \\ \qquad \sum_{j:j<i} x_{ji} + \sum_{j:j>i} x_{ij} \geq 1, \quad i,j=1,2,\ldots,8 \\ \qquad \sum_{j:j<i} \eta_{ji}x_{ji} + \sum_{j:j>i} \eta_{ij}x_{ij} \geq \eta_j, \quad i,j=1,2,\ldots,8 \\ \qquad x_{ij} \in \{0,1\}, \quad i,j=1,2,\ldots,8. \end{cases} \tag{8}$$

By using the mathematical software LINGO, we can get the following optimal solution of (8) as

$$(x_{12}, x_{13}, x_{14}, x_{23}, x_{25}, x_{34}, x_{36}, x_{47}, x_{56}, x_{58}, x_{67}, x_{68}, x_{78}) = (1,0,0,0,0,1,0,0,1,0,0,0,1)^T,$$

that is to say, the expected minimum edge cover is $\{(1,2),(3,4),(5,6),(7,8)\}$. The minimum weight edge cover is shown in Figure 2(a) (denoted by solid lines).

According to model (4), the 0.9-minimum weight edge covering problem can be formulated as follows:

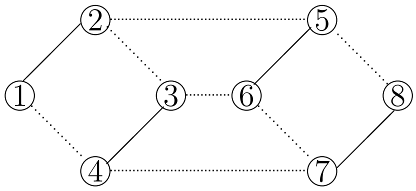
$$\begin{cases} \min_{\mathbf{x}} \quad \overline{W} \\ \text{subject to:} \\ \qquad M\left\{\sum_{i=1}^{8}\sum_{j=1}^{8} \xi_{ij}x_{ij} \leq \overline{W}\right\} \geq 0.9 \\ \qquad \sum_{j:j<i} x_{ji} + \sum_{j:j>i} x_{ij} \geq 1, \quad i,j=1,2,\ldots,8 \\ \qquad \sum_{j:j<i} \eta_{ji}x_{ji} + \sum_{j:j>i} \eta_{ij}x_{ij} \geq \eta_j, \quad i,j=1,2,\ldots,8 \\ \qquad x_{ij} \in \{0,1\}, \quad i,j=1,2,\ldots,8. \end{cases} \tag{9}$$

According to Theorem 3.2, model (9) is equivalent to the deterministic programming model:

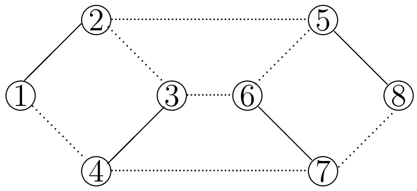
$$\begin{cases} \min_{\mathbf{x}} \quad \sum_{i=1}^{8}\sum_{j=1}^{8} x_{ij}\Phi_{ij}^{-1}(0.9) \\ \text{subject to:} \\ \qquad \sum_{j:j<i} x_{ji} + \sum_{j:j>i} x_{ij} \geq 1, \quad i,j=1,2,\ldots,8 \\ \qquad \sum_{j:j<i} \eta_{ji}x_{ji} + \sum_{j:j>i} \eta_{ij}x_{ij} \geq \eta_j, \quad i,j=1,2,\ldots,8 \\ \qquad x_{ij} \in \{0,1\}, \quad i,j=1,2,\ldots,8. \end{cases} \tag{10}$$

By using the mathematical software LINGO, we can get the following optimal solution of the (10) as $\mathbf{x}^* = (1,0,0,0,0,1,0,0,0,1,0,0,1)^T$. The 0.95-minimum weight edge cover is $\{(1,2),(3,4),(5,8),(6,7)\}$. The minimum weight edge cover is shown in Figure 2(b) (denoted by solid lines).

5. **Conclusions and Future Research.** In this paper, we have investigated the minimum weight edge covering problem with vertex weight constraint under an uncertain environment. With different criteria, two types of uncertain programming models were



(a) Model (8)



(b) Model (10)

FIGURE 2. Uncertain minimum weight edge cover

constructed for the problem, including expected value model and chance-constrained programming model. The proposed models were transformed into their corresponding deterministic forms by taking advantage of some properties of the uncertainty theory. At last, numerical experiments were presented to show the performance of the models.

It is worth saying that there are several other types of indeterminacy environment in the real systems, such as random uncertain environment and uncertain random environment. This paper only researches the minimum weight edge covering problem with vertex weight constraint under an uncertain environment, and the problem in other more complex environment (Gao et al. [6], Gao and Yao [7]) may become new topics in our further research.

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Appendix: Preliminaries on Uncertainty Theory. Let Γ be a nonempty set and L a σ-algebra over Γ. For any $\Lambda \in L$, Liu [10] presented an axiomatic uncertain measure $M\{\Lambda\}$ to indicate the belief degree that uncertain event Λ occurs. The uncertain measure M satisfies the following three axioms:
Axiom 1. (**Normality Axiom**) $M\{\Gamma\} = 1$ for the universal set Γ;
Axiom 2. (**Duality Axiom**) $M\{\Lambda\} + M\{\Lambda^c\} = 1$ for any event $\Lambda \in L$, where Λ^c is a complement of Λ;
Axiom 3. (**Subadditivity Axiom**) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have

$$M\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} M\{\Lambda_i\}.$$

Definition A.1. (Liu [10]) *An uncertain variable is a measurable function ξ from an uncertainty space (Γ, L, M) to the set of real numbers, i.e., for any Borel set B of real numbers, the set*

$$\{\xi \in B\} = \{\gamma \in \Gamma \,|\, \xi(\gamma) \in B\}$$

is an event.

An uncertain variable ξ is called linear if it has a linear uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x \leq a \\ \dfrac{x-a}{b-a}, & \text{if } a \leq x \leq b \\ 1, & \text{if } x \geq b, \end{cases}$$

denoted by $L(a, b)$, where a and b are real numbers with $a < b$.

Definition A.2. (Liu [10]) *Let ξ be an uncertain variable. Then the expected value of ξ is defined by*

$$E[\xi] = \int_0^{+\infty} M\{\xi \geq r\} dr - \int_{-\infty}^{0} M\{\xi \leq r\} dr$$

provided that at least one of the two integrals is finite.

Theorem A.1. (Liu [12]) *Let ξ be an uncertain variable with regular uncertainty distribution Φ. Then*

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha) d\alpha.$$

It follows from Theorem A.1 that the expected value of the linear uncertain variable $\xi \sim L(a, b)$ is $E[\xi] = \dfrac{a+b}{2}$.