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INFERENCE ON THE DISTRIBUTION QUANTILES USING NONPARAMETRIC METHODS

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Abstract. *Quantile information is useful in business and engineering applications, but the exact sampling distribution of sample quantile is often unknown. In this paper, we study the performance of four nonparametric methods, the kernel density estimation (KDE), bootstrap percentile (BP), bootstrap-t (BT) and accelerated bias-correction bootstrap (BCa) methods, through Monte Carlo simulations for conducting interval inference on the quantiles of normal and generalized Pareto distributions. Simulation results show that the BCa and BP methods outperform the BT and KDE methods. Sample sizes to implement the recommended nonparametric methods for inferring a range of upper quantiles are also studied based on the coverage probability.*
Keywords: Accelerated bias-correction bootstrap, Bootstrap percentile, Coverage probability, Quantile

1. **Introduction.** Let $X_1, X_2, \ldots, X_n$ be a random sample that is taken from a continuous distribution $F(x|\Theta)$, where $\Theta = (\theta_1, \theta_2, \ldots, \theta_m)$ is a vector of parameters. The probability density function of $F(x|\Theta)$ is denoted by $f(x|\Theta)$. The pth quantile $(0 < p < 1)$, named q_p, is defined by $p = \int_{-\infty}^{q_p} f(t|\Theta)dt = F(q_p|\Theta)$, where q_p is a function of Θ. Denote the estimate of q_p by $\hat{q}_p$. Two popular distributions on business or engineering applications are the normal distribution (ND(μ, σ)) and generalized Pareto distribution (GPD(μ, σ, ξ)). The μ and σ in the ND are the mean and standard deviation, respectively; and the μ, σ, and ξ in the GPD are the location, scale and shape parameters, respectively. The pth quantiles of the ND(μ, σ) and GPD(μ, σ, ξ) can be defined, respectively, by

$$p = \int_{-\infty}^{q_p} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx \tag{1}$$

and

$$p = \int_{-\infty}^{q_p} \frac{1}{\sigma} \left(1 + \xi\frac{x-\mu}{\sigma}\right)^{-\left(1+\frac{1}{\xi}\right)} dx. \tag{2}$$

The ND and GPD cover a lot of distribution shapes for model fitting. For example, if X is lognormal distributed, then $Y = \log(X)$ has an ND. The GPD can have different kurtosis and skewness through varying the shape parameter. Applying Fisher information matrix with maximum likelihood estimation method or bootstrap methods for inferring distribution parameters can be found in [3-18]. Most of the existing studies focus on studying the algorithms for evaluating the mean value or the maximum likelihood estimates (MLEs) of the distribution parameters, and some existing studies focus on providing algorithms to evaluate the MLEs of median. In practical applications, the KDE method often is used for large sample cases to establish the confidence interval (CI) of distribution parameters, and bootstrap methods are used to establish the CI of distribution parameters for small sample cases. However, reference sample sizes for applying the KDE, BP, BT and BCa methods on inferring quantiles, especially on inferring a range of upper quantiles, lack clear suggestions in the aforementioned existing studies.

Parametric methods for inferring the distribution parameters or quantiles can be found in [4,7,11,15,18]. Nonparametric methods for inferring the distribution parameters or quantiles can be found in [2,3,5,6,9,12-14,16,17]. Efron and Tibshirani [9] mentioned that a good nonparametric method should give dependably accurate coverage probabilities for all situations, and they also concluded that BCa CIs can conserve second-order accuracy and transformation respecting on inferring the distribution parameters. To infer the quantiles, it is not clear if the BCa method can outperform the BT and BP methods on inferring a range of upper quantiles, and the performance study of the KDE method on inferring the upper quantiles is also concerned. Another important issue is how large sample size is enough for implementing and applying the nonparametric methods to infer quantiles with a guaranteed coverage probability (CP). Hence, this study has two goals. The first goal is to study the performance of the popular nonparametric methods, the KDE, BP, BT and BCa methods, for conducting interval inference on q_p, with different levels of p, when the exact sampling distribution of $\hat{q}_p$ cannot be obtained. The second goal is to study the sample sizes of applying the nonparametric methods to inferring quantiles with a guaranteed CP. These two goals are studied through Monte Carlo simulations.

The rest of this paper is organized as follows. In Section 2, we review the KDE, BT, BP and BCa methods, and all four nonparametric methods are used to construct the approximate CIs (ACIs) of distribution quantiles. Monte Carlo simulations are conducted in Section 3 to evaluate the performance of the KDE, BT, BP and BCa methods on estimating a range of upper quantiles for the ND(μ, σ) and GPD(μ, σ, ξ) with different sample sizes. In Section 4, some conclusions are given.

2. Nonparametric Methods.

Procedures to implement the KDE, BT, BP and BCa methods are addressed as the following subsections.

2.1. The KDE method.

Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ be a random sample that is drawn from $F(x|\Theta)$. According to Bahadur representation in [2], the approximate sampling distribution of $\hat{q}_p$ is ND$(q_p, \sigma_*/\sqrt{n})$, where $\sigma_* = \sqrt{p(1-p)}/f(q_p|\Theta)$ and $f(.|\Theta)$ is the probability density function of $F(.|\Theta)$. The $100 \times (1-\alpha)\%$ CI of q_p, named the KDE-ACI, can be obtained by

$$\hat{q}_p \pm z_{1-\alpha/2}\frac{\sigma_*}{\sqrt{n}} \tag{3}$$

through using the normal approximation method, where z_δ is the δth quantile of ND$(0, 1)$. In practice, it is difficult to obtain the standard deviation σ_* due to the $f(x)$ could be unknown. Based on the KDE method proposed by [1] for the realizations of $x_1, x_2, \ldots, x_n$, the kernel estimate of $f(x|\Theta)$ can be obtained by

$$\hat{f}(x) = \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x - x_i}{h}\right), \tag{4}$$

where K is the kernel, which is nonnegative with mean 0 and integrates to one. $h > 0$ is a smoothing bandwidth parameter.

Several kernel functions are widely used to implement the KDE method, for example, the uniform, triangular, biweight, triweight, Epanehnikov and normal. The normal kernel of $K(u) = \phi(u) = \frac{1}{\sqrt{2\pi}}\exp\{-u^2/2\}$ is popular for applying the kernel estimate in (4). The bandwidth parameter h exhibits a strong influence on the resulting estimate. One common optimality criterion to select h is to minimize the mean integrated squared error: $\mathrm{MISE}(h) = \mathrm{E}\left[\int \left(\hat{f}(x) - f(x)\right)^2 dx\right]$.

The selection of h is more important than the selection of $K(.)$. Using too small or too large value of h is unsuitable. Small h leads to very spiky estimates (less smoothing), while larger h leads to over-smoothing and could blind important information for the density estimate. The selections of h and $K(.)$ will be further discussed in Section 3.

2.2. **Bootstrap methods.** Three algorithms are reported to implement the BP, BT and BCa methods, in which the simplest one for practical applications is the BP method. The BP method can be implemented through using Algorithm 1. The BT method can be established through using a studentized pivot, in which the second bootstrap samples for evaluating the standard error of $\hat{q}_p$ are needed. The BT method can be implemented via using Algorithm 2. The BCa method includes the bias correction and acceleration procedures for bootstrap computation. We summarize the steps to implement the BCa method in Algorithm 3.

Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ be a random sample for parameter estimation. Denote $\hat{q}_p$ as the MLE of q_p based on $\boldsymbol{x}$. Let $\hat{F}(\boldsymbol{x})$ denote the empirical distribution of $F(\boldsymbol{x}|\Theta)$, and $\boldsymbol{y_i} = (x_1^{*i}, x_2^{*i}, \ldots, x_n^{*i})$ denote the ith bootstrap sample generated from $\hat{F}(\boldsymbol{x})$. The ACIs through using the BP, BT and BCa methods are denoted by the BP-ACI, BT-ACI and BCa-ACI, respectively.

Algorithm 1: The BP-ACI of q_p.

Step 1. Generate a bootstrap sample $\boldsymbol{y}$ from $\hat{F}(\boldsymbol{x})$. Then evaluate $\hat{q}_p$ based on $\boldsymbol{y}$ and denote it by $\hat{q}_p^*$.

Step 2. Repeat Step 1 N times (N is a big positive integer), and label the sample quantiles by $\hat{q}_p^{*i}$ for $i = 1, 2, \ldots, N$. Let $\hat{G}_B$ denote the bootstrap empirical distribution of $\hat{q}_p^{*i}$, $i = 1, 2, \ldots, N$.

Step 3. The $100 \times (1-\alpha)\%$ ACI of q_p can be obtained by $\left(\hat{q}^*_{p(\alpha/2)}, \hat{q}^*_{p(1-\alpha/2)}\right)$, where $\hat{q}^*_{p(\delta)}$ is the δth quantile of $\hat{G}_B$. In practice, $\hat{q}^*_{p(\delta)}$ can be evaluated by $\hat{q}^*_{p[\delta B]}$, where $[\delta B]$ denotes the largest integer less than or equal to $\delta \times B$.

Algorithm 2: The BT-ACI of q_p.

Step 1. Generate N_1 bootstrap samples, $\boldsymbol{y_i}$, $i = 1, 2, \ldots, N_1$, from $\hat{F}(\boldsymbol{x})$. Then evaluate sample quantile based on $\boldsymbol{y_i}$ and denote it by $\hat{q}_p^{*i}$ for $i = 1, 2, \ldots, N_1$.

Step 2. For each bootstrap sample in Step 1, we generate second bootstrap samples, each has entries $\boldsymbol{y^*_{i,j}}$, $j = 1, 2, \ldots, N_2$ based on the empirical distribution of $\boldsymbol{y_i}$ and denote sample quantiles based on q_p based on the second bootstrap samples by $\hat{q}_p^{*i,j}$ for $j = 1, 2, \ldots, N_2$. Let $\widehat{se}(i) = \sqrt{\sum_{j=1}^{N_2}\left(\hat{q}_p^{*i,j} - \bar{q}^{*i}\right)^2/N_2}$, where $\bar{q}^{*i} = \sum_{j=1}^{N_2}\hat{q}_p^{*i,j}/N_2$. Compute $t_i^* = \left(\hat{q}_p^{*i} - \hat{q}_p\right)/\widehat{se}(i)$ for $i = 1, 2, \ldots, N_1$ and $\hat{\sigma}_q = \sqrt{\sum_{j=1}^{N_1}(\hat{q}_p^{*i} - \bar{q}_p)^2/N_1}$, where $\bar{q}_p$ is the sample mean of $\hat{q}_p^{*i}$ for $i = 1, 2, \ldots, N_1$.

Step 3. Let $\hat{H}$ be the empirical distribution of t_i^* for $i = 1, 2, \ldots, N_1$. Define $\tilde{q}_{bt}(\delta) = \hat{q}_p - \hat{H}^{-1}(\delta)\hat{\sigma}_q$ for $0 < \delta < 1$. The $100 \times (1-\alpha)\%$ BT-ACI of q_p can be obtained by $(\tilde{q}_{bt}(1-\alpha/2), \tilde{q}_{bt}(\alpha/2))$.

Algorithm 3: The BCa-ACI of q_p.

Step 1. Implement Steps 1-2 of Algorithm 1 to obtain the sample quantiles $\hat{q}_p^{*i}$, $i = 1, 2, \ldots, N$.

Step 2. Let $\hat{z}_0 = \Phi^{-1}\left(\hat{G}_B(\hat{q}_p)\right)$, where $\Phi(.)$ is the standard ND function. Compute the following quantities: $\overline{\hat{q}_p^*} = \sum_{i=1}^{N} \hat{q}_p^{*i}/N$, $a_1 = \sum_{i=1}^{N}\left(\hat{q}_p^{*i} - \overline{\hat{q}_p^*}\right)^3$, $a_2 = \sum_{i=1}^{N}\left(\hat{q}_p^{*i} - \overline{\hat{q}_p^*}\right)^2$, $\hat{a} = a_1/6a_2^{3/2}$, $\alpha_1 = \Phi\left(\hat{z}_0 + \frac{\hat{z}_0 + z_{\alpha/2}}{1-\hat{a}\times(\hat{z}_0+z_{\alpha/2})}\right)$, and $\alpha_2 = \Phi\left(\hat{z}_0 + \frac{\hat{z}_0 + z_{1-\alpha/2}}{1-\hat{a}\times(\hat{z}_0+z_{1-\alpha/2})}\right)$.

Step 3. Use the quantities in Step 2 to obtain the $100 \times (1-\alpha)\%$ BCa-ACI of q_p by $\left(\hat{q}^*_{p(\alpha_1)}, \hat{q}^*_{p(\alpha_2)}\right)$.

3. **Monte Carlo Simulations.** An intensive simulation study is conducted to evaluate the performance of the KDE, BP, BT and BCa methods on obtaining the ACIs of the quantiles of the ND and GPD with different sample sizes. Consider $p = 0.5$, 0.65, 0.75, 0.85, 0.90, 0.95 and 0.99 with the sample sizes of $n = 5$, 10, 15, 20, 25, 30, 35, 40, 45, 50, 60, 70, 80, 90, 100, 200, 300, 400, 500, 600, 700, 800, 900 and 1000 to use the KDE method for estimating q_p. Consider the same values of p with those for the KDE method but use the sample size of $n = 5$, 10, 15, 20, 25, 30, 35, 40, 45, 50, 60, 70, 80, 90, 100, 200, 300, 400 and 500 for implementing the BP, BT and BCa methods to estimate q_p. All combinations of p and n are run for the distributions of ND$(3, 2)$, GPD$(0, 1, -0.4)$ and GPD$(0, 1, 0.4)$. The GPD$(0, 1, -0.4)$ is a thin-tailed distribution, and the GPD$(0, 1, 0.4)$ is a heavy-tailed distribution.

The normal kernel is considered to implement the KDE method. The SJ-ste method, which was proposed by Sheather and Jones [8] and Jones et al. [10], is used to determine the bandwidth parameter h for implementing the KDE method. The simulations of bootstrap methods are run with 10000 repetitions for the nominal CP of 95%. R source codes are prepared to obtain the simulation results. Because the simulation results are huge, only some of them are reported in Tables 1-3 to save pages. Table 1 reports the CPs of q_p for ND$(3, 2)$, and Tables 2 and 3 report the CPs for GPD$(0, 1, 0.4)$ and GPD$(0, 1, -0.4)$, respectively. The smallest sample size to reach the condition that at

TABLE 1. Coverage probabilities of the pth quantiles for ND$(3, 2)$

p	n	**5**	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**200**
0.50	BCa-CP	0.782	0.924	0.936	0.944	0.946	0.947	0.945	0.944	0.948	0.947	0.946	0.951
	BT-CP	0.948	0.89	0.866	0.867	0.879	0.877	0.878	0.879	0.888	0.884	0.888	0.904
	BP-CP	0.938	0.942	0.939	0.946	0.948	0.948	0.948	0.948	0.949	0.949	0.946	0.953
	KDE-CP	0.787	0.911	0.937	0.949	0.95	0.953	0.954	0.955	0.958	0.96	0.959	0.957
0.75	BCa-CP	0.754	0.916	0.928	0.939	0.937	0.939	0.945	0.943	0.946	0.943	0.943	0.947
	BT-CP	0.862	0.865	0.854	0.854	0.857	0.866	0.87	0.871	0.875	0.872	0.878	0.896
	BP-CP	0.745	0.914	0.925	0.941	0.938	0.939	0.945	0.943	0.945	0.945	0.944	0.947
	KDE-CP	0.75	0.883	0.924	0.929	0.938	0.939	0.946	0.949	0.944	0.947	0.951	0.948
	n	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**200**	**300**	**400**	**500**
0.90	BCa-CP	0.932	0.918	0.925	0.93	0.93	0.935	0.94	0.937	0.942	0.943	0.949	0.944
	BT-CP	0.805	0.82	0.821	0.839	0.835	0.847	0.847	0.857	0.868	0.877	0.898	0.895
	BP-CP	0.862	0.911	0.899	0.93	0.923	0.92	0.931	0.931	0.937	0.94	0.945	0.946
	KDE-CP	0.861	0.878	0.897	0.897	0.899	0.903	0.91	0.915	0.926	0.932	0.936	0.937
0.95	BCa-CP	0.785	0.867	0.91	0.93	0.923	0.915	0.926	0.931	0.936	0.941	0.941	0.939
	BT-CP	0.791	0.784	0.812	0.771	0.812	0.809	0.833	0.814	0.845	0.858	0.863	0.872
	BP-CP	0.788	0.872	0.881	0.847	0.875	0.91	0.914	0.896	0.932	0.937	0.939	0.939
	KDE-CP	0.787	0.793	0.831	0.834	0.846	0.848	0.86	0.865	0.887	0.905	0.915	0.919
0.99	BCa-CP	0.265	0.33	0.39	0.454	0.5	0.546	0.594	0.639	0.859	0.902	0.902	0.892
	BT-CP	0.64	0.658	0.67	0.691	0.676	0.691	0.685	0.67	0.758	0.751	0.789	0.798
	BP-CP	0.266	0.33	0.39	0.456	0.501	0.548	0.596	0.641	0.864	0.832	0.903	0.887
	KDE-CP	0.466	0.531	0.588	0.612	0.63	0.648	0.647	0.634	0.732	0.783	0.807	0.824

most 1% error is allowed between the estimated CP and its nominal value 0.95, denote this condition by |est. CP-0.95| < 0.01, was searched and summarized in Table 4.

TABLE 2. Coverage probabilities of the pth quantiles for GPD$(0, 1, 0.4)$

p	n	**5**	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**200**
0.50	BCa-CP	0.781	0.928	0.94	0.943	0.942	0.942	0.944	0.948	0.952	0.944	0.948	0.945
	BT-CP	0.946	0.924	0.908	0.904	0.909	0.913	0.911	0.921	0.921	0.916	0.924	0.926
	BP-CP	0.937	0.935	0.943	0.94	0.947	0.946	0.943	0.947	0.952	0.944	0.95	0.948
	KDE-CP	0.762	0.881	0.904	0.913	0.918	0.916	0.919	0.924	0.925	0.934	0.928	0.93
0.75	BCa-CP	0.745	0.922	0.93	0.933	0.942	0.935	0.943	0.943	0.945	0.938	0.945	0.948
	BT-CP	0.819	0.895	0.891	0.892	0.905	0.907	0.906	0.908	0.91	0.912	0.916	0.924
	BP-CP	0.738	0.923	0.928	0.937	0.939	0.935	0.944	0.943	0.945	0.94	0.944	0.946
	KDE-CP	0.605	0.774	0.825	0.847	0.86	0.865	0.876	0.88	0.886	0.896	0.893	0.913
	n	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**200**	**300**	**400**	**500**
0.90	BCa-CP	0.932	0.927	0.93	0.94	0.932	0.937	0.937	0.938	0.944	0.943	0.945	0.943
	BT-CP	0.835	0.859	0.851	0.872	0.878	0.877	0.88	0.883	0.901	0.91	0.912	0.914
	BP-CP	0.874	0.915	0.905	0.929	0.926	0.92	0.929	0.934	0.942	0.943	0.945	0.945
	KDE-CP	0.69	0.721	0.747	0.754	0.768	0.784	0.793	0.794	0.841	0.862	0.88	0.892
0.95	BCa-CP	0.779	0.861	0.909	0.938	0.928	0.916	0.926	0.928	0.934	0.939	0.944	0.941
	BT-CP	0.774	0.797	0.833	0.807	0.837	0.841	0.852	0.839	0.875	0.887	0.891	0.9
	BP-CP	0.782	0.868	0.887	0.854	0.876	0.91	0.916	0.894	0.928	0.931	0.94	0.941
	KDE-CP	0.676	0.526	0.689	0.578	0.716	0.618	0.726	0.642	0.733	0.77	0.799	0.822
0.99	BCa-CP	0.254	0.33	0.389	0.445	0.505	0.545	0.588	0.627	0.863	0.9	0.905	0.889
	BT-CP	0.492	0.527	0.569	0.626	0.65	0.664	0.665	0.64	0.757	0.763	0.797	0.812
	BP-CP	0.256	0.332	0.392	0.448	0.508	0.548	0.592	0.631	0.868	0.838	0.905	0.884
	KDE-CP	0.573	0.7	0.768	0.788	0.77	0.735	0.621	0.178	0.204	0.243	0.277	0.308

TABLE 3. Coverage probabilities of the pth quantiles for GPD$(0, 1, -0.4)$

p	n	**5**	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**200**
0.50	BCa-CP	0.787	0.927	0.942	0.943	0.946	0.944	0.946	0.948	0.942	0.944	0.941	0.948
	BT-CP	0.935	0.866	0.852	0.849	0.858	0.859	0.871	0.877	0.876	0.877	0.876	0.899
	BP-CP	0.943	0.936	0.943	0.946	0.95	0.945	0.951	0.95	0.946	0.946	0.944	0.949
	KDE-CP	0.747	0.871	0.908	0.921	0.926	0.926	0.933	0.926	0.932	0.935	0.936	0.94
0.75	BCa-CP	0.767	0.923	0.932	0.938	0.942	0.938	0.94	0.938	0.944	0.94	0.945	0.945
	BT-CP	0.822	0.845	0.837	0.848	0.848	0.851	0.856	0.86	0.868	0.872	0.876	0.892
	BP-CP	0.757	0.921	0.927	0.94	0.941	0.94	0.938	0.939	0.945	0.941	0.946	0.945
	KDE-CP	0.682	0.831	0.875	0.889	0.902	0.907	0.913	0.914	0.919	0.92	0.925	0.929
	n	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**200**	**300**	**400**	**500**
0.90	BCa-CP	0.415	0.642	0.871	0.923	0.916	0.922	0.934	0.933	0.936	0.941	0.936	0.942
	BT-CP	0.838	0.766	0.813	0.792	0.823	0.812	0.829	0.833	0.839	0.84	0.844	0.868
	BP-CP	0.415	0.646	0.877	0.852	0.912	0.894	0.926	0.926	0.917	0.932	0.934	0.938
	KDE-CP	0.72	0.732	0.803	0.84	0.863	0.869	0.883	0.889	0.891	0.896	0.899	0.923
0.95	BCa-CP	0.226	0.392	0.64	0.776	0.86	0.907	0.927	0.918	0.914	0.928	0.927	0.938
	BT-CP	0.81	0.781	0.735	0.791	0.794	0.816	0.782	0.814	0.81	0.831	0.801	0.845
	BP-CP	0.226	0.393	0.643	0.778	0.866	0.877	0.84	0.88	0.912	0.916	0.885	0.933
	KDE-CP	0.51	0.678	0.718	0.794	0.804	0.831	0.842	0.858	0.859	0.866	0.868	0.904
0.99	BCa-CP	0.047	0.096	0.19	0.26	0.329	0.393	0.454	0.51	0.553	0.594	0.627	0.863
	BT-CP	0.773	0.736	0.731	0.73	0.738	0.744	0.741	0.744	0.737	0.728	0.699	0.777
	BP-CP	0.047	0.096	0.19	0.26	0.329	0.394	0.454	0.511	0.554	0.596	0.629	0.866
	KDE-CP	0.126	0.266	0.444	0.542	0.611	0.651	0.696	0.715	0.727	0.718	0.71	0.79

TABLE 4. Sample size for implementing the recommended methods

Distributions	Quantiles	Recommended methods	Sample size
ND(3, 2)	0.5	BP\KDE	20
	0.65	BP\BCa\KDE	35
	0.75	BP\BCa\KDE	60
	0.85	BCa	80
		KDE	300
	0.9	BCa	200
		KDE	700
	0.95	BCa	300 (unstable)
		KDE	> 1000
	0.99	BCa	> 500
GPD with $(\mu, \sigma, \xi) = (0, 1, -0.4)$	0.5	BP	20
	0.65	BP\BCa	25-30
	0.75	BP\BCa	80
	0.85	BCa\BP	200
	0.9	BCa	200
	0.95	BCa	300 (unstable)
	0.99	BCa\BP	> 500
GPD with $(\mu, \sigma, \xi) = (0, 1, 0.4)$	0.5	BP	30
		KDE	600
	0.65	BP\BCa	35-40
		KDE	700
	0.75	BP\BCa	60
		KDE	> 1000
	0.85	BCa\BP	200
	0.9	BCa	200
	0.95	BCa\BP	400
	0.99	BCa\BP	> 500

From Tables 1-3 we find that the BT method works worse, compared with the BP, BCa and KDE methods. All BT-CPs underestimate the nominal CP. The BP, BCa and KDE methods are competitive for normal or thin-tailed distributions in Table 1 and Table 3. The simulation results for heavy-tailed distribution are reported in Table 2.

To estimate the median for thin-tailed distributions of ND$(3, 2)$ or GPD$(0, 1, -0.4)$, Table 4 shows that the BP and KDE methods perform better than the BCa and BT methods and only ask sample size 20 to satisfy the condition of $|\text{est. CP-}0.95| < 0.01$ in simulation. For the heavy-tailed distribution of GPD$(0, 1, 0.4)$, the BP method performs best among all nonparametric methods when sample size is 30, while the KDE method requires sample size 600 to satisfy the condition of $|\text{est. CP-}0.95| < 0.01$.

For $p \in (0.5, 0.75]$, the BP performs best with sample size 60 for estimating q_p. If $p \in (0.75, 0.90]$, the BCa method performs best among all nonparametric methods for estimating q_p, but this method requires more sample resource. For example, the BCa method asks sample sizes 200 and 300 for estimating q_p when $p = 0.90$ and 0.95, respectively, to satisfy the condition of $|\text{est. CP-}0.95| < 0.01$. Moreover, we find that if sample size is less than 500, all methods are out of work for estimating q_p if p is in $(0.95, 0.99]$.

The KDE method can also be used to infer the quantiles for thin-tailed distributions of ND$(3, 2)$ or GPD$(0, 1, -0.4)$, but the KDE method requires larger sample sizes for estimating q_p than that used by the BCa and BP methods to satisfy the condition of $|\text{est. CP-}0.95| < 0.01$. The BT method looks to perform best for estimating q_p when the sample size is very small such as 5, but we also find that the estimation results are unstable by operating bootstrap methods for estimating q_p when the sample size is extremely small. Hence, we do not recommend BT method in this study.

Overall, we conclude that practitioners can implement the BP method with sample size 20-80 to estimate q_p if $0.5 \leq p \leq 0.75$, and implement the BCa method with sample size 100-200 to estimate q_p if $0.75 < p \leq 0.90$, and implement the BCa method with sample size 300-500 to estimate q_p if $0.90 < p \leq 0.95$. When the sample size is less than 500, all KDE, BP, BT and BCa methods cannot perform well to estimate q_p if $0.95 < p \leq 0.99$ whatever the underlying distribution is ND or GPD.

4. **Conclusions.** In the paper, the KDE, BP, BT and BCa methods are used to obtain the ACIs of the quantiles for the ND and GPD. The estimation performance of all the applied nonparametric methods is evaluated through Monte Carlo simulations. The simulations were conducted for symmetric and asymmetric distributions with a wide range of quantiles. Simulation results are generated by using R source codes. We find that the BCa and BP methods outperform the other two nonparametric methods for estimating the quantiles if $0.50 \leq p \leq 0.90$, and the BCa method is best for the estimation of quantiles, if $0.90 < p \leq 0.95$, in terms of the CP. We recommend proper sample sizes to implement the KDE, BP and BCa methods for estimating q_p in Table 4. Table 4 can be a guideline to implement these nonparametric methods on inferring quantiles for a range of values of p.

It is noticed that when the sample size is less than 500, all KDE, BP, BT and BCa methods cannot perform well to estimate the extremely high quantiles of $p \geq 0.99$. This topic can be studied in the future.

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