

SHAPE SYNCHRONIZATION OF A NOVEL FOUR-DIMENSIONAL CHAOTIC SYSTEM

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Received March 2017; accepted June 2017

ABSTRACT. *This paper investigates the synchronization problem of a novel four-dimensional chaotic system from the view point of the chaotic attractor's shape. To extract the shape information of the chaotic attractor, the four-dimensional chaotic attractor is projected onto the coordinate plane. A group of controllers are designed according to the shape information of the chaotic attractor. A secure communication scheme is proposed based on the shape synchronization of the novel four-dimensional chaotic system. Numerical simulation results show the effectiveness of the proposed method.*

Keywords: Shape synchronization, Four-dimensional chaotic system, Secure communication

1. Introduction. In 1990, Pecora and Carroll proposed the drive-response synchronization concept of chaotic systems in [1] for the first time. During the past two decades, drive-response chaotic synchronization problem has been a hot topic in academia. So far, many research findings about drive-response synchronization of chaotic systems have been discovered. These findings vary with the definition of chaotic synchronization. Different definitions come to different conclusions. Complete synchronization [1-5], partial synchronization [6], generalized synchronization [7], projective synchronization [8-10], phase synchronization [11], lag synchronization [12,13] are commonly accepted by the academia. As we can see, these researches focus on the state of the chaotic systems and ignore the shape characters of the chaotic attractors.

Under a given initial condition, the trajectory of a chaotic system that is described by dynamic equations is a regular spatial curve with special characteristics in shape. Different chaotic attractors have different characteristics in shape. Thus, the shape of the chaotic attractors not only has geometric intuition, but also contains the inherent feature of the chaotic systems. Therefore, it is well worth paying attention to the synchronization problem of chaotic systems from the aspect of chaotic attractor's geometry shape.

As is known, the attractor of a two dimensional chaotic system is a smooth plane curve. According to the theory of plane curve differential geometry [14,15], two plane curves possess the same shape when their signed curvature is equivalent while choosing the same arc-length parameter. [16,17] investigated the shape synchronization problem of two and three dimensional chaotic systems, but the chaotic systems with higher dimension are not involved. Unlike two and three dimensional chaotic systems, the dynamics of higher dimensional chaotic systems is more complex and the shape of the chaotic attractors is invisible. It means that the shape of the higher chaotic attractors cannot be described

by a single physical quality. On the other hand, in information transmission, higher dimension means stronger information carrying capacity. Therefore, it is very necessary to study the shape synchronization problem of higher dimensional chaotic systems. This paper studies the shape synchronization of the four-dimensional Memristive Device based chaotic shape.

The rest of the paper is organized as follows. Section 2 introduces the prior knowledge of the plane curve that is involved in this paper. The description of drive system and information of the drive system chaotic attractor is shown in Section 3. Section 4 gives out the design of the controlled response system and the design of the shape synchronization controller. A secure communication scheme based on the shape synchronization of four-dimensional Memristive Device based chaotic shape is proposed in Section 5. The conclusions are drawn in the last section.

2. Priori Knowledge of Plane Curve. Let $r = r(t) \in R^n$, $t \in R^+$, $R^+ \in [0, +\infty)$ be a regular curve in R^n . When $n = 2$, it is called a plane curve. ‘Regular’ means to any $t \in R^+$, $r'(t) = dr(t)/dt$ does not vanish. By the way, all curves that are mentioned in this paper are regular curves.

The arc-length parameter s of curve $r = r(t)$ is defined by the equation $s(t_1) = \int_{t_0}^{t_1} \|r'(\tau)\| d\tau$, which indicates the unit length of the curve between the point $r(t_0)$ and $r(t_1)$. Obviously, for any $t \in R^+$, $s'(t) > 0$ and thus there must be an anti-function $t = t(s)$.

Consider the plane curve $r(t) = (x(t), y(t))^T \in R^2$ on the Cartesian right hand Frame. The unit tangent is defined as:

$$T(t) = \frac{r'(t)}{|r'(r)|} = \frac{1}{\sqrt{(x'(t))^2 + (y'(t))^2}} \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} \quad (1)$$

The unit normal vector is defined as:

$$N(t) = \frac{1}{\sqrt{(x'(t))^2 + (y'(t))^2}} \begin{bmatrix} -y'(t) \\ x'(t) \end{bmatrix} \quad (2)$$

It is easy to figure out that $\|T(t)\| = 1$, $\|N(t)\| = 1$, $(T(y), N(t)) = 0$, $t \in R^+$. $\|*\|$ denotes Euclidean norm, and $(*)$ denotes the inner product. The signed curvature $\kappa(t)$ is defined as follows.

$$\kappa(t) = \frac{x'(t)y''(t) - x''(t)y'(t)}{[(x'(t))^2 + (y'(t))^2]^{3/2}} \quad (3)$$

Extraordinarily, if we choose the arc-length s as the parameter instead of time t , the relationship between the signed curvature $\kappa(s)$, unit tangent vector $T(s)$ and unit normal vector $N(s)$ can be written as follows.

$$\dot{T}(s) = \kappa(s)N(s), \quad \dot{N}(s) = -\kappa(s)T(s) \quad (4)$$

It is the so called Frenet-Serret formula. At the same time, we have the signed curvature $\kappa(s)$, unit tangent $T(s)$ and unit normal vector $N(s)$ as below.

$$\begin{aligned} \kappa(s) &= \dot{x}(s)\ddot{y}(s) - \ddot{x}(s)\dot{y}(s) \\ T(s) = \dot{r}(s) &= \begin{bmatrix} \dot{x}(s) \\ \dot{y}(s) \end{bmatrix}, \quad N(s) = \begin{bmatrix} -\dot{y}(s) \\ \dot{x}(s) \end{bmatrix} \end{aligned} \quad (5)$$

Suppose there are two arc-length parameterized regular curves $r_1(s)$ and $r_2(s)$. If their signed curvatures $\kappa_1(s)$ and $\kappa_2(s)$ are equivalent everywhere and do not vanish, then the curves $r_1(s)$ and $r_2(s)$ can transform to each other by a rotation and a translation. It

means that $r_1(s) = Ar_2(s) + \Upsilon_0$, where matrix $A = \begin{bmatrix} \cos \theta_0 & -\sin \theta_0 \\ \sin \theta_0 & \cos \theta_0 \end{bmatrix}$, θ_0 is the rotation angle, and $\Upsilon_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ is a vector which gives the translation.

Definition 2.1. Consider two t -parameterized regular plane curves $r_1(t)$ and $r_2(t)$. Make arc-length s be the common parameter from initial time t_0 . If there exist a matrix $A = \begin{bmatrix} \cos \theta_0 & -\sin \theta_0 \\ \sin \theta_0 & \cos \theta_0 \end{bmatrix}$ and a vector $\Upsilon_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ such that $r_1(s) = Ar_2(s) + \Upsilon_0$, then the two plane curves $r_1 = r_1(t)$ and $r_2 = r_2(t)$ are called to share the same shape [14,15].

3. Description of Drive-Response Systems.

3.1. Description of drive system. Drive system is the following four-dimensional chaotic system based on Memristive Device [18].

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = -x_3 - ax_4 - bx_3x_4 - x_2 + x_1x_2 \end{cases} \tag{6}$$

where $x = [x_1, x_2, x_3, x_4]^T$ is the state vector of the drive system and $a = 0.5, b = 0.4$.

Under a given initial condition, the attractor of the drive system is a four-dimensional spatial curve. As we all know, four-dimensional space is invisible in geometric intuition, so it is difficult to describe the shape of the chaotic attractors. However, when we refer to [17], this problem can be solved by projecting the chaotic attractor onto the coordinate plane. The projections of the chaotic attractor are regular plane curves. According to the theory of plane curve [14,15], the shape of the plane can be described by using the signed curvature. If there is a four-dimensional coordinate system, x_1, x_2, x_3, x_4 are the axes and O is the origin, then the projection on the coordinate x_1ox_2 can be denoted as $r_1 = (x_1(t), x_2(t))^T$. The projection on the coordinate plane x_3ox_4 can be denoted as $r_2 = (x_3(t), x_4(t))^T$. It is noticed that the curves r_1, r_2 contain all of state of drive system (6).

Therefore, the shape information of the projection chaotic attractor of drive system can be extracted as follows.

$$\begin{aligned} s_1 &= \int_{t_0}^t \sqrt{\dot{x}_1^2 + \dot{x}_2^2} dt \\ \kappa_1 &= \frac{\dot{x}_1\ddot{x}_2 - \ddot{x}_1\dot{x}_2}{(\dot{x}_1^2 + \dot{x}_2^2)^{3/2}} \end{aligned} \tag{7}$$

$$\begin{aligned} s_2 &= \int_{t_0}^t \sqrt{\dot{x}_3^2 + \dot{x}_4^2} dt \\ \kappa_2 &= \frac{\dot{x}_3\ddot{x}_4 - \ddot{x}_3\dot{x}_4}{(\dot{x}_3^2 + \dot{x}_4^2)^{3/2}} \end{aligned} \tag{8}$$

Remark 3.1. (7) is the shape information of the projection on the coordinate plane (namely r_1) x_1ox_2 . (8) is the shape information of the projection on the coordinate plane x_3ox_4 . (7) combined with (8) is the shape information of the chaotic attractor of the drive system. The shape information of the attractor will be used in the design of the controllers.

3.2. Description of response system. The controlled response system is designed as below.

$$\begin{cases} \dot{y}_1^i = \sin(y_3^i)u_1^i \\ \dot{y}_2^i = \cos(y_3^i)u_1^i \\ \dot{y}_3^i = u_2^i \end{cases} \tag{9}$$

where $i = I, II$, $y^i = (y_1^i, y_2^i, y_3^i)^T$ is the state vector of the i th subsystem and $u^i = (u_1^i, u_2^i)^T$ is the control input.

Response system (9) including two subsystems, the plane curves determined by the front two dimension of subsystem can be denoted as $\tilde{r}_1 = (y_1^I(t), y_2^I(t))^T$, $\tilde{r}_2 = (y_1^{II}(t), y_2^{II}(t))^T$. The shape information of \tilde{r}_1 can be extracted as

$$\begin{aligned} \tilde{s}_1 &= \int_{t_0}^t \sqrt{(\dot{y}_1^I)^2 + (\dot{y}_2^I)^2} dt \\ \tilde{\kappa}_1 &= \frac{\dot{y}_1^I \ddot{y}_2^I - \dot{y}_2^I \ddot{y}_1^I}{\left((\dot{y}_1^I)^2 + (\dot{y}_2^I)^2 \right)^{3/2}} \end{aligned} \tag{10}$$

The shape information of \tilde{r}_2 can be extracted as

$$\begin{aligned} \tilde{s}_2 &= \int_{t_0}^t \sqrt{(\dot{y}_1^{II})^2 + (\dot{y}_2^{II})^2} dt \\ \tilde{\kappa}_2 &= \frac{\dot{y}_1^{II} \ddot{y}_2^{II} - \dot{y}_2^{II} \ddot{y}_1^{II}}{\left((\dot{y}_1^{II})^2 + (\dot{y}_2^{II})^2 \right)^{3/2}} \end{aligned} \tag{11}$$

Definition 3.1. If \tilde{r}_1, \tilde{r}_2 have the same shape with the projective curves r_1, r_2 respectively, then the response system (9) and drive system (6) are called as synchronized in shape.

Remark 3.2. The two subsystems in (9) are designed to restore the projections of drive system (r_1, r_2). It means the shapes of the curves \tilde{r}_1, \tilde{r}_2 will be the same as shapes of r_1, r_2 via a group of well-designed controllers, respectively. Here comes the design process of the controllers.

4. Synchronization of the Chaotic Systems. According to the theory of classic differential geometry, two plane curves possess the same shape if their arc-length parameter and signed curvature are the same. Inspired by this, the controller can be designed as follows.

$$\begin{aligned} u_1^I &= \sqrt{\dot{x}_1^2 + \dot{x}_2^2} \\ u_2^I &= \frac{\dot{x}_1 \ddot{x}_2 - \ddot{x}_1 \dot{x}_2}{\dot{x}_1^2 + \dot{x}_2^2} \end{aligned} \tag{12}$$

$$\begin{aligned} u_1^{II} &= \sqrt{\dot{x}_3^2 + \dot{x}_4^2} \\ u_2^{II} &= \frac{\dot{x}_3 \ddot{x}_4 - \ddot{x}_3 \dot{x}_4}{\dot{x}_3^2 + \dot{x}_4^2} \end{aligned} \tag{13}$$

Theorem 4.1. Under the controller of (12), the curves r_1, \tilde{r}_1 have the same arc-length parameter and signed curvatures.

Proof: Substituting (9) and (12) into (10), then we have

$$\begin{aligned} \tilde{s}_1 &= \int_{t_0}^t \sqrt{(u_1^I)^2} dt = \int_{t_0}^t \sqrt{\dot{x}_1^2 + \dot{x}_2^2} dt = s_1 \\ \tilde{\kappa}_1 &= -\frac{u_2^I}{u_1^I} = \frac{\dot{x}_1 \ddot{x}_2 - \ddot{x}_1 \dot{x}_2}{(\dot{x}_1^2 + \dot{x}_2^2)^{3/2}} = \kappa_1 \end{aligned} \tag{14}$$

It is seen that via controllers in (12), $\tilde{s}_1 = s_1$, $\tilde{\kappa}_1 = \kappa_1$. This completes the proof of Theorem 4.1.

Theorem 4.2. *Under the controller of (13), the curves r_2 , \tilde{r}_2 have the same arc-length parameter and signed curvatures.*

Proof: Substituting (9) and (13) into (11), then we have

$$\begin{aligned} \tilde{s}_2 &= \int_{t_0}^t \sqrt{(u_1^{II})^2} dt = \int_{t_0}^t \sqrt{\dot{x}_3^2 + \dot{x}_4^2} dt = s_2 \\ \tilde{\kappa}_2 &= -\frac{u_2^{II}}{u_1^{II}} = \frac{\dot{x}_3\ddot{x}_4 - \ddot{x}_1\dot{x}_3}{(\dot{x}_3^2 + \dot{x}_4^2)^{3/2}} = \kappa_2 \end{aligned} \tag{15}$$

It is seen that via controllers in (12), $\tilde{s}_2 = s_2$, $\tilde{\kappa}_2 = \kappa_2$. This completes the proof of Theorem 4.2.

According to aforementioned theory of plane curve and Definition 2.1, two plane curves possess the same shape, if they have the same arc-length parameter and signed curvature, and they can transform to each other through a rotation and a translation. r_1, r_2 contain the state of drive system (6) and \tilde{r}_1, \tilde{r}_2 contain the state of response system (9). That is to say, the state of drive system (6) can be restored in the above curves, if we do the same transformation to the state that these curves correspond to.

When the initial state of drive system (6) is chosen as $x(t_0) = [0.06, 1e-6, 0.001, 0.001]^T$, the initial state of the response system is chosen as $y^I(t_0) = [0.06, 1e-6, 0]$ and $y^{II}(t_0) = [0.001, 0.001, 0]^T$. The projections of drive system (6) on the coordinate plane x_1ox_2, x_3ox_4 (namely r_1, r_2) and the curves determined by response system (9) (namely \tilde{r}_1, \tilde{r}_2) are shown in Figure 1.

As is shown in Figure 1, \tilde{r}_1, r_1 and \tilde{r}_2, r_2 respectively have the same shape, but their positions are different. According to Definition 3.1, response system (9) is synchronized with drive system (6) in shape.

According to Definition 2.1, a transformation is applied on the curves \tilde{r}_1, \tilde{r}_2 . We can find that after the transformation, \tilde{r}_1 is completely the same with r_1 and \tilde{r}_2 is completely the same with r_2 , as shown in Figure 2.

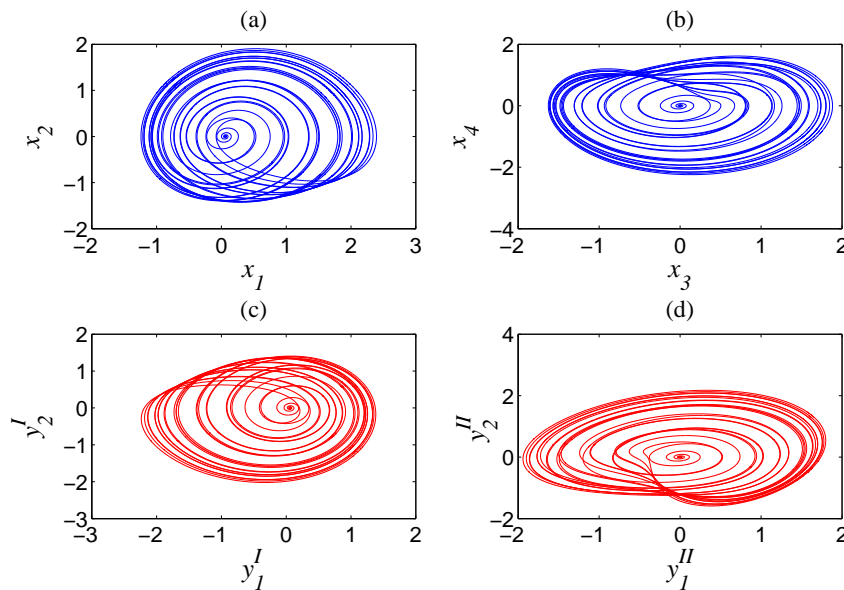


FIGURE 1. Curves determined by drive and response system, where (a) is the curve r_1 ; (b) is the curve r_2 ; (c) is the curve \tilde{r}_1 ; (d) is the curve \tilde{r}_2

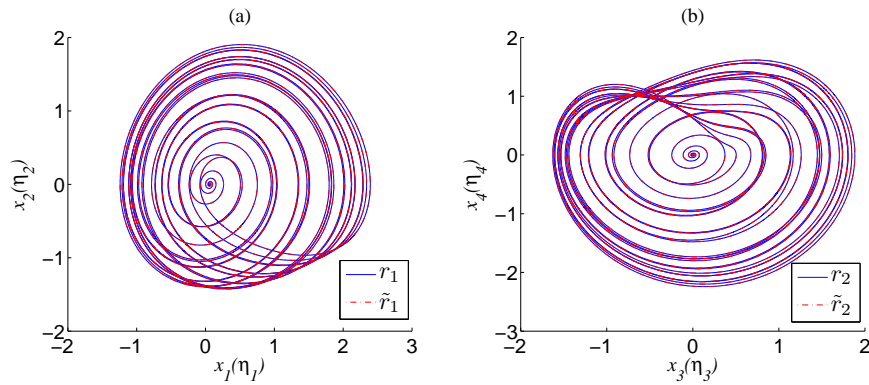


FIGURE 2. Curves after a transformation, where (a) the solid one is r_1 ; the dashed one is \tilde{r}_1 after a transformation; (b) the solid one is r_2 ; the dashed one is \tilde{r}_2 after a transformation

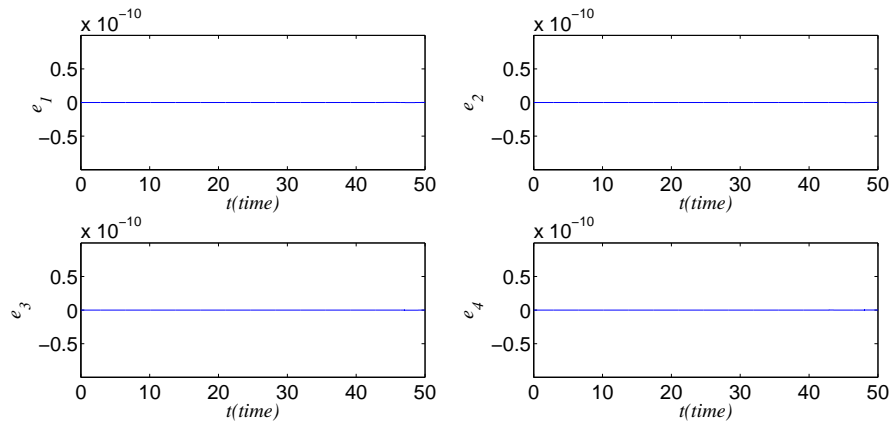


FIGURE 3. State errors between drive system and recovered state

If we do the same transformation to the state that r_1, r_2 correspond to, the state after transformation is combined as $\eta = (\eta_1, \eta_2, \eta_3, \eta_4)^T$. The state error between the state of the drive system (6) and η is shown in Figure 3. The state error is defined as $e_1 = x_1 - \eta_1$, $e_2 = x_2 - \eta_2$, $e_3 = x_3 - \eta_3$, $e_4 = x_4 - \eta_4$.

From Figure 3, it can be seen that the state error between the state of drive system (6) and the state of response system (9) after a transformation is very small. That is to say, response system (9) is completely synchronized with drive system (6).

5. Chaotic Secure Communication Scheme. A secure communication scheme is shown in Figure 4 which is designed based on the shape synchronization of the four-dimensional Memristive Device based chaotic system.

The secure communication scheme in Figure 4 is based on Chaotic Masking encryption technique. The function of the modules is explained as follows.

At the transmitter side:

- 1) *Drive system*: This module is designed based on Equation (6) and is used to generate chaotic signals x_1, x_2, x_3, x_4 . The chaotic signals are used to mask the original information signals $m_1(t), m_2(t)$.
- 2) *Shape analyzer*: The module is used to extract the shape information (namely $s_1, \kappa_1, s_2, \kappa_2$) of the drive system's chaotic attractor.
- 3) *Encryption module*: The initial condition ($\theta(t)$) of the drive system is encrypted by the module.

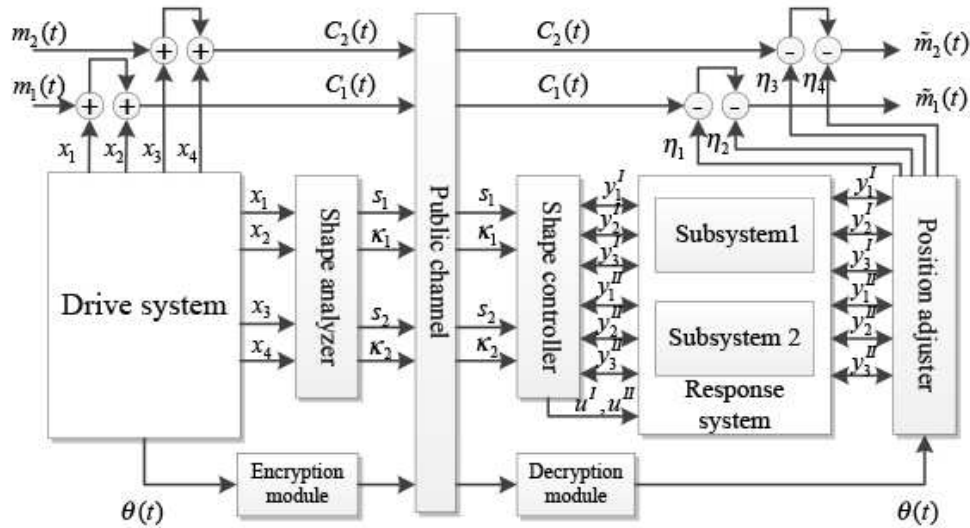


FIGURE 4. Secure communication scheme

At the receiver side:

- 1) *Shape controller*: The function of this module is to achieve shape synchronization between the drive and response system.
- 2) *Response system*: Response system consists of two subsystems which are designed based on Equation (9).
- 3) *Position adjuster*: The main responsibility of this module is to conduct a transformation (as shown in Figure 2) to the state of response system according to the initial condition of the drive system.
- 4) *Decryption module*: this module is designed to decrypt the initial condition of the drive system, which is supplied to the position adjuster.

To verify the feasibility of the proposed secure communication scheme, we choose $m_1(t) = 0.01 \sin(2\pi t)$, $m_2(t) = 0.1 \sin(\pi t)$ as original information signal to conduct a secure communication. The original information signal is masked by the chaotic signal, and the combined signal is called as secret signal. In Figure 4, $C_1(t)$ and $C_2(t)$ are secret signals and $C_1(t) = x_1 + x_2 + m_1(t)$, $C_2(t) = x_3 + x_4 + m_2(t)$. At the receiver side, the recovered information signals are denoted as $\tilde{m}_1(t)$ and $\tilde{m}_2(t)$. Figure 5(a) and Figure 6(a) are original information signals. Figure 5(b) and Figure 6(b) are secret signals. Figure 5(c) and Figure 6(c) are recovered information signals.

From Figure 5 and Figure 6, we can draw the conclusion that the original information signals can be recovered accurately at the receiver side.

6. Conclusions. Shape synchronization of the four-dimensional Memristive Device based chaotic shape has been achieved in this paper. A secure communication is proposed based on the shape synchronization of the Memristive Device based chaotic shape. The information delivered in the public channel is the shape information of the drive systems chaotic attractor, which makes the communication system hard to be attacked. In addition, the original information is overlapped with two chaotic masking signals respectively, which enhances the security of the system. Therefore, the proposed secure communication scheme has a better performance in the data encryption field. Nevertheless, there are still many problems that need to be solved. Our future work will focus on the designing of communication system circuital implementation and expanding our study into integer and fractional order chaotic systems.

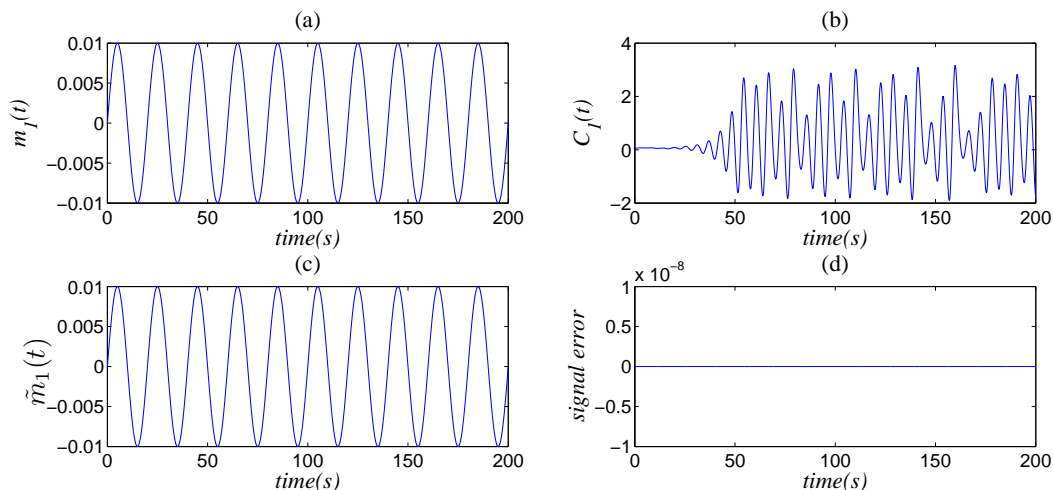


FIGURE 5. (a) The original information signal $m_1(t)$; (b) the secret signal $C_1(t)$; (c) the recovered information signal $\tilde{m}_1(t)$; (d) signal error between $m_1(t)$ and $\tilde{m}_1(t)$

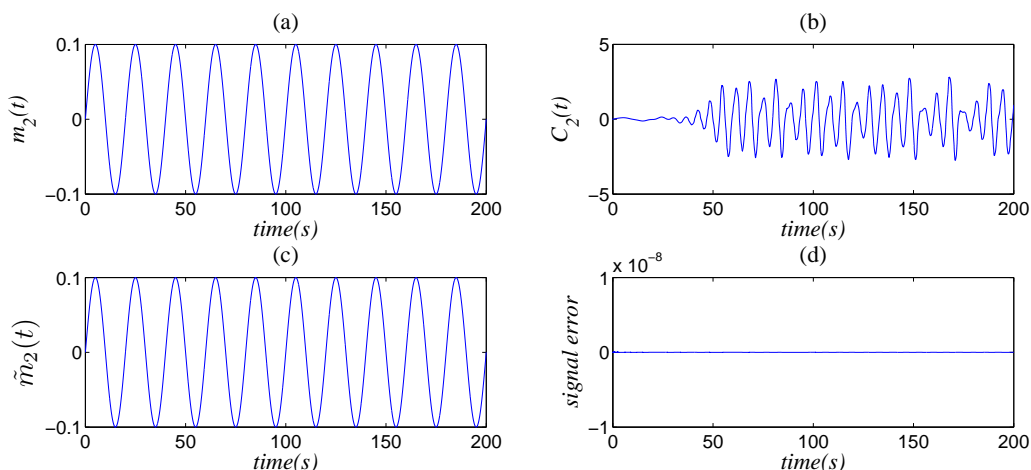


FIGURE 6. (a) The original information signal $m_2(t)$; (b) the secret signal $C_2(t)$; (c) the recovered information signal $\tilde{m}_2(t)$; (d) signal error between $m_2(t)$ and $\tilde{m}_2(t)$

Acknowledgment. This work was supported by the National Natural Science Foundation of China (61673120, 61273219); The Specialized Research Fund for the Doctoral Program of Higher Education of China (20134420110003); The Scientific and Technological Research Program of Chongqing Municipal Education Commission (KJ1710244, KJ1710257, KJ1601003, KJ1401010).

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