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# NUMERICAL SOLUTION OF FUZZY VOLTERRA INTEGRAL EQUATIONS BASED ON LEAST SQUARES APPROXIMATION

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Abstract. *The purpose of this article is solving the fuzzy linear Volterra integral equations by a numerical method. The proposed approach is based on least squares approximation method, which is based on a polynomial of degree n to compute an approximation to the solution of fuzzy Volterra integral equations. At first, by using the parametric form of fuzzy numbers the linear fuzzy Volterra integral equation is transformed into two crisp linear Volterra integral equations, then the least squares approximation method is used to find approximate solution of the obtained system, and hence, obtain an approximation for fuzzy solutions of linear Volterra integral equation. Several numerical examples, in which the exact solutions are known, are provided for better showing the accuracy of least squares approximation method. The results obtained through numerical procedure show that the method is effective and reliable, and also numerical results reveal that the mentioned method is very easy to implement.*
**Keywords:** Fuzzy Volterra integral equation, Least squares approximation method, Parametric form of fuzzy function

1. **Introduction.** The integral equations have been one of the principal tools in various areas of applied mathematics, physics, engineering, geographics and biology.

Usually in many applications, some of the parameters in our problems are represented by fuzzy number rather than crisp one, and hence it is important to develop mathematical models and numerical procedures that would appropriately treat general fuzzy integral equations and solve them [7-9,14,18,19].

There are several methods for solving fuzzy Fredholm integral equations by using parametric form of fuzzy number and converting a fuzzy Fredholm integral equation to two linear systems of integral equations in deterministic case [2,5,6,10-12]. Moreover, there are some computational and analytical methods to obtain solution of fuzzy Volterra integro-differential, fuzzy Volterra-Fredholm and fuzzy Volterra integral equations [3,4,16]. This paper applies the least squares method to the fuzzy linear Volterra integral equations of the form

$$u(x,r) = f(x,r) + \lambda \int_a^x (k(x,t)u(t,r))dt, \tag{1}$$

where $u(x,r) = (\underline{u}(x,r), \overline{u}(x,r))$ is a fuzzy-valued function to be solved for given known fuzzy function $f(x,r) = \left(\underline{f}(x,r), \overline{f}(x,r)\right)$ and given known real-valued function $k(x,t)$. Also $\lambda \in \mathbb{R}$ and $x \in [a,b]$, $b < \infty$, where $a$, $b$ are constants.

This paper is organized as follows. Section 2 presents some preliminaries of fuzzy calculus which will be used later. Section 3 is focused on least squares approximation method for fuzzy Volterra equation (1). Then in Section 4, for implementing the proposed approach, some numerical examples are brought. Finally conclusion is drawn in Section 5.

## 2. **Preliminaries.**

**Definition 2.1.** [8] *A fuzzy number $u$ in parametric form is a pair of functions $(\underline{u}(r), \overline{u}(r))$, $0 \leq r \leq 1$, which satisfy the following requirements:*

1) *$\underline{u}(r)$ is a bounded non-decreasing left continuous function in $(0,1]$, and right continuous at 0.*
2) *$\overline{u}(r)$ is a bounded non-increasing left continuous function in $(0,1]$, and right continuous at 0.*
3) *$\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.*

*Let $\mathbb{E}$ be the set of all upper semicontinuous normal convex fuzzy numbers with bounded $r$-level intervals. It means that if $u \in \mathbb{E}$ then $r$-level set,*

$$[u]^r = \{s \mid u(s) \geq r\},\ 0 \leq r \leq 1$$

*is a closed bounded interval which is denoted by*

$$[u]^r = [\underline{u}(r), \overline{u}(r)]\,.$$

*For arbitrary $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r))$ and $k \in \mathbb{R}$, we have:*

- *Addition: $\underline{u(r)+v(r)} = \underline{u}(r) + \underline{v}(r)$ and $\overline{u(r)+v(r)} = \overline{u}(r) + \overline{v}(r)$*
- *Subtraction: $\underline{u(r)-v(r)} = \underline{u}(r) - \underline{v}(r)$ and $\overline{u(r)-v(r)} = \overline{u}(r) - \overline{v}(r)$*
- *Scalar product: $k \cdot u(r) = \begin{cases} (k\underline{u}(r), k\overline{u}(r)) & k \geq 0, \\ (k\overline{u}(r), k\underline{u}(r)) & k < 0. \end{cases}$*

**Definition 2.2.** [8] *The Hausdorff distance between fuzzy numbers given by $D : \mathbb{E} \times \mathbb{E} \to \mathbb{R}_+ \cup \{0\}$,*

$$D(u,v) = \sup_{r\in[0,1]} \max\{|\underline{u}(r) - \underline{v}(r)|\,, |\overline{u}(r) - \overline{v}(r)|\}.$$

*$D$ is a metric in $\mathbb{E}$ and has the following properties* [15],

(i) *$D(u+w, v+w) = D(u,v)$, $\forall u,v,w \in \mathbb{E}$,*
(ii) *$D(k \cdot u, k \cdot v) = |k| D(u,v)$, $\forall k \in \mathbb{R}, u,v \in \mathbb{E}$,*
(iii) *$D(u+v, w+e) \leq D(u,w) + D(v,e)$, $\forall u,v,w,e \in \mathbb{E}$,*
(iv) *$(D,\mathbb{E})$ is a complete metric space.*

**Definition 2.3.** [10] *A fuzzy function $f : \mathbb{R} \to \mathbb{E}$ is said to be continuous for arbitrary fixed $x_0 \in [a,b]$, if for every one $\epsilon > 0$ there exists $\delta > 0$ such that if $|x - x_0| < \delta$ then $D(f(x), f(x_0)) < \epsilon$.*

**Definition 2.4.** [13] *A mapping $f : I \subset \mathbb{R} \to \mathbb{E}$ is called levelwise continuous at $a \in I$ if the set-valued mapping $f_r(x) = [f(x)]^r$ is continuous at $x = a$ with respect to the Hausdorff metric $D$ for all $r \in [0,1]$.*

**Theorem 2.1.** [13] *Assume that*

(1) *$f(x)$ is a levelwise continuous mapping on $[a, a+x_0]$, $x_0 > 0$;*
(2) *$k(x,t)$ is a levelwise continuous mapping on $\Delta : a \leq t \leq x \leq a+x_0$, $D(u(x), f(x)) \leq y_0$, $y_0 > 0$;*
(3) *for any $(x,t,u(t)) \in \Delta$, $(x,t,v(t)) \in \Delta$, we have $D([k(x,t,u(t))]^r, [k(x,t,v(t))]^r) \leq LD([u(t)]^r, [v(t)]^r)$, where $L > 0$ is a given constant for any $r \in [0,1]$.*

*Then there exists a unique levelwise continuous solution $u(x)$ of* (1) *defined for $x \in (a, a+\delta)$, $\delta = \min\{x_0, \frac{y_0}{M}\}$, where $M = D(k(x,t,u(t)), 0)$, $(x,t,v(t)) \in \Delta$.*

3. **Fuzzy Least Squares Approximation Method.** Assuming that the function $k(x,t)$ satisfies some conditions (Theorem 2.1) such that the solution of (1) exists and is unique. Without loss of generality, we assume that $k(x,t) \geqslant 0$ for any $a \leqslant x, t \leqslant b$, and then according to Section 2, the parametric form of Equation (1) is as follows

$$\underline{u}(x,r) = \underline{f}(x,r) + \lambda \int_a^x (k(x,t)\underline{u}(t,r))\,dt, \tag{2}$$

$$\overline{u}(x,r) = \overline{f}(x,r) + \lambda \int_a^x (k(x,t)\overline{u}(t,r))\,dt. \tag{3}$$

Now, we define the following operator:

$$T(x,u(x,r)) = u(x,r) - f(x,r) - \lambda \int_a^x (k(x,t)u(t,r))dt. \tag{4}$$

For positive integer $n > 0$, suppose $\varphi_0(x), \varphi_1(x), \ldots, \varphi_n(x)$ are linearly independent functions on the interval $[a,b]$ and $\Phi_n = span\{\varphi_0(x), \varphi_1(x), \ldots, \varphi_n(x)\}$.

Let $u_n = (\underline{u}_n, \overline{u}_n)$ where $\underline{u}_n, \overline{u}_n \in \Phi_n$, and then there exist numbers $c_i = (\underline{c}_i, \overline{c}_i)$, $i = 0, 1, \ldots, n$, such that

$$u_n(x,r) = \sum_{i=0}^{n} c_i(r)\varphi_n(x). \tag{5}$$

Now with placement (5) into (4), for any $x \in [a,b]$, $i = 0, 1, \ldots, n$ we have:

$$\begin{aligned} T(x,u_n(x,r)) &= u_n(x,r) - f(x,r) - \lambda \int_a^x (k(x,t)u_n(t,r))dt \\ &= \sum_{i=0}^{n} c_i(r) \left[\varphi_i(x) - \lambda \int_a^x (k(x,t)\varphi_i(t))dt\right] - f(x,r) \\ &= \sum_{i=0}^{n} c_i(r)\alpha_i(x) - f(x,r), \end{aligned} \tag{6}$$

where $\alpha_i(x) = \varphi_i(x) - \lambda \int_a^x (k(x,t)\varphi_i(t))dt$.

Also, for any $a \leq x \leq b$, we define $R_n(x,r) = T(x,u_n(x,r)) - T(x,u(x,r))$, i.e.,

$$R_n(x,r) = (u_n(x,r) - u(x,r)) - \lambda \int_a^x k(x,t)(u_n(t,r) - u(t,r))dt.$$

In the following, let

$$\begin{aligned} I &= I(c_0, c_1, \ldots, c_n) \\ &= \left(\underline{I}(\underline{c}_0, \underline{c}_1, \ldots, \underline{c}_n), \overline{I}(\overline{c}_0, \overline{c}_1, \ldots, \overline{c}_n)\right) \\ &= \left(\int_a^b \underline{T}^2\left(x, \underline{u}_n(x,r)\right)dx, \int_a^b \overline{T}^2\left(x, \overline{u}_n(x,r)\right)dx\right). \end{aligned} \tag{7}$$

The purpose of this step is to find coefficients $c_i$, $i = 0, 1, \ldots, n$ that will minimize $I$.

A necessary condition for minimizing $I$ by numbers $c_i$, $i = 0, 1, \ldots, n$ is that $\dfrac{\partial \underline{I}}{\partial \underline{c}_i} = 0$ and $\dfrac{\partial \overline{I}}{\partial \overline{c}_i} = 0$ for each $i = 0, 1, \ldots, n$.

By the relation (7) we have:

$$\begin{aligned} \frac{\partial \underline{I}}{\partial \underline{c}_i} &= 2\int_a^b \underline{T}\left(x, \underline{u}_n(x,r)\right) \cdot \frac{\partial \underline{I}\left(x, \underline{u}_n(x,r)\right)}{\partial \underline{c}_i} dx \\ &= 2\int_a^b \left\{\sum_{j=0}^{n} \left[\varphi_j(x) - \lambda \int_a^x k(x,t)\varphi_j(t)dt\right]\underline{c}_j - \underline{f}(x,r)\right\} \end{aligned}$$

$$\cdot\left[\varphi_i(x)-\lambda\int_a^x k(x,t)\varphi_j(t)dt\right]dx=0, \tag{8}$$

or, for $j=0,1,\ldots,n$

$$\sum_{i=0}^{n}\underline{c}_j\int_a^b \alpha_j(x)\cdot\alpha_i(x)dx=\int_a^b \underline{f}(x,r)\alpha_i(x)dx. \tag{9}$$

Similarly,

$$\begin{aligned}\frac{\partial \overline{I}}{\partial \overline{c}_i}&=2\int_a^b \overline{T}\left(x,\overline{u}_n(x,r)\right)\cdot\frac{\partial \overline{T}\left(x,\overline{u}_n(x,r)\right)}{\partial \overline{c}_i}dx\\&=2\int_a^b\left\{\sum_{j=0}^{n}\left[\varphi_j(x)-\lambda\int_a^x k(x,t)\varphi_j(t)dt\right]\overline{c}_j-\overline{f}(x,r)\right\}\\&\quad\cdot\left[\varphi_i(x)-\lambda\int_a^x \kappa(x,t)\varphi_j(t)dt\right]dx=0,\end{aligned} \tag{10}$$

or

$$\sum_{i=0}^{n}\overline{c}_j\int_a^b \alpha_j(x)\cdot\alpha_i(x)dx=\int_a^b \overline{f}(x,r)\alpha_i(x)dx \quad j=0,1,\ldots,n. \tag{11}$$

To find $\underline{u}_n$ and $\overline{u}_n$, systems of Equations (9) and (11) must be solved for $2n+2$ unknown $\underline{c}_j$ and $\overline{c}_j$.

The systems (9) and (11) can be written in the form:

$$B.\underline{\mathbf{c}}=\underline{\mathbf{d}},$$

and

$$B.\overline{\mathbf{c}}=\overline{\mathbf{d}},$$

where $B=[b_{ij}]_{(n+1)\times(n+1)}$, $b_{ij}=\int_a^b\alpha_j(x).\alpha_i(x)dx$ for $i,j=0,1,\ldots,n$, also $\underline{\mathbf{d}}=[\underline{d}_0,\underline{d}_1,\ldots,\underline{d}_n]$ and $\overline{\mathbf{d}}=\left[\overline{d}_0,\overline{d}_1,\ldots,\overline{d}_n\right]$ where $\underline{d}_i=\int_a^b\underline{f}(x)\alpha_i(x)dx$ and $\overline{d}_i=\int_a^b\overline{f}(x)\alpha_i(x)dx$.

4. **Numerical Examples.** In this section, three examples are given to illustrate the application of the proposed method. Also, the computed errors $\delta_n$ are defined by,

$$\delta_n\approx\left[\int_a^b e_n^2(x,r)dx\right]^{\frac{1}{2}},$$

where $e_n(x,r)=u(x,r)-u_n(x,r)$.

For easy calculation, we assume $\Phi_n=span\{\varphi_0(x),\varphi_1(x),\ldots,\varphi_n(x)\}$ to be an $n$th polynomial function space.

**Example 4.1.** *Let us consider the following fuzzy Volterra integral equation:*

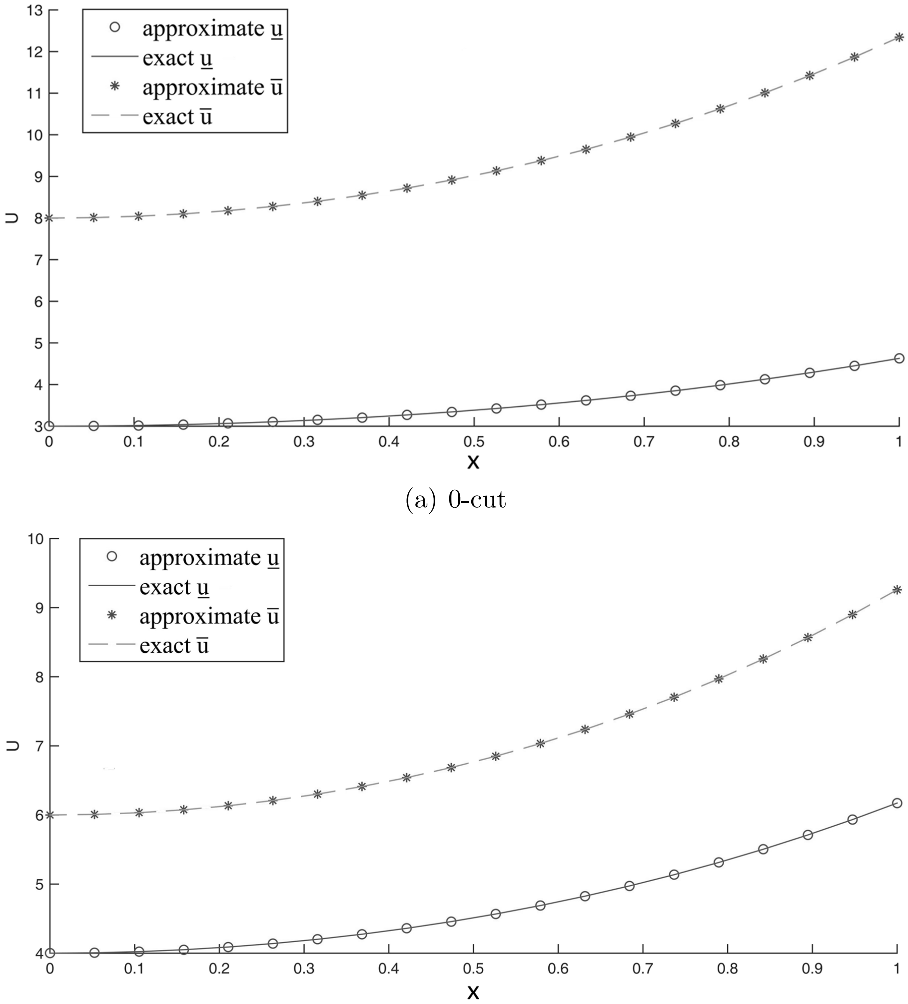
$$u(x)=f(x)+\int_0^x(x-t)u(t)dt,$$

*where* $f(x,r)=[3+r,8-2r]$. *This integral equation has the exact solution* $u(x,r)=[3+r,8-2r]\cdot\cosh(x)$.

We present the error of the proposed method in Table 1 for $r=0.1,0.9$ and different values of $n$. Also, in Figure 1, we plot the obtained solution by $n=8$ and exact solution based on $r$-cuts for $r=0,1$. Also, in Figure 2, we see that the approximate solution obtained by the present method has a good accuracy on the whole interval.

TABLE 1. The error of $\underline{u}$ and $\overline{u}$ for $r = 0.1, 0.9$ and $n = 2, 3, \ldots, 8$

| | $\underline{\delta}_n$ | $\overline{\delta}_n$ | $\underline{\delta}_n$ | $\overline{\delta}_n$ |
|---|---|---|---|---|
| $n$ | $r = 0.1$ | $r = 0.1$ | $r = 0.9$ | $r = 0.9$ |
| 2 | 5.20e-3 | 1.31e-2 | 6.55e-3 | 1.04e-2 |
| 3 | 7.01e-4 | 1.76e-3 | 8.82e-4 | 1.40e-3 |
| 4 | 1.63e-5 | 4.10e-5 | 2.05e-5 | 3.26e-5 |
| 5 | 1.46e-6 | 3.69e-6 | 1.84e-6 | 2.93e-6 |
| 6 | 2.43e-8 | 6.12e-8 | 3.06e-8 | 4.86e-8 |
| 7 | 1.64e-9 | 4.13e-9 | 2.06e-9 | 3.28e-9 |
| 8 | 2.12e-11 | 1.57e-10 | 7.86e-11 | 4.32e-11 |




(a) 0-cut

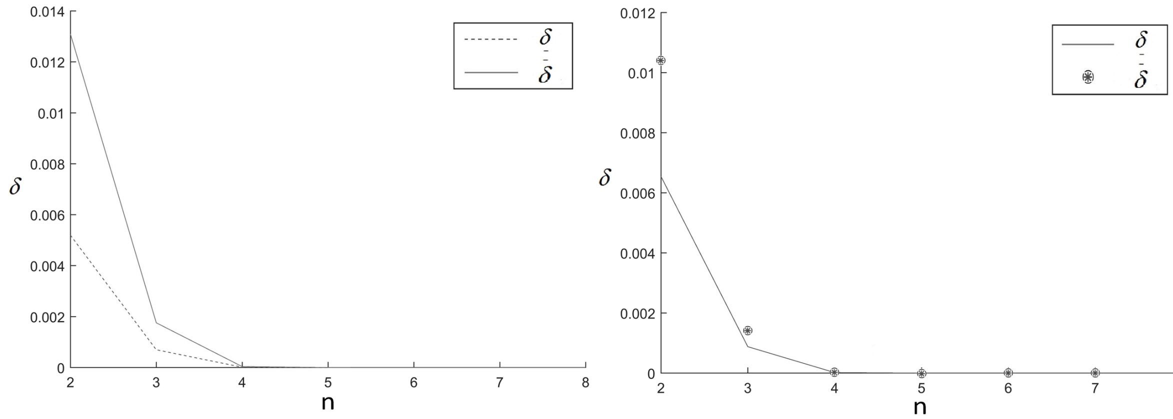
(b) 1-cut

FIGURE 1. Exact and approximate solution ($n = 8$) for Example 4.1

**Example 4.2.** *Consider the following fuzzy Volterra integral equation:*

$$u(x) = f(x) + \int_0^x (x - t)u(t)dt,$$

*where* $f(x, r) = \left(1 - x - \dfrac{x^2}{2}\right) \cdot [r, 2 - r]$*. In this example, the exact solution is* $[r, 2 - r] \cdot (1 - \sinh(x))$*.*




(a) 0.1-cut (b) 0.9-cut

FIGURE 2. The error of $\underline{u}$ and $\overline{u}$ for $r = 0.1, 0.9$ and $n = 2, 3, \ldots, 8$

TABLE 2. The error of $\underline{u}$ and $\overline{u}$ for $r = 0.1, 0.9$ and $n = 2, 3, \ldots, 8$

| | $\underline{\delta}_n$ | $\overline{\delta}_n$ | $\underline{\delta}_n$ | $\overline{\delta}_n$ |
|---|---|---|---|---|
| $n$ | $r = 0.1$ | $r = 0.1$ | $r = 0.9$ | $r = 0.9$ |
| 2 | 3.60e-4 | 6.84e-3 | 3.24e-3 | 3.96e-3 |
| 3 | 1.05e-5 | 1.99e-4 | 9.46e-5 | 1.15e-4 |
| 4 | 1.13e-6 | 2.15e-5 | 1.02e-5 | 1.24e-5 |
| 5 | 2.19e-8 | 4.17e-7 | 1.97e-7 | 2.41e-7 |
| 6 | 1.69e-9 | 3.22e-8 | 1.52e-8 | 1.86e-8 |
| 7 | 2.45e-11 | 4.66e-10 | 2.21e-10 | 2.70e-10 |
| 8 | 1.48e-12 | 2.90e-11 | 1.33e-11 | 1.66e-11 |

*We have the numerical results in Table 2 for $r = 0.1$, $r = 0.9$ and $n = 2, 3, \ldots, 8$.*

Figure 3 shows the exact solution and the resulted solution by the presented method ($n = 8$) based on 0-cut and 1-cut. Also, Figure 4 shows error of $\underline{u}$ and $\overline{u}$ for $r = 0.1, 0.9$ and $n = 2, 3, \ldots, 8$.

**Example 4.3.** *Let us consider the following fuzzy Volterra integral equation:*

$$u(x) + \int_0^x \left(e^{-(t-x)} \cdot u(t)\right) = c \cdot f(x),$$

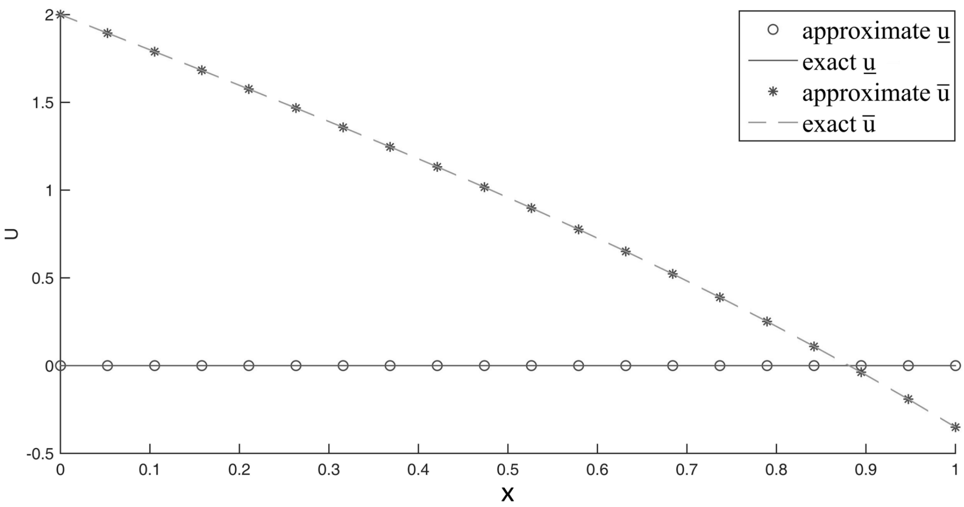
*where $c \in \mathbb{E}$ with $c(r) = [1, 2-r]$, $f(x) = \cosh(x)$, $0 \le r \le 1$. In this example, the exact solution is $u(x, r) = [1, 2-r] \cdot e^{-x}$.*

*We have the numerical results in Table 3 for $r = 0.1$, $r = 0.9$ and $n = 2, 3, \ldots, 8$.*

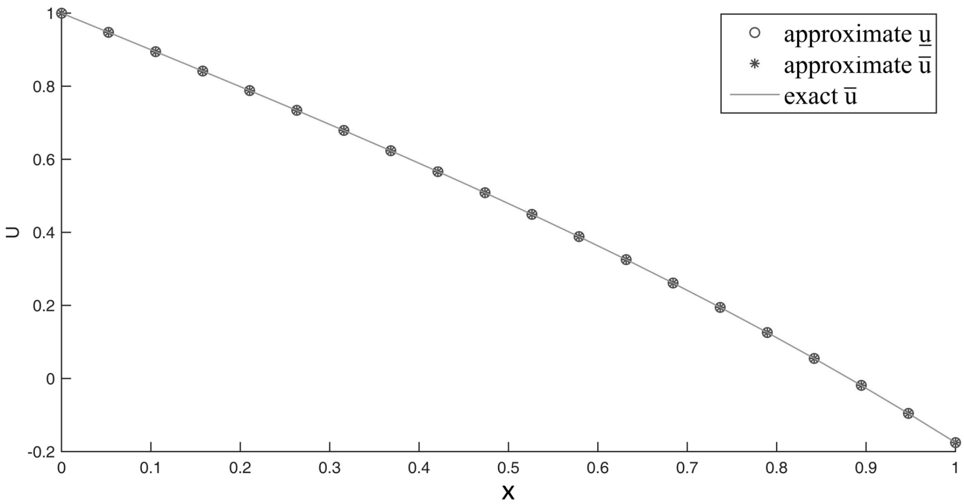
Figure 5 shows the exact solution and the numerical solution based on least square approximation method for $n = 8$ based on 0-cut and 1-cut, and Figure 6 shows error of $\underline{u}$ and $\overline{u}$ for $r = 0.1, 0.9$ and $n = 2, 3, \ldots, 8$.

According to examples we show that the proposed method is effective and gives solution with high accuracy.

5. **Conclusion.** The fuzzy integral equations are important for solving a large proportion of the problems in many topics in applied mathematics.

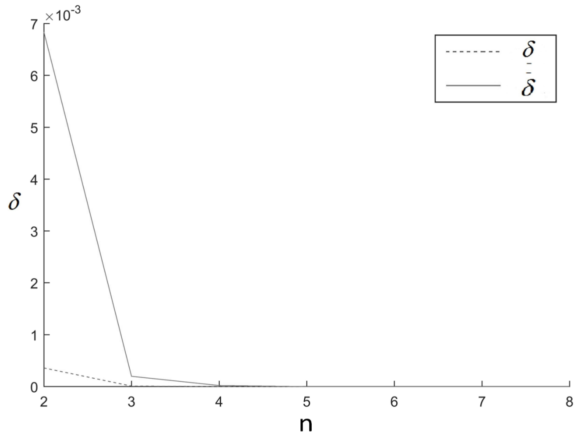



(a) 0-cut

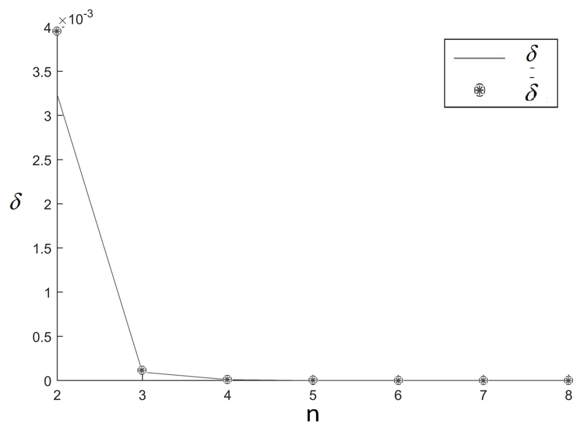



(b) 1-cut

FIGURE 3. Exact and approximate solution ($n = 8$) for Example 4.2




(a) 0.1-cut

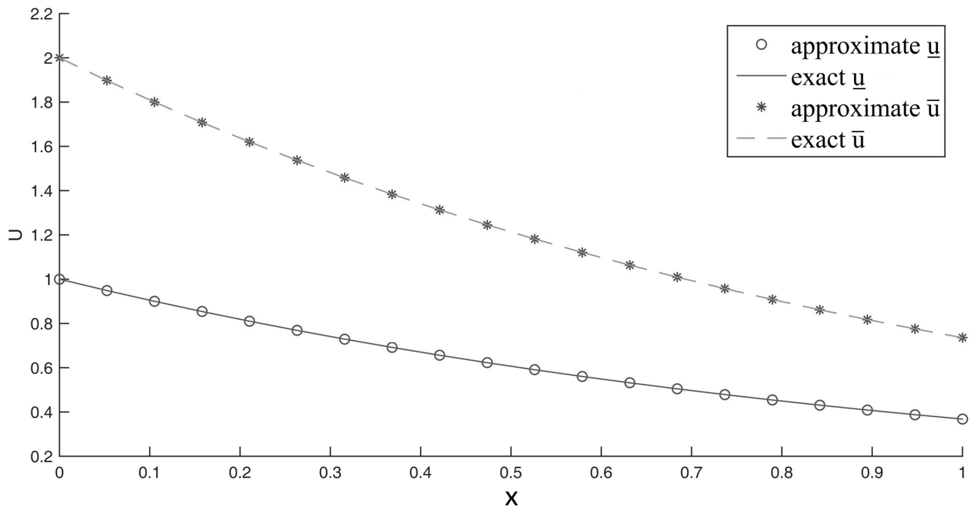



(b) 0.9-cut

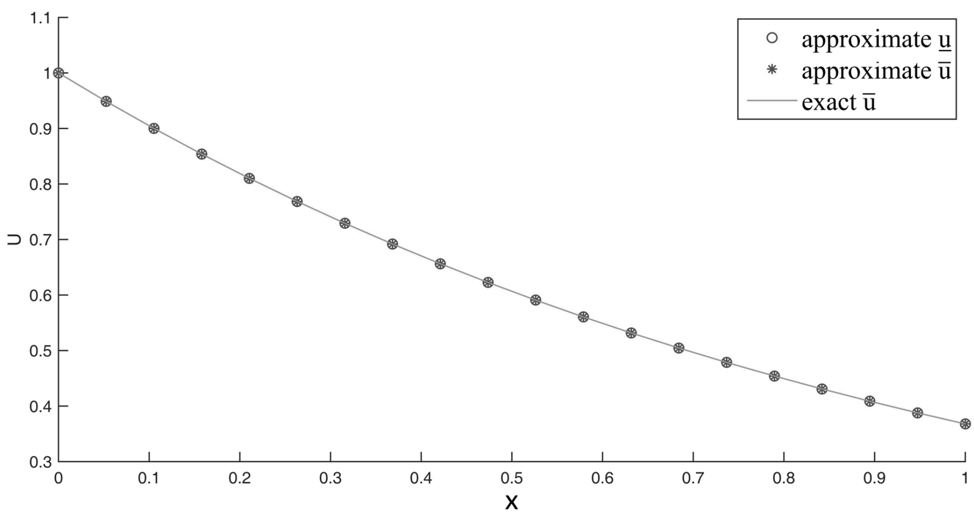
FIGURE 4. The error of $\underline{u}$ and $\overline{u}$ for $r = 0.1, 0.9$ and $n = 2, 3, \ldots, 8$

TABLE 3. The error of $\underline{u}$ and $\overline{u}$ for $r = 0.1, 0.9$ and $n = 2, 3, \ldots, 8$

| | $\underline{\delta}_n$ | $\overline{\delta}_n$ | $\underline{\delta}_n$ | $\overline{\delta}_n$ |
|---|---|---|---|---|
| $n$ | $r = 0.1$ | $r = 0.1$ | $r = 0.9$ | $r = 0.9$ |
| 2 | 1.94e-3 | 3.68e-3 | 1.94e-3 | 2.13e-3 |
| 3 | 1.22e-4 | 2.31e-4 | 1.21e-4 | 1.34e-4 |
| 4 | 6.11e-6 | 1.16e-5 | 6.11e-6 | 6.72e-6 |
| 5 | 2.55e-7 | 4.84e-7 | 2.55e-7 | 2.80e-7 |
| 6 | 9.12e-9 | 1.73e-8 | 9.12e-9 | 1.00e-8 |
| 7 | 2.85e-10 | 5.42e-10 | 2.85e-10 | 3.14e-10 |
| 8 | 2.13e-11 | 8.33e-11 | 2.13e-11 | 3.53e-11 |



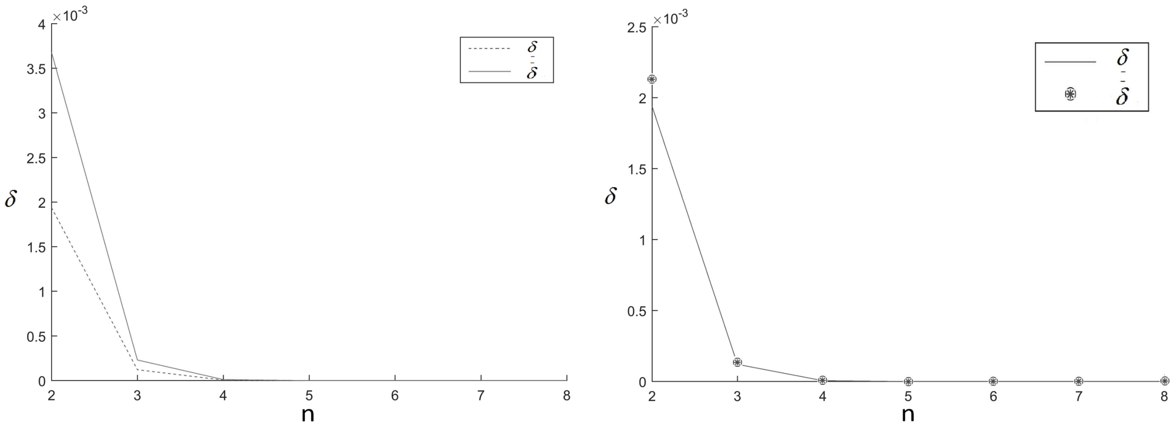

(a) 0-cut




(b) 1-cut

FIGURE 5. Exact and approximate solution ($n = 8$) for Example 4.3

In this paper, we used the least squares approximation method to obtain a numerical approximation of linear fuzzy Volterra integral equations. The accuracy of the proposed



(a) 0.1-cut (b) 0.9-cut

FIGURE 6. The error of $\underline{u}$ and $\overline{u}$ for $r = 0.1, 0.9$ and $n = 2, 3, \ldots, 8$

method has been demonstrated by the numerical examples. This method is efficient that it gives approximations of high accuracy.

In the near future, the least squares approximation method will be used on non-linear fuzzy Volterra integral equations.

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