ICIC Express Letters
Volume **12**, Number **12**, December **2018**
ICIC International ©2018 ISSN 1881-803X
pp. 1285–1291

STABILITY FOR A CLASS OF DYNAMIC SYSTEMS WITH AN INTERNAL CAUSE CONDITION

Peng Li[1], Yuan Yuan[2], Hongjiu Yang[1], Hui Li[1] and Min Shi[1]

[1]Institute of Electrical Engineering
Yanshan University
No. 438, West Hebei Street, Qinhuangdao 066004, P. R. China
{ lipeng_2017; yandalihui; minminxiaomu }@163.com; yanghongjiu@ysu.edu.cn

[2]School of Astronautics
Northwestern Polytechnical University
No. 127, West Youyi Road, Xi'an 710072, P. R. China
snowkey@aliyun.com

Received May 2018; accepted August 2018

Abstract. *This paper generalizes stability for a dynamic system with an internal cause condition. An internal factor which determines system state information is described as internal cause for the dynamic system. Some stability definitions are presented for the dynamic system with an internal cause condition. Moreover, some criteria are also derived to guarantee that the dynamic system is globally asymptotically stable and quadratically stable. A numerical result is provided to demonstrate the effectiveness of the proposed techniques in this paper.*

Keywords: Dynamic systems, Internal cause, Global asymptotical stability, Quadratical stability

1. **Introduction.** Researches on dynamic systems have arisen in various disciplines of science and engineering in recent years [1, 2, 3]. Dynamic systems always experience abrupt changes in their structures or parameters caused by phenomena such as component failure or repair, subsystem interconnection changing, and abrupt environmental disturbance [4]. Such phenomena have been modeled as operations in different forms in [5]. Furthermore, the dynamic systems with structural changes have been modeled as a class of novel dynamic systems in [6], where an internal factor which determines system state information has been described as internal cause. Moreover, the dynamic system model with internal cause has been effectively applied to cyber-physical power systems. It is widely known that stability is a basic structural characteristic to guarantee a normal operation for dynamic systems. Therefore, stability analysis is an important topic in the research on dynamic systems. Although the dynamic system with internal cause has been investigated preliminarily, stability analysis has not been considered in [6], which motivated us to carry on this research work.

A Lyapunov's direct method provides a way to analyze stability of dynamic systems without explicitly solving differential equations [7]. Moreover, the Lyapunov's direct method generalizes an idea which shows that dynamic systems are stable if there exist some appropriate Lyapunov function candidates. In [8], some piece-wise Lyapunov functions have been applied to considering stability of some nonimpulsive dynamic systems. In [9], sufficient stability conditions have been derived for hybrid dynamic systems without perturbations, in which Lyapunov function was only required to be nonincreasing along a subsequence of switchings. Moreover, an asymptotic stability problem has been investigated for a class of impulsive switched dynamic systems with time invariant

DOI: 10.24507/icicel.12.12.1285

delays based on linear matrix inequality (LMI) approach in [10]. An exponential stability criterion has also been derived in terms of LMI based on a Lyapunov method and a convex optimization approach in [11]. Although stability has been investigated for dynamic systems, global asymptotical stability and quadratical stability have not been generalized for dynamic systems with an internal cause. This motivates our current research work to investigate stability for the dynamic systems with an internal cause condition.

In this paper, stability is considered for a class of dynamic systems with internal cause. Some stability definitions are developed for the dynamic systems with internal cause. Furthermore, some criteria are also obtained to guarantee global asymptotical stability and quadratical stability. A numerical simulation is presented to show the effectiveness of the proposed results in this paper.

In Section 2, problem statement is presented. In Section 3, our main results are given. Simulation results are shown in Section 4, and conclusions are presented in Section 5.

2. **Problem Statement.** A dynamic system with internal cause shown in [6] is given as the following form

$$\dot{e}(t) = \mathcal{E}(t, e(t), g'(t)), \tag{1}$$

$$e'(t) = \mathcal{I}(t, e(t), p(t)), \tag{2}$$

$$y(t) = \overline{\mathcal{E}}(t, e'(t)), \tag{3}$$

where $g'(t) \in R^S$ is the control input, $p(t) \in R^\Phi$ is the internal cause, $e(t) \in R^\Phi$ is the state before changing the internal cause, $e'(t) \in R^\Phi$ is the state after changing the internal cause, $y(t) \in R$ is the output, $\mathcal{E}(\cdot)$, $\mathcal{I}(\cdot)$ and $\overline{\mathcal{E}}(\cdot)$ are a series of nonlinear functions. Note that there exists an inverse function $e(t) = \mathcal{I}^{-1}(t, e'(t), p(t))$ for function $\mathcal{I}(t, e(t), p(t))$. Then dynamic system (1)-(3) is rewritten as

$$\dot{e}'(t) = F(t, e'(t), g'(t)), \tag{4}$$

$$y(t) = \overline{\mathcal{E}}(t, e'(t)), \tag{5}$$

where function $F(\cdot)$ is shown as

$$F(t, e'(t), g'(t)) = \frac{\partial \mathcal{I}}{\partial t} + \frac{\partial \mathcal{I}}{\partial e(t)} \cdot \mathcal{E}\left(t, \mathcal{I}^{-1}(t, e'(t), p(t)), g'(t)\right) + \frac{\partial \mathcal{I}}{\partial p(t)} \cdot \dot{p}(t).$$

Taking Taylor expansion of dynamic system (4) and (5) at a neighborhood of (e'_0, g'_0) and ignoring high order terms, a linear dynamic system is obtained as

$$\dot{e}'(t) = A(t)e'(t) + B(t)g'(t), \tag{6}$$

$$y(t) = C(t)e'(t), \tag{7}$$

where

$$A(t) = \left(\frac{\partial F}{\partial e'^T}\right)_{e'_0, g'_0}, \quad B(t) = \left(\frac{\partial F}{\partial g'^T}\right)_{e'_0, g'_0}, \quad C(t) = \left(\frac{\partial \overline{\mathcal{E}}}{\partial e'^T}\right)_{e'_0, g'_0}.$$

For dynamic system (6) and (7), a corresponding autonomous system is shown as

$$\dot{e}'(t) = A(t)e'(t), \quad e'(t_0) = e'_0, \quad t \in [t_0, \infty). \tag{8}$$

Moreover, an equilibrium state of the autonomous system (8) is expressed as state e'_e such that $\dot{e}'_e(t) = 0$ holds for all $t \in [t_0, \infty)$. Disturbed motions of system (6) and (7) are a class of state motions which are caused by initial state disturbance e'_0 of system (8).

Definition 2.1. *Giving a positive scalar ε, if there exists positive scalar $\delta(\varepsilon, t_0)$ dependent on ε and t_0 such that a disturbed motion $\phi(t; e'_0, t_0)$ starting from any initial state e'_0 which is satisfied with*

$$\|e'_0 - e'_e\| \leq \delta(\varepsilon, t_0) \tag{9}$$

satisfies inequality $\|\phi(t; e'_0, t_0) - e'_e\| \leq \varepsilon$ for all $t \geq t_0$, then isolated equilibrium state $e'_e = 0$ of the autonomous system (8) is stable in sense of Lyapunov at instant t_0.

Definition 2.2. *An isolated equilibrium state* $e'_e = 0$ *of the autonomous system (8) is asymptotically stable at instant* t_0 *if the following two conditions are satisfied.*

i) The isolated equilibrium state $e'_e = 0$ *is stable in sense of Lyapunov at instant* t_0*.*

ii) For positive scalars $\delta(\varepsilon, t_0)$ *and* μ*, there exists a positive scalar* $T(\mu, \delta, t_0)$ *such that a disturbed motion starting from initial state* e'_0 *which satisfies inequality (9) is also satisfied with the following inequality*

$$\|\phi(t; e'_0, t_0) - e'_e\| \le \mu, \quad \forall t \ge t_0 + T(\mu, \delta, t_0). \tag{10}$$

Definition 2.3. *An equilibrium state* $e'_e = 0$ *of the autonomous system (8) is uniformly asymptotically stable if the following two conditions are satisfied.*

i) There exists a positive scalar $\delta(\varepsilon)$ *uncorrelated with* t_0 *for any positive scalar* ε*.*

ii) There exists a positive scalar $T(\mu, \delta)$ *uncorrelated with* t_0 *such that disturbed motion* $\phi(t, e'_0, t_0)$ *is bounded for equilibrium state* $e'_e = 0$ *and satisfied with inequality (10).*

Definition 2.4. *If equilibrium state* $e'_e = 0$ *is asymptotically stable for any initial state* $e'_0 \neq 0$*, then equilibrium state* $e'_e = 0$ *of system (8) is globally asymptotically stable.*

3. **Main Results.**

In this section, two criteria are derived to guarantee global asymptotical stability and quadratical stability.

Theorem 3.1. *An equilibrium state* $e' = 0$ *of system (8) is globally asymptotically stable if there exists a scalar function* $V(e', t)$ *which is continuous first-order partial derivation with variables* e' *and* t*, and the following three conditions are satisfied.*

i) The scalar function $V(e', t)$ *is positive definite and bounded, i.e., there exist two continuous nondecreasing functions* $\alpha(\|e'\|)$ *and* $\beta(\|e'\|)$*, in which* $\alpha(0) = 0$ *and* $\beta(0) = 0$*, such that inequality* $\beta(\|e'\|) \ge V(e', t) \ge \alpha(\|e'\|) > 0$ *holds for all* $t \in [t_0, \infty)$ *and* $e' \neq 0$*.*

ii) The derivative $\dot{V}(e', t)$ *is negative definite and bounded, i.e., there exists a continuous nondecreasing function* $\gamma(\|e'\|)$ *satisfying* $\gamma(0) = 0$ *such that inequality* $\dot{V}(e', t) \le \gamma(\|e'\|) < 0$ *holds for all* $t \in [t_0, \infty)$ *and* $e' \neq 0$*.*

iii) There exists $\alpha(\|e'\|) \to \infty$ *when* $\|e'\| \to \infty$*.*

Proof: According to conditions i) and ii), one has the following inequality

$$V(\phi(t; e'_0, t_0), t) - V(e'_0, t_0) = \int_{t_0}^{t} \dot{V}(\phi(\tau; e'_0, t_0), \tau) d\tau \le 0$$

for all $t \in [t_0, \infty)$. For any initial instant t_0 and non-zero initial state e'_0 with $\|e'_0\| \le \delta(\varepsilon)$, it is obtained that

$$\alpha(\varepsilon) \ge \beta(\delta) \ge V(e'_0, t_0) \ge V(\phi(t; e'_0, t_0), t) \ge \alpha(\|\phi(t; e'_0, t_0)\|). \tag{11}$$

Note that $\alpha(0) = 0$ and $\alpha(\|e'\|)$ is continuous and nondecreasing. It is obtained from inequality (11) that the following inequality

$$\|\phi(t; e'_0, t_0)\| \le \varepsilon, \ \forall t \ge t_0 \tag{12}$$

also holds for any initial instant t_0 and non-zero initial state e'_0 with $\|e'_0\| \le \delta(\varepsilon)$. Therefore, there exists a positive scalar $\delta(\varepsilon)$ for any positive scalar ε such that disturbed motions starting from any initial instant t_0 and non-zero initial state e'_0 satisfied with $\|e'_0\| \le \delta(\varepsilon)$ satisfy inequality (12), and $\delta(\varepsilon)$ is uncorrelated with the initial instant t_0. Therefore, one has that equilibrium state $e' = 0$ is uniformly stable.

For any positive scalars μ and $\delta(\varepsilon)$, one constructs a positive scalar $T(\mu, \delta)$ by setting $0 < \mu \le \|e'_0\|$ for any initial state t_0 and non-zero initial state e'_0 with $\|e'_0\| \le \delta(\varepsilon)$. Based on the boundedness of $V(e', t)$, one obtains a positive scalar $\nu(\mu)$ satisfying $\beta(\nu) \le \alpha(\mu)$

for a given positive scalar μ. Note that $\gamma(\|e'\|)$ is a continuous and nondecreasing function. Let $\rho(\mu,\delta)$ be a minimum value of $\gamma(\|e'\|)$ in interval $\nu(\mu) \leq \|e'\| \leq \varepsilon$. Then one has

$$T(\mu,\delta) = \frac{\beta(\delta)}{\rho(\mu,\delta)} \tag{13}$$

uncorrelated with initial instant t_0 for any positive scalar μ. Give a counter assumption that $\phi(t; e_0', t_0) > \nu(\mu)$ holds for all $t_0 \leq t \leq t_1$, where $t_1 = t_0 + T(\mu,\delta)$. Then one has

$$\begin{aligned} 0 < \alpha(\nu) &\leq V(\phi(t_1; e_0', t_0), t_1) \leq V(e_0', t_1) \leq V(e_0', t_0) - (t_1 - t_0)\rho(\mu,\delta) \\ &\leq \beta(\delta) - T(\mu,\delta)\rho(\mu,\delta) = \beta(\delta) - \beta(\delta) = 0 \end{aligned} \tag{14}$$

by condition (13). Obviously, inequality (14) is contradictory. Therefore, the counter assumption is not held, i.e., there exists instant t_2 satisfying $\phi(t_2; e_0', t_0) = \nu(\mu)$ in time interval $t_0 \leq t \leq t_1$. Based on $\phi(t_2; e_0', t_0) = \nu(\mu)$ and the boundedness of $V(e', t)$, one has

$$\alpha(\|\phi(t; e_0', t_0)\|) \leq V(\phi(t; e_0', t_0), t) \leq V(\phi(t_2; e_0', t_0), t_2) \leq \beta(\nu) \leq \alpha(\mu) \tag{15}$$

for all $t \geq t_2$. Furthermore, it is obtained from inequality (15) that inequality

$$\|\phi(t; e_0', t_0)\| \leq \mu \tag{16}$$

holds for all $t \geq t_2$. Then inequality (16) also holds for all $t \geq t_0 + T(\mu,\delta)$. Therefore, disturbed motions starting from initial state e_0' satisfying $\|e_0'\| \leq \delta(\varepsilon)$ converge to equilibrium state $e' = 0$ when $t \to \infty$ for any initial instant t_0.

Note that $\|e'\| \to \infty$ when $\alpha(\|e'\|) \to \infty$. There exists a finite positive scalar $\varepsilon(\delta)$ such that inequality $\beta(\delta) < \alpha(\varepsilon)$ holds for any large finite positive scalar δ. Based on the boundedness of $V(e', t)$, there exists the following inequality

$$\alpha(\varepsilon) > \beta(\delta) \geq V(e_0', t_0) \geq V(\phi(t; e_0', t_0), t) \geq \alpha(\|\phi(t; e_0', t_0)\|)$$

for all $t \in [t_0, \infty)$ and $x_0 \in \mathbb{R}^n$. It is noted that function $\alpha(e')$ is continuous and nondecreasing. Therefore, one has the following inequality

$$\|\phi(t; e_0', t_0)\| \leq \varepsilon(\delta), \ \forall t \geq t_0, \ \forall e_0' \in \mathbb{R}^n,$$

where $\varepsilon(\delta)$ is uncorrelated with initial instant t_0. It is obtained that $\phi(t; e_0', t_0)$ is uniform bounded for any non-zero initial state $e_0' \in \mathbb{R}^n$. This completes the proof.

Note that the autonomous system (8) is a time-varying system. It is very difficult to judge stability of system (8) by the criterion shown in Theorem 3.1. For the autonomous system (8), a polyhedron system matrix is given as

$$\mathcal{A} = \left\{ A(\alpha(t)) : A(\alpha(t)) = \sum_{j=1}^{N} \alpha_j(t) A_j, \ \sum_{j=1}^{N} \alpha_j(t) = 1, \ \alpha_j(t) \geq 0, \ j = 1, \ldots, N \right\}.$$

Then the autonomous system (8) is rewritten as

$$\dot{e}'(t) = A(\alpha(t)) e'(t), \quad e'(t_0) = e_0', \quad t \in [0, \infty), \quad t_0 \in [0, \infty). \tag{17}$$

Furthermore, a low conservative stability condition is shown through linear matrix inequalities (LMIs) in the following theorem.

Theorem 3.2. *If there exist symmetric positive definite matrices $P_j \in \mathbb{R}^{n\times n}$ and positive scalars ρ_i with $i = 1, \ldots, N$ such that*

$$A_j^T P_j + P_j A_j \pm \rho_1 P_1 \pm \rho_2 P_2 \pm \cdots \pm \rho_1 N P_N < 0, \tag{18}$$

$$A_j^T P_k + P_k A_j + A_k^T P_j + P_j A_k \pm 2\rho_1 P_1 \pm \cdots \pm 2\rho_N N P_N < 0, \tag{19}$$

where $j = 1, \ldots, N-1$, $k = j+1, \ldots, N$. Then system (17) is asymptotically stable for all time-varying uncertain parameters such that

$$|\dot{\alpha}_j(t)| \leq \rho_j, \quad j = 1, \ldots, N. \tag{20}$$

Moreover, the parameter dependent Lyapunov matrix is given by

$$P(\alpha(t)) = \sum_{j=1}^{N} \alpha_j(t) P_j, \tag{21}$$

where $\sum_{j=1}^{N} \alpha_j(t) = 1$*,* $\alpha_j(t) \geq 0$*,* $j = 1, \ldots, N$*.*

Proof: Consider a parameter dependent Lyapunov function $V(x) = x^T P(\alpha(t))x > 0$ where $P(\alpha(t)) = P^T(\alpha(t)) > 0$ with $P(\alpha(t))$ given by equality (21). Then the derivative is obtained as

$$\dot{V}(x) = x^T A^T(\alpha(t))P(\alpha(t)) + P(\alpha(t))A(\alpha(t)) + \dot{P}(\alpha(t)) = x^T Q(\alpha(t))x,$$

where $P(\alpha(t))$ is given by equality (21). It is shown that

$$Q(\alpha(t)) = \sum_{j=1}^{N} \alpha_j(t)^2 \left(A_j^T P_j + P_j A_j\right) + \sum_{j=1}^{N-1} \sum_{k=j+1}^{N} \alpha_j(t)\alpha_k(t) \Bigg(\big(A_j^T P_k + P_k A_j + A_k^T P_j + P_j A_k\big) + \sum_{j=1}^{N} \dot{\alpha}_j(t) P_j \Bigg),$$

which is rewritten as

$$Q(\alpha(t)) = \sum_{j=1}^{N} \alpha_j(t)^2 \left(A_j^T P_j + P_j A_j + \sum_{j=1}^{N} \dot{\alpha}_j(t) P_j \right) + \sum_{j=1}^{N-1} \sum_{k=j+1}^{N} \alpha_j(t)\alpha_k(t) \left(\big(A_j^T P_k + P_k A_j + A_k^T P_j + P_j A_k\big) + 2\sum_{j=1}^{N} \dot{\alpha}_j(t) P_j \right).$$

Considering that the time-derivatives $\dot{\alpha}_j(t)$ with $j = 1, \ldots, N$ satisfy inequality (20), one has $Q(\alpha(t)) < 0$ by imposing conditions (18) and (19) for all $\alpha_j(t) \geq 0$ with $j = 1, \ldots, N$ and $\sum_{j=1}^{N} \alpha_j(t) = 1$. The proof is completed.

Remark 3.1. *Note that Theorem 3.2 encompasses the quadratic stability analysis in the sense that if* $A_j^T P + P A_j < 0$ *with* $j = 1, \ldots, N$ *holds, then there exist values of* ρ_j *with* $j = 1, \ldots, N$ *such that* $P_1 = P_2 = \cdots = P_N = P$ *is a feasible solution of* (18) *and* (19)*. On the other hand, for linear time-invariant systems, i.e.,* $\rho_j = 0$ *with* $j = 1, \ldots, N$*, conditions* (18) *and* (19) *become equivalent to the results of* [12]*, which generally provides better robust stability evaluations than other LMI bases results.*

4. **Numerical Example.** A numerical example is given to illustrate our results. The autonomous system (8) is described by

$$\dot{e}'(t) = \begin{bmatrix} 0 & 1 \\ -2 - \dfrac{1}{1+t} & -1 \end{bmatrix} e'(t). \tag{22}$$

Based on Theorem 3.2, system (22) is rewritten as

$$\dot{e}'(t) = \alpha_1(t) A_1 e'(t) + (1 - \alpha_1(t)) A_2 e'(t),$$

where

$$A_1 = \begin{bmatrix} 0 & 1 \\ -3 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}, \quad \alpha_1(t) = \frac{1}{1+t}.$$

It is obtained from Theorem 3.2 that system (22) is asymptotically stable if there exist a scalar ρ_1 and symmetric positive definite matrices P_1 and P_2 satisfying the following LMIs

$$A_1^T P_1 + P_1 A_1 \pm \rho_1 (P_1 - P_2) < 0, \tag{23}$$

$$A_2^T P_2 + P_2 A_2 \pm \rho_1 (P_1 - P_2) < 0, \quad (24)$$

$$A_1^T P_2 + P_2 A_1 + A_2^T P_1 + P_1 A_2 \pm 2\rho_1 (P_1 - P_2) < 0. \quad (25)$$

Note that ρ_1 is chosen as 1 for $\dot{\alpha}(t) = 1/(1+t)^2$. By solving LMIs (23)-(25), one has

$$P_1 = \begin{bmatrix} 44.9713 & 4.3069 \\ 4.3069 & 14.6201 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 43.6378 & 4.7650 \\ 4.7650 & 17.0338 \end{bmatrix}.$$

Two state trajectories of system (22) are plotted in Figure 1, in which the two states converge to zero point. Therefore, the autonomous system (22) is asymptotically stable. The numerical example illustrates the validity of the proposed techniques in this paper.

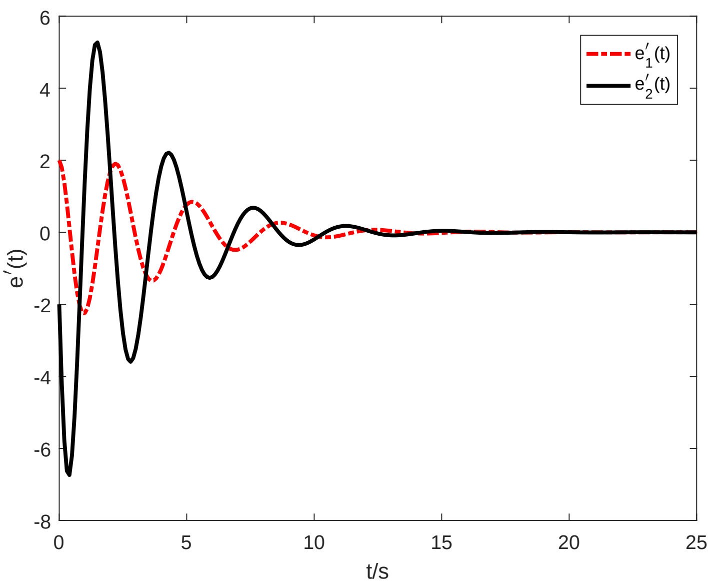

FIGURE 1. State trajectories of system (22)

5. **Conclusions.** In this paper, stability has been investigated for a dynamic system with an internal cause. Some stability definitions have been presented for the dynamic system with an internal cause. Some criteria have also derived to guarantee global asymptotical stability and quadratical stability for the dynamic system. A numerical result has been shown to demonstrate the effectiveness of the proposed techniques. Some researches on controllability and observability will be investigated for dynamic system with an internal cause in the further work.

REFERENCES

[1] H. Yu, T. Xie, S. Paszczynski and B. M. Wilamowski, Advantages of radial basis function networks for dynamic system design, *IEEE Trans. Industrial Electronics*, vol.58, no.12, pp.5438-5450, 2011.
[2] H. Yang, Y. Xia and H. Li, An overview of Delta operator systems, *Control Theory and Applications*, vol.32, no.5, pp.569-578, 2015.
[3] X. Yuan, Y. Wang, C. Yang, Z. Ge and Z. Song, Weighted linear dynamic system for feature representation and soft sensor application in nonlinear dynamic industrial processes, *IEEE Trans. Industrial Electronics*, vol.65, no.2, pp.1508-1517, 2017.
[4] X. Liu, X. Shen and Y. Zhang, Stability analysis of a class of hybrid dynamic systems, *Dynamics of Continuous, Discrete and Impulsive Systems*, vol.8, no.3, pp.359-373, 2001.
[5] B. D. O. Anderson and J. B. Moore, *Linear Optimal Control*, Prentice-Hall, Englewood, 1971.
[6] Y. Yuan, H. Yang, Y. Xia and Y. Yuan, Modeling research for cyber-physical system based on cause-effect logic relation, *Information and Control*, vol.47, no.1, pp.119-128, 2018.
[7] Y. Li, Y. Chen and I. Podlubny, Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag-Leffler stability, *Computers and Mathematics with Applications*, vol.59, no.5, pp.1810-1821, 2010.

[8] M. Johansson and A. Rantzer, Computation of piecewise quadratic Lyapunov functions for hybrid systems, *IEEE Trans. Automatic Control*, vol.43, no.4, pp.555-559, 1998.
[9] Z. Li, C. Soh and X. Xu, Lyapunov stability of a class of hybrid dynamic systems, *Automatica*, vol.36, no.2, pp.297-302, 2000.
[10] H. Xu, X. Liu and K. L. Teo, A LMI approach to stability analysis and synthesis of impulsive switched systems with time delays, *Nonlinear Analysis Hybrid Systems*, vol.2, no.1, pp.38-50, 2008.
[11] O. M. Kwon, J. H. Park and S. M. Lee, Exponential stability for uncertain dynamic systems with time-varying delays: LMI optimization approach, *Journal of Optimization Theory and Applications*, vol.137, no.3, pp.521-532, 2008.
[12] D. C. W. Ramos and P. L. D. Peres, An LMI condition for the robust stability of uncertain continuous-time linear systems, *IEEE Trans. Automatic Control*, vol.47, no.4, pp.675-678, 2002.