ICIC Express Letters
Volume **12**, Number **2**, February **2018**

ICIC International ©2018 ISSN 1881-803X
pp. 145–150

ALL-PAIRWISE MULTIPLE COMPARISON FOR MULTIVARIATE NORMAL COVARIANCE MATRICES BASED ON THE SINGLE STEP PROCEDURE

Tsunehisa Imada

School of Management
Tokai University
9-1-1, Toroku, Higashi-ku, Kumamoto 862-8652, Japan
timada@ktmail.tokai-u.jp

Received August 2017; accepted November 2017

Abstract. *In this study we discuss multiple comparisons for checking differences among multivariate normal covariance matrices. Specifically, we consider the all-pairwise multiple comparison. We determine the conservative critical value for pairwise comparison for a specified significance level using asymptotic distribution and Bonferroni's inequality. Finally, we give some numerical examples regarding critical values and power of the test.*

Keywords: Asymptotic distribution, Bonferroni's inequality, Conservation

1. **Introduction.** When we test whether two normal means are equal or not, the method for testing depends on whether corresponding two normal variances are equal or not. Specifically, we use the canonical t-test when two normal variances are equal, and we use Welch's test when they are different. More generally, when we test whether plural normal means are equal or not by the analysis of variance, the assumption that corresponding normal variances are uniformly equal is necessary. The assumption is also necessary for multiple comparisons for checking differences among plural normal means. Therefore, it is occasionally necessary to test whether plural normal variances are uniformly equal or not. If plural normal variances are not equal, we occasionally want to find the pair consisting of different normal variances. Therefore, we need multiple comparison procedures for plural normal variances. The multiple comparison with a control and the all-pairwise multiple comparison for plural normal variances were discussed in [3] (Refer to [1] and [7] for more).

On the other hand, when we test whether two multivariate normal means are equal or not, the assumption that corresponding two multivariate normal covariance matrices are equal is necessary for applying the canonical Hotelling's T^2-distribution to the test. More generally, when we test whether plural multivariate normal means are uniformly equal or not by the analysis of variance, the assumption that corresponding multivariate normal covariance matrices are uniformly equal is necessary. The assumption is also necessary for multiple comparisons for checking differences among plural multivariate normal means. Although many researchers proposed all-pairwise multiple comparison procedures for plural multivariate normal means like [2] and [6], their procedures are available under the assumption that corresponding plural multivariate normal covariance matrices are uniformly equal. Therefore, before we carry out the all-pairwise multiple comparison procedure for plural multivariate normal means, we should check whether the plural multivariate normal covariance matrices are uniformly equal. If the plural multivariate normal covariance matrices are not uniformly equal, we should find the pair consisting of different normal covariance matrices. The aim of this study is to construct the all-pairwise multiple comparison procedure for plural normal covariance matrices. This is the development of [2]. Here, we focus on the single step multiple comparison procedure [7]. We determine the conservative critical value for pairwise comparison for a specified significance level using asymptotic distribution and Bonferroni's inequality. Finally, we give some numerical examples regarding critical values and power of the test.

2. **Asymptotic Distribution.** In this section we discuss an asymptotic distribution which is necessary for constructing the all-pairwise multiple comparison procedure for plural multivariate normal covariance matrices.

There are independent normal random variable vectors $\boldsymbol{X}_1$, $\boldsymbol{X}_2$ satisfying

$$\boldsymbol{X}_1 \sim N(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1),\ \boldsymbol{X}_2 \sim N(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2).$$

We consider testing the difference between $\boldsymbol{\Sigma}_1$ and $\boldsymbol{\Sigma}_2$. We set up a null hypothesis and its alternative hypothesis as

$$H_0 : \boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2 \text{ vs. } H_1 : \boldsymbol{\Sigma}_1 \neq \boldsymbol{\Sigma}_2. \tag{1}$$

For a sample $\boldsymbol{X}_{i1}, \boldsymbol{X}_{i2}, \ldots, \boldsymbol{X}_{in_i}$ from $N_p(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)$ for $i = 1, 2$, let

$$\bar{\boldsymbol{X}}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} \boldsymbol{X}_{ij},\quad \boldsymbol{A}_i = \sum_{j=1}^{n_i} \left(\boldsymbol{X}_{ij} - \bar{\boldsymbol{X}}_i\right)\left(\boldsymbol{X}_{ij} - \bar{\boldsymbol{X}}_i\right)'$$

and $\boldsymbol{A} = \boldsymbol{A}_1 + \boldsymbol{A}_2$. The likelihood ratio test for (1) is discussed in [4] (Chap. 8, Sec. 8·3, Chap. 10, Sec. 10·1). By the likelihood ratio test criteria for (1) we reject H_0 when

$$L_2 = \frac{\prod_{i=1}^{2} |\boldsymbol{A}_i|^{\frac{n_i-1}{2}}}{|\boldsymbol{A}|^{\frac{N-2}{2}}} \cdot \frac{(N-2)^{\frac{p(N-2)}{2}}}{\prod_{i=1}^{2}(n_i-1)^{\frac{p(n_i-1)}{2}}} > c$$

for a specified critical value c. Herein $N = n_1 + n_2$. Although we should determine c so that

$$P(L_2 > c) = \alpha$$

for a specified significance level α under H_0, it is difficult to determine it, because it is difficult to determine the distribution of L_2 under H_0. On the other hand, by

$$P(-2\log L_2 \leq c) = P\left(\chi^2_{f_2} \leq c\right) + O\left(N^{-1}\right)$$

$-2\log L_2$ is asymptotically distributed according to χ^2-distribution with f_2 degrees of freedom where $f_2 = p(p+1)/2$. Specifically, under H_0

$$-2\log L_2 \asymp \chi^2_{f_2}. \tag{2}$$

More precise asymptotic distribution

$$-2\rho_2 \log L_2 \asymp \chi^2_{f_2} \tag{3}$$

is determined by

$$P(-2\rho_2\log L_2 \leq c) = P\left(\chi^2_{f_2} \leq c\right) + O\left(N^{-2}\right),$$

where

$$\rho_2 = 1 - \left(\sum_{i=1}^{2} \frac{1}{n_i - 1} - \frac{1}{N-2}\right)\frac{2p^2 + 3p - 1}{6(p+1)}.$$

We investigate the closeness of the approximation of (2) and (3). Letting $p = 2$, $f = 3$ and the upper 0.05-point of χ^2_3 is $c = 7.815$. Table 1 gives the probabilities $P(-2\log L_2 \leq c)$ and $P(-2\rho_2 \log L_2 \leq c)$ for $n_1 = n_2 = 10, 20, 50, 100$.

Letting $p = 3$, $f = 6$ and the upper 0.05-point of χ^2_3 is $c = 12.592$. Table 2 gives the probabilities $P(-2\log L_2 \leq c)$ and $P(-2\rho_2\log L_2 \leq c)$ for $n_1 = n_2 = 10, 20, 50, 100$.

They are calculated by Monte Carlo simulation with 1,000,000 times of experiments. The tables show that the approximation (3) is closer to χ^2_f compared to (2). Even if n is small, the approximation (3) is precise.

Table 1. Comparisons of the closeness of the approximation ($p = 2$, $c = 7.815$)

n	$P(-2\log L_2 \le c)$	$P(-2\rho_2 \log L_2 \le c)$
10	0.0761	0.0500
20	0.0608	0.0499
50	0.0540	0.0501
100	0.0519	0.0498

Table 2. Comparisons of the closeness of the approximation ($p = 3$, $c = 12.592$)

n	$P(-2\log L_2 \le c)$	$P(-2\rho_2 \log L_2 \le c)$
10	0.1143	0.0515
20	0.0739	0.0502
50	0.0582	0.0500
100	0.0538	0.0499

3. **Multiple Comparison.** Assume there are independent p-dimensional normal random variable vectors $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_K$. Assume

$$\boldsymbol{X}_k \sim N_p(\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k) \ (k = 1, 2, \ldots, K).$$

3.1. **Analysis of variance.** We consider testing whether $\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2 = \cdots = \boldsymbol{\Sigma}_K$ or not. We set up a null hypothesis and its alternative hypothesis as

$$H_0 : \boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2 = \cdots = \boldsymbol{\Sigma}_K \quad \text{vs.} \quad H_1 : \boldsymbol{\Sigma}_i \ne \boldsymbol{\Sigma}_j \text{ for some } i, j \ (1 \le i < j \le K). \tag{4}$$

For a sample $\boldsymbol{X}_{k1}, \boldsymbol{X}_{k2}, \ldots, \boldsymbol{X}_{kn_k}$ from $N_p(\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$ for $k = 1, 2, \ldots, K$, let

$$\bar{\boldsymbol{X}}_k = \frac{1}{n_k} \sum_{i=1}^{n_k} \boldsymbol{X}_{ki}, \quad \boldsymbol{A}_k = \sum_{i=1}^{n_k} \left(\boldsymbol{X}_{ki} - \bar{\boldsymbol{X}}_k\right)\left(\boldsymbol{X}_{ki} - \bar{\boldsymbol{X}}_k\right)'$$

and $\boldsymbol{A} = \sum_{k=1}^K \boldsymbol{A}_k$. By the likelihood ratio test criteria for (4) we reject H_0 when

$$L_K = \frac{\prod_{k=1}^K |\boldsymbol{A}_k|^{\frac{n_k-1}{2}}}{|\boldsymbol{A}|^{\frac{N-K}{2}}} \cdot \frac{(N-K)^{\frac{p(N-K)}{2}}}{\prod_{k=1}^K (n_k - 1)^{\frac{p(n_k-1)}{2}}} > c$$

for a specified critical value c. Herein $N = \sum_{k=1}^K n_k$. It is difficult to determine the distribution of L_K under H_0. On the other hand, by

$$P(-2\log L_K \le c) = P\left(\chi^2_{f_K} \le c\right) + O\left(N^{-1}\right)$$

$-2\log L_K$ is asymptotically distributed according to χ^2-distribution with f_K degrees of freedom where $f_K = p(p+1)(K-1)/2$. Specifically, under H_0

$$-2\log L_K \asymp \chi^2_{f_K}. \tag{5}$$

More precise asymptotic distribution

$$-2\rho_K \log L_K \asymp \chi^2_f \tag{6}$$

is determined by

$$P(-2\rho_K \log L_K \le c) = P\left(\chi^2_{f_K} \le c\right) + O\left(N^{-2}\right),$$

where

$$\rho_K = 1 - \left(\sum_{i=1}^{2} \frac{1}{n_i - 1} - \frac{1}{N-K}\right) \frac{2p^2 + 3p - 1}{6(p+1)(K-1)}.$$

If H_0 is rejected, we occasionally want to find the pair $\boldsymbol{\Sigma}_i$, $\boldsymbol{\Sigma}_j$ $(i < j)$ satisfying $\boldsymbol{\Sigma}_i \ne \boldsymbol{\Sigma}_j$. Then, we use multiple comparison procedures. Here, we construct the all-pairwise multiple comparison procedure for finding the pair $\mathbf{\Sigma}_i$, $\mathbf{\Sigma}_j$ $(i < j)$ satisfying $\mathbf{\Sigma}_i \neq \mathbf{\Sigma}_j$. Here we focus on the single step multiple comparison procedure [7].

3.2. **Single step multiple comparison procedure.** Intended to compare $\mathbf{\Sigma}_i$ and $\mathbf{\Sigma}_j$ $(i < j)$, we set up a null hypothesis and its alternative hypothesis as

$$H_{ij} : \mathbf{\Sigma}_i = \mathbf{\Sigma}_j \quad \text{vs.} \quad H_{ij}^A : \mathbf{\Sigma}_i \neq \mathbf{\Sigma}_j$$

and consider the simultaneous test of H_{ij}s. We consider the single step multiple comparison procedure for H_{ij}s [7]. Let

$$S_{ij} = -2\rho_{ij} \log L_{ij},$$

where

$$L_{ij} = \frac{|\boldsymbol{A}_i|^{\frac{n_i-1}{2}} |\boldsymbol{A}_j|^{\frac{n_j-1}{2}}}{|\boldsymbol{A}_{ij}|^{\frac{N_{ij}-2}{2}}} \cdot \frac{(N_{ij}-2)^{\frac{p(N_{ij}-2)}{2}}}{(n_i-1)^{\frac{p(n_i-1)}{2}} (n_j-1)^{\frac{p(n_j-1)}{2}}}$$

and

$$\rho_{ij} = 1 - \left(\frac{1}{n_i - 1} + \frac{1}{n_j - 1} - \frac{1}{N_{ij} - 2} \right) \frac{2p^2 + 3p - 1}{6(p+1)}.$$

Herein $\boldsymbol{A}_{ij} = \boldsymbol{A}_i + \boldsymbol{A}_j$ and $N_{ij} = n_i + n_j$. If $S_{ij} > c$ for a specified critical value c, H_{ij} is rejected. Otherwise, it is retained. We should determine c so that

$$P\left(\max_{i<j} S_{ij} > c \right) = \alpha \tag{7}$$

for a specified significance level α when all H_{ij}s are true. Since it is difficult to determine the distribution of $\max_{i<j} S_{ij}$, we cannot obtain c satisfying (7). Each S_{ij} is asymptotically distributed according to χ^2-distribution $\chi^2_{f_2}$ with f_2 degrees of freedom under H_{ij}. If we determine c so that

$$P\left(\chi^2_{f_2} > c \right) = \frac{2\alpha}{K(K-1)}, \tag{8}$$

we obtain approximately

$$P\left(\max_{i<j} S_{ij} > c \right) \leq \alpha$$

by Bonferroni's inequality

$$P\left(\max_{i<j} S_{ij} > c \right) \leq \sum_{i<j} P(S_{ij} > c).$$

We obtain conservative critical value by (8).

4. **Simulation Results.** First, we give critical values for a specified significance level α. Let $\alpha = 0.05$, $p = 2, 3$ and $K = 4, 5$. We set up the balanced sample size $n = 10, 20, 50, 100$ for each multivariate normal. If $p = 2$, $f_2 = 3$. If $p = 3$, $f_2 = 6$.

Table 3 gives conservative critical values of the single step procedure determined by the asymptotic distribution and Bonferroni's inequality. Table 4 gives Type I error by using the critical value in Table 3 calculated by Monte Carlo simulation. The results of Table 4 are obtained by 1,000,000 times of experiments. Table 4 shows that the critical value for $K = 5$ is more conservative compared to that for $K = 4$ and the critical value is less conservative as the sample size n is larger.

Table 3. Conservative critical values of the single step procedure determined by Bonferroni's inequality

K	4	5
$p = 2$	11.739	12.839
$p = 3$	17.273	18.548

Table 4. Type I error using conservative critical values in Table 3 for $n = 10, 20, 50, 100$

n	10		20		50		100	
K	4	5	4	5	4	5	4	5
$p = 2$	0.0384	0.0352	0.0406	0.0383	0.0418	0.0404	0.0430	0.0412
$p = 3$	0.0410	0.0385	0.0416	0.0398	0.0431	0.0413	0.0430	0.0422

Next, we consider the power of the test. Let $K = 4$ and $p = 2$. Letting $\rho > 0$, we consider three cases as follows.
Case 1.

$$\mathbf{\Sigma}_1 = \mathbf{\Sigma}_2 = \mathbf{\Sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbf{\Sigma}_4 = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

Case 2.

$$\mathbf{\Sigma}_1 = \mathbf{\Sigma}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbf{\Sigma}_3 = \mathbf{\Sigma}_4 = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

Case 3.

$$\mathbf{\Sigma}_1 = \mathbf{\Sigma}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbf{\Sigma}_3 = \begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}, \ \mathbf{\Sigma}_4 = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

We focus on the all pairs power defined by Ramsey [5]. In Case 1 the power is the probability that H_{14}, H_{24}, H_{34} are rejected. In Case 2 the power is the probability that H_{13}, H_{14}, H_{23}, H_{24} are rejected. In Case 3 the power is the probability that H_{13}, H_{14}, H_{23}, H_{24}, H_{34} are rejected. Table 5 gives the power of Cases 1 to 3 for $\rho = 0.3, 0.6, 0.9$. They are calculated by Monte Carlo simulation with 100,000 times of experiments. The table shows that the power increases as ρ increases. The power increases as n is larger. The power decreases as the number of hypotheses which should be rejected increases.

Table 5. Power comparison ($p = 2$, $K = 4$, $\alpha = 0.05$)

			Case 1	Case 2	Case 3
$n = 10$	ρ	0.3	0.001	0.000	0.000
		0.6	0.007	0.001	0.000
		0.9	0.246	0.158	0.089
$n = 20$	ρ	0.3	0.002	0.000	0.000
		0.6	0.039	0.012	0.003
		0.9	0.900	0.875	0.859
$n = 50$	ρ	0.3	0.008	0.001	0.000
		0.6	0.407	0.313	0.218
		0.9	1.000	1.000	1.000
$n = 100$	ρ	0.3	0.043	0.016	0.002
		0.6	0.925	0.905	0.894
		0.9	1.000	1.000	1.000

5. **Conclusions.** In this study we discussed the all-pairwise multiple comparison for multivariate normal covariance matrices. Specifically, we constructed the single step multiple comparison procedure using a conservative critical value derived by Bonferroni's inequality. We gave simulation results regarding critical values and power of the test and investigated their characteristics.

Although it is difficult to determine a critical value for pairwise comparison satisfying a specified significance level exactly, we should construct less conservative critical value by various devices. Furthermore, we should construct stepwise multiple comparison procedures like step down procedure and step up procedure which enable us to obtain higher power.

Furthermore, we should discuss the multiple comparison with a control for multivariate normal covariance matrices.

Acknowledgement. The author is deeply grateful to the referee and the editors for their valuable comments and suggestions.

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