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STRUCTURE ANALYSIS OF FUZZY GRAPH AND ITS APPLICATION

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Abstract. *It should be interesting to analyse the sequence of nodes in fuzzy graph. It is an important issue for instruction design process in instruction analysis and for relation structure analysis in sociometry analysis and so on. Nishida et al. proposed an one-sided connectivity method. Generally, applying the method, the ordering structure may be a partially ordered set, and we could not sequence nodes totally in fuzzy graph. In this paper, we would propose a fuzzy core index method that can sequence nodes totally in fuzzy graph. Moreover, we show the effectiveness of this proposed method through the case study.*
Keywords: Fuzzy graph, Sequence analysis, One-sided connectivity method, Fuzzy core index method, Fuzzy core value

1. **Introduction.** Fuzzy information such as human behavior and cognition is represented by fuzzy graph. In general, the information structure is very complicated. Approximation method (Yamashita [5] and others) and fuzzy Shapley value (Matsui and Hirashima [11] and others) could significantly clarify the characteristics of the fuzzy graph. One of the remaining challenges is its sequence analysis among nodes of the fuzzy graph.

This challenge is an important issue for instruction design process in instruction analysis and for relation structure analysis in sociometry analysis and so on. For example, in teaching material structure analysis, it is important to consider the order of teaching. Also, in sociometry analysis for elementary school students, it is important for creating a communication network.

As to this problem, Nishida and Takeda [1] proposed a one-sided connectivity method. This method considers the sequencing problem based on the α cut. In general, this ordering structure may be a partially ordered set and we could not sequence nodes totally in fuzzy graph. Matsui and Hirashima [11] introduced the method applying GA (Genetic Algorithm) and fuzzy reasoning. These methods could be effectively analyzed for the sequencing problem using the not nonlinear model but linear model.

In this paper, we introduce the new index "Fuzzy Core Index" to represent the importance of each node in sequence analysis and summarize the properties of the index. Moreover, we discuss the relationship between AHP (Analytic Hierarchy Process) method and characteristic analysis of fuzzy graph concerning the fuzzy core index. According to this proposed index, we can extend the AHP method.

At first, we present the approximate analysis and fuzzy Shapley value concerning fuzzy graph. Secondly, we introduce the new index "Fuzzy Core Index" for sequence analysis. Finally, we show the effectiveness of the proposed index through case study which could be applied even to nonlinear model.

2. **Approximate Analysis of Fuzzy Graph.**

2.1. **Approximate analysis.**

A fuzzy graph G is defined by

$$G = (V, T),\ V = \{v_i | i = 1, 2, \dots, n\},\ T = \{t_{ij} | 0 < t_{ij} \le 1\} \tag{1}$$

where V is the set of the nodes and T is the $n \times n$ matrix whose t_{ij} represents the fuzziness of the arc from the node v_i to the node v_j. Here, the matrix T can be considered to be equivalent to the fuzzy structure graph T.

Applying approximate analysis [5] to the fuzzy structure graph T, we can obtain approximate N-nary graph T^* which can be able to represent the approximate structure of the graph T.

2.2. **Fuzzy Shapley value.**

Applying similarity analysis to the fuzzy structure graph $T = (t_{ij})$, we can obtain the fuzzy similarity graph $S = (s_{ij})$ whose s_{ij} represents the fuzziness of the similarity between the node v_i and the node v_j. Here, the importance of each node from the similarity point of view could be measured by the fuzzy Shapley value σ_i as defined [11] by

$$\sigma_i = \frac{\sum\limits_j s_{ij}}{\sum\limits_i \sum\limits_j s_{ij}} \tag{2}$$

Furthermore, by the cluster analysis of the fuzzy similarity graph S, we can obtain the partition tree P. If we summarize the approximate N-nary graph T^*, the partition tree P and the importance of each node σ_i, then we can obtain the approximate structure graph ϕ which significantly clarifies the global characteristics of the graph T.

3. **Sequence Analysis of Fuzzy Graph.**

Another challenge is its sequence analysis among nodes of the fuzzy graph. According to the graph ϕ, we can roughly order the nodes by one-sided connectivity method [1]; however, it is not precise. In other words, this ordering structure may be a partially ordered set. For this reason, we need a more detailing sequence analysis which the ordering structure would be totally ordered set.

3.1. **One-sided connectivity method.**

An α cut matrix T^α of the matrix T can be defined by

$$T^\alpha = \left(t_{ij}^\alpha\right), \quad t_{ij}^\alpha = \begin{cases} 1, & t_{ij} \ge \alpha \\ 0, & t_{ij} < \alpha \end{cases} \tag{3}$$

Here, the walk from the node v_i to v_j is defined as $v_i \mapsto v_j$ and one-sided connectivity graph could be defined by:

$$\forall v_i, v_j \in V, \quad \exists v_i \mapsto v_j \vee v_j \mapsto v_i \tag{4}$$

One-sided connectivity level $\mu(T)$ of the one-sided connectivity graph can be defined by

$$\mu(T) = \min_{i,j} \left\{ \max \left\{ \widehat{t_{ij}}, \widehat{t_{ji}} \right\} \right\} \tag{5}$$

Here, the transitive closure of the matrix T is $\widehat{T} = \left(\widehat{t_{ij}}\right)$. We can roughly order the nodes by the $\mu(T)$ cut matrix $T^{\mu(T)}$.

3.2. **Fuzzy core index.** We introduce the new index "Fuzzy Core Index" c_{ij}, which can represent the important level of the node v_i related to the node v_j

$$c_{ij} = \begin{cases} \dfrac{t_{ji}}{t_{ij}}, & \dfrac{t_{ji}}{t_{ij}} < 9 \\ 9, & \dfrac{t_{ji}}{t_{ij}} \geq 9 \end{cases}, \quad c_{ji} = \frac{1}{c_{ij}} \text{ if } t_{ij} \leq t_{ji} \tag{6}$$

$$c_{ji} = \begin{cases} \dfrac{t_{ij}}{t_{ji}}, & \dfrac{t_{ij}}{t_{ji}} < 9 \\ 9, & \dfrac{t_{ij}}{t_{ji}} \geq 9 \end{cases}, \quad c_{ij} = \frac{1}{c_{ji}} \text{ if } t_{ij} > t_{ji} \tag{7}$$

Here, if the values of t_{ij} and t_{ji} are $0.1, 0.2, \ldots, 0.9$, Figure 1 shows the values of the fuzzy core indexes.

t_{ij} \ t_{ji}	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	1.00	2.00	3.00	4.00	5.00	6.00	7.00	8.00	9.00
0.2	0.50	1.00	1.50	2.00	2.50	3.00	3.50	4.00	4.50
0.3	0.33	0.67	1.00	1.33	1.67	2.00	2.33	2.67	3.00
0.4	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25
0.5	0.20	0.40	0.60	0.80	1.00	1.20	1.40	1.60	1.80
0.6	0.17	0.33	0.50	0.67	0.83	1.00	1.17	1.33	1.50
0.7	0.14	0.29	0.43	0.57	0.71	0.86	1.00	1.14	1.29
0.8	0.13	0.25	0.38	0.50	0.63	0.75	0.88	1.00	1.13
0.9	0.11	0.22	0.33	0.44	0.56	0.67	0.78	0.89	1.00

FIGURE 1. Fuzzy core index c_{ij}

By the way, in the analytic hierarchy process proposed by Saaty [2], a_{ij} which represents the important level of the node v_i related to the node v_j by pairwise comparison is performed with reference to the following table.

From Figure 1 and Table 1, we can support that the fuzzy core index c_{ij} is an index that includes a_{ij} based on the AHP method. This is because c_{ij} can be obtained from data, but a_{ij} is a subjective indicator.

TABLE 1. Paired comparison value a_{ij}

a_{ij}	Definition
1	v_i and v_j are equally important
3	v_i is moderate more important than v_j
5	v_i is more important than v_j
7	v_i is considerably more important than v_j
9	v_i is absolutely more important than v_j
2, 4, 6, 8	We use complementarily
Rationals	a_{ji} represents the reciprocal of the a_{ij}

3.3. **Fuzzy core index method.** According to the $C = (c_{ij})$, we can obtain fuzzy core value $c(v_i)$ which shows the importance of each node v_i from the connectivity point of view by eigenvector method as follows.

$$Cv = \lambda_{\max} v,\ v = (c(v_i)),\ \lambda_{\max} \geq n \tag{8}$$

Here, $\lambda_{\max}$ represents the maximum eigenvalue of the fuzzy core matrix C and v represents the eigenvector corresponding to $\lambda_{\max}$. We can totally order the nodes v_i based on the v eigenvector corresponding to $\lambda_{\max}$.

By the way, we would describe the reasons for focusing on the maximum eigenvalue and eigenvector. Let the original important level of the node v_i be the w_i. The important level c^*_{ij} of the node v_i related to the node v_j could be defined by

$$c^*_{ij} = \frac{w_i}{w_j} \tag{9}$$

When $v^* = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$ is multiplied from the right side of the pairwise comparison matrix $C^* = \left(c^*_{ij}\right)$, nv^* can be obtained as follows.

$$C^*v^* = \begin{pmatrix} \frac{w_1}{w_1} & \frac{w_1}{w_2} & \cdots & \frac{w_1}{w_n} \\ \frac{w_2}{w_1} & \frac{w_2}{w_2} & \cdots & \frac{w_2}{w_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{w_n}{w_1} & \frac{w_n}{w_2} & \cdots & \frac{w_n}{w_n} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = n \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = nv^* \tag{10}$$

Here, estimating unknown c^*_{ij} with c_{ij}, we can obtain (6), (7).

Regarding the $\lambda_{\max}$,

$$\lambda_{\max} \geq n \tag{11}$$

holds. Since λ can be obtained as

$$\begin{gathered} \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}, \\ \lambda = \sum_{j=1}^{n} c_{1j}\frac{v_j}{v_1} = \sum_{j=1}^{n} c_{2j}\frac{v_j}{v_2} = \cdots = \sum_{j=1}^{n} c_{nj}\frac{v_j}{v_n} \end{gathered} \tag{12}$$

we have

$$n\lambda = n + \left(c_{12}\frac{v_2}{v_1} + c_{21}\frac{v_1}{v_2}\right) + \left(c_{13}\frac{v_3}{v_1} + c_{31}\frac{v_1}{v_3}\right) + \cdots + \left(c_{(n-1)n}\frac{v_n}{v_{n-1}} + c_{n(n-1)}\frac{v_{n-1}}{v_n}\right) \tag{13}$$

As a result, $n\lambda \geq n + 2 \times \frac{n(n-1)}{2}$ and $\lambda_{\max} \geq n$.

4. **Case Study.** We would present an illustrative case study. From the fuzzy structure matrix $T = (t_{ij})$ (Figure 2), we have the fuzzy structure graph T (Figure 3).

Applying approximate analysis and similarity analysis to the fuzzy structure graph T, we can obtain the approximate ternary graph T^* (Figure 4), the fuzzy similarity matrix $S = (s_{ij})$ (Figure 5) and the fuzzy similarity graph S (Figure 6). Here, the importance of each node from the similarity point of view could be measured by the fuzzy Shapley value σ_i. This time, each fuzzy Shapley value could be shown on the right side of Figure

1	1.00	0.56	0.96	0.73	0.84
0.68	1	0.37	0.95	0.49	0.62
1.00	1.00	1	1.00	0.94	0.96
0.63	0.93	0.35	1	0.43	0.59
0.98	1.00	0.72	0.90	1	0.74
0.94	1.00	0.60	1.00	0.62	1

Figure 2. Fuzzy structure matrix T

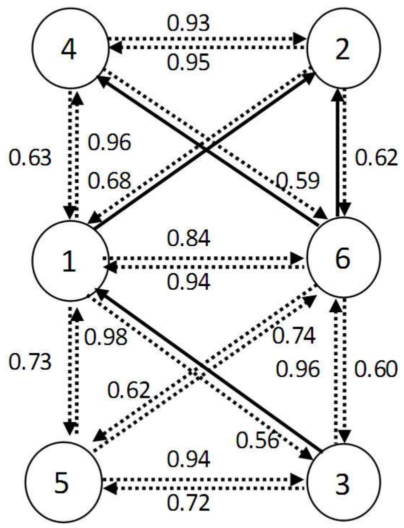

Figure 3. Fuzzy structure graph T

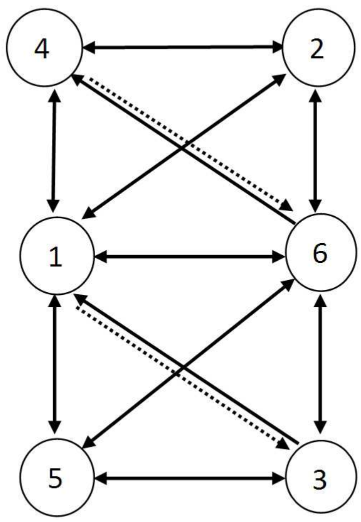

Figure 4. Approximate ternary graph T^*

5. This time, the highest value is $\sigma_1 = 0.179$. So, we can say v_1 is strengthen point of similarity.

Furthermore, by executing the cluster analysis of the fuzzy similarity graph S, we can obtain the partition tree P (Figure 7). If we summarize the approximate ternary graph T^*, the partition tree P and the fuzzy Shapley value, then we can obtain the approximate

						FSV
1	0.81	0.72	0.76	0.84	0.89	0.179
0.81	1	0.54	0.94	0.66	0.77	0.169
0.72	0.54	1	0.52	0.82	0.74	0.155
0.76	0.94	0.52	1	0.58	0.73	0.162
0.84	0.66	0.82	0.58	1	0.68	0.164
0.89	0.77	0.74	0.73	0.68	1	0.172

Figure 5. Fuzzy similarity matrix S

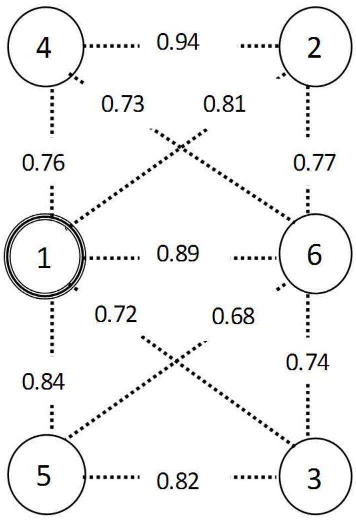

Figure 6. Fuzzy similarity graph S

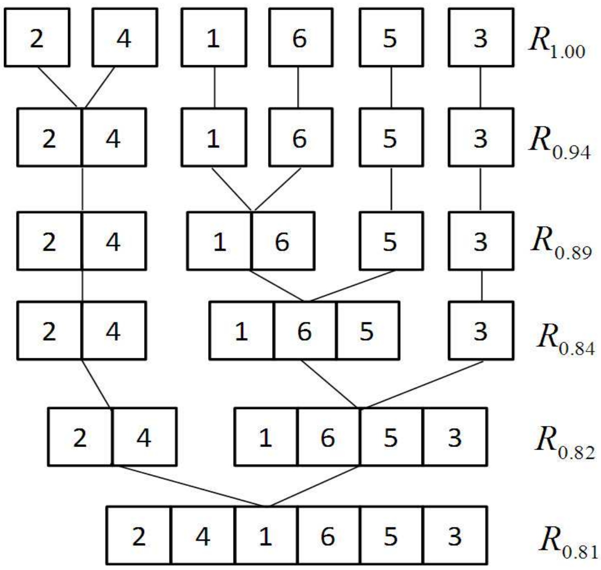

Figure 7. Partition tree P

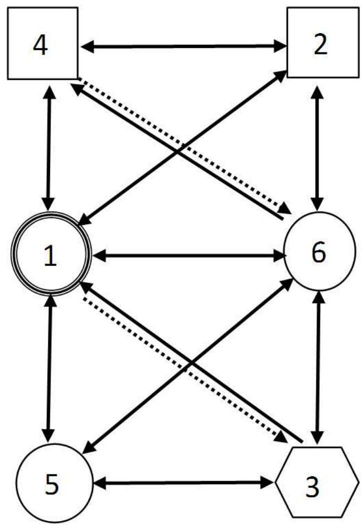

FIGURE 8. Approximate structure graph ϕ

						FCV
1	0.68	1.79	0.66	1.34	1.12	0.161
1.47	1	2.70	0.98	2.04	1.61	0.239
0.56	0.37	1	0.35	0.77	0.63	0.089
1.52	1.02	2.86	1	2.09	1.69	0.247
0.74	0.49	1.31	0.48	1	0.84	0.119
0.89	0.62	1.60	0.59	1.19	1	0.144

FIGURE 9. Fuzzy core matrix C

structure graph ϕ (Figure 8) which can be able to significantly clarify the approximate structure of the graph T.

One of the remaining challenges is a sequence analysis among nodes of the fuzzy graph. For example, according to the graph ϕ, we can only order the nodes broadly since there exists the crisp spanning walk $v_3 \rightarrow \{v_1, v_6, v_5\} \rightarrow \{v_2, v_4\}$. For this reason, we need a more precise sequence analysis.

As to this problem, we apply the new index "Fuzzy Core Index" c_{ij} described above. For example, from the fuzzy connectivity matrix T (Figure 2), we can obtain the fuzzy core matrix $C = (c_{ij})$ (Figure 9). According to C, we can obtain the fuzzy core value $c(v_i)$ which shows the importance of each node v_i from the connectivity point of view by eigenvector method.

This time, each fuzzy core value could be shown on the right side of Figure 9. Here, the value $c(v_i)$ indicates the descending order v_4, v_2, v_1, v_6, v_5, v_3.

As a result, we can totally order the nodes since there exists the only crisp spanning walk $v_3 \rightarrow v_5 \rightarrow v_6 \rightarrow v_1 \rightarrow v_2 \rightarrow v_4$ (Figure 10). In this way, we could enable more precise sequence analysis.

Furthermore, we can obtain the c_{ij} and c_{ji} from not pairwise comparisons but data. In this sense, fuzzy core index could be the extension of the analytic hierarchy process.

5. **Conclusion.** In this paper we proposed a fuzzy core index method that can sequence nodes totally in fuzzy graph. Moreover, we show the effectiveness of this proposed method

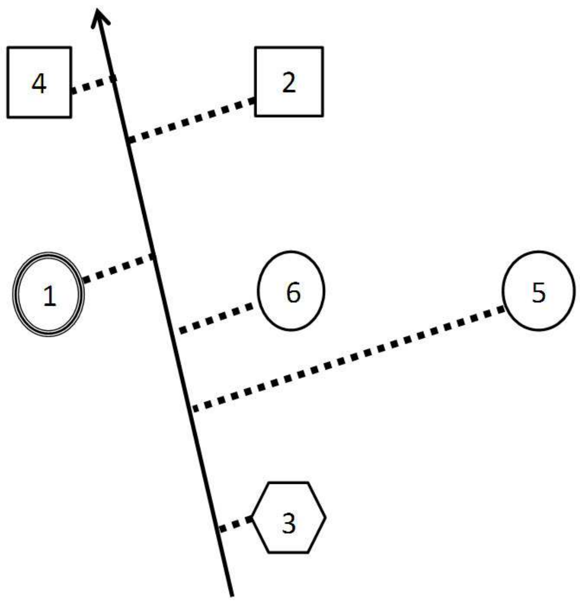

Figure 10. Image of projection to eigenvector

through the case study. These should be useful for nonlinear problems. In the future, various case studies for nonlinear model such as sociometry analysis will be examined.

REFERENCES

[1] T. Nishida and E. Takeda, *Fuzzy Set and Its Application*, Morikita-Shuppan, 1978 (in Japanese).
[2] T. L. Saaty, *The Analytic Hierarchy Process*, RWS Publications, Pittsburgh, 1996.
[3] T. L. Saaty, *Decision Making for Leaders: The Analytical Hierarchy Process for Decisions in a Complex World*, RWS Publications, Pittsburgh, 1999.
[4] T. L. Saaty, How to make a decision: The analytic hierarchy process, *Interfaces*, vol.24, no.6, pp.19-43, 1994.
[5] H. Yamashita, Approximation algorithm of fuzzy graph and its application, *Int'l Congress of Fuzzy System Association III*, 1989.
[6] H. Yamashita and E. Tsuda, Sociogram analysis applying fuzzy graph, *North American Fuzzy Information Processing Sociometry IX*, 1990.
[7] T. L. Saaty, *Theory and Applications of the Analytic Network Process*, RWS Publications, Pittsburgh, PA, 2005.
[8] K. Shinkai, Fuzzy cluster analysis and its evaluation method, *Int'l Journal of Biomedical Soft Computing and Human Sciences*, vol.13, no.2, 2008.
[9] K. Shinkai and K. Motegi, Fuzzy hierarchical cluster analysis on international stock markets, *Waseda Journal of Mathematics Education*, vol.28, pp.38-46, 2010 (in Japanese).
[10] H. Yamashita and T. Takizawa, *Fuzzy Theory and Its Application*, Kyoritsu Shuppan, Tokyo, 2010 (in Japanese).
[11] T. Matsui and T. Hirashima, Sequencing of learning materials and exercise problems, *Journal of Japanese Society for Artificial Intelligence*, vol.25, no.2, pp.259-267, 2010 (in Japanese).
[12] E. Tsuda and H. Yamashita, *Shapley Value Analysis of Fuzzy Graph and Its Application I-III*, Mathematics Society of Japan, Applied Mathematics Division, 1999-2007 (in Japanese).
[13] E. Tsuda, H. Yamashita and K. Nagashima, Opinion survey applying fuzzy theory, *Int'l Journal of Biomedical Soft Computing and Human Sciences*, vol.16, no.1, pp.57-62, 2011.
[14] K. Shinkai, E. Tsuda and H. Yamashita, Characteristic analysis of fuzzy graph and its application, *Proc. of the Annual Conference of Biomedical Fuzzy Systems Association*, Kumamoto, 2015.
[15] K. Shinkai, E. Tsuda and H. Yamashita, Structure analysis of fuzzy graph and its application – Fuzzy extension of AHP method –, *Proc. of the Annual Conference of Soft Science Workshop*, Yamagata, 2016 (in Japanese).