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THE CONSTRUCTION OF ORTHOGONAL SYMMETRIC MULTI-WAVELETS WITH DILATION FACTOR 3

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Abstract. *In this paper, we present a method to construct compactly supported symmetric-antisymmetric orthogonal multi-wavelets with dilation 3. By studying the relationship between matrix sequences of multi-scaling function and matrix sequences of multi-wavelets, we first show that matrix sequences of multi-wavelets with four coefficients can be constructed from the associated matrix sequences of multi-scaling function. Next, the explicit formulations of orthogonal multi-wavelets functions with dilation 3 are obtained. Finally, we give an example to illustrate our general constructive scheme.*
Keywords: Multi-scaling, Multi-wavelets, Symmetry, Dilation factor

1. **Introduction.** As we know, the compactly supported orthonormal real-valued scalar wavelet with dilation 2 cannot have symmetry except Haar wavelet. However, orthogonality and symmetry are two important properties in practical application. Symmetry of an orthonormal wavelet can be obtained by considering multi-wavelets or complex-valved wavelets. The multi-wavelets can simultaneously possess orthogonality, shorter supported, symmetry, and higher approximation order. So, in recent years, the multi-wavelets have received more attention both in theory and in application.

Geronimo et al. [1] presented the earliest multi-wavelets by using fractal interpolation. Based on the multi-resolution analysis, by showing the relationship between length-$2N$ and length-$(2N+1)$ multi-wavelets, the symmetric-antisymmetric orthonormal multi-wavelets were constructed from their associated multi-scaling functions in [2, 3, 6]. Furthermore, several explicit algorithms for constructing multi-wavelets with high approximation order and symmetry were discussed in [4, 5]. A finitely supported real-valued 2-orthogonal filter cannot be symmetric about a point. It has motivated many researchers to find alternatives for achieving both symmetry and orthogonality. The symmetric orthogonal multi-wavelets with dilation factor $a > 2$ were constructed in [7-10]. Moreover, due to wavelets transform time-frequency localisation characteristic, they have been widely used in signal analysis, image processing and many other areas [14-20]. The multi-wavelets are more extensive than scalar wavelets in application of wavelet analysis. Therefore, the construction of compactly supported symmetric-antisymmetric orthogonal multi-wavelets is very important. In this paper, based on the results of [6-8, 21], we shall further study symmetric orthogonal multi-wavelets with dilation factor $a > 2$. By studying the relationship between matrix sequences of multi-scaling function and matrix sequences of multi-wavelets, we will consider the construction of univariate symmetric orthogonal multi-wavelets with dilation factor 3.

The organization of this paper is as follows. The background introduction is given in Section 1. The summary of multi-wavelets is introduced in Section 2. In Section 3, we will present a set of explicit formulas to construct the orthogonal symmetric multi-wavelets

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with dilation factor 3. Section 4 providers an example to validate our results. Section 5 concludes this paper.

2. **Problem Statement and Preliminaries.** In this section, we shall recall some basic definitions and lemmas. To facilitate our discussion on symmetry and antisymmetry, we only consider multi-scaling functions with finite and real valued matrix sequences. For the compactly supported orthogonal complex wavelets with dilation a ($a \geq 2$), we can refer to [9, 10].

2.1. **Several definitions and theorems.** Let $\Phi(x) = (\phi_1(x), \phi_2(x), \ldots, \phi_r(x))^{\mathrm{T}}$ be a compactly support orthogonal scaling vector-valued function with multiplicity r and dilation factor 3, $\phi_i(x) \in L^2(\mathbb{R})$, $i = 1, \ldots, r$ and satisfy the following matrix scaling equation

$$\Phi(x) = 3 \sum_{k \in \mathbb{Z}} P(k)\Phi(3x - k), \tag{1}$$

where $\{P(k)\}$ is an $r \times r$ real-value matric sequence. $\Psi_l(x) = \left(\psi_1^l(x), \psi_2^l(x), \ldots, \psi_r^l(x)\right)^{\mathrm{T}}$, $l = 1, 2$ is the corresponding orthogonal multi-wavelets satisfying

$$\Psi_l(x) = 3 \sum_{k \in \mathbb{Z}} Q_l(k)\Phi(3x - k), \quad l = 1, 2, \tag{2}$$

where $\{Q_l(k)\}$ is an $r \times r$ real-value matrix sequence. Applying Fourier transform to (1) and (2), respectively, we obtain

$$\hat{\Phi}(\omega) = P(\omega/3)\hat{\Phi}(\omega/3), \quad \hat{\Psi}_l(\omega) = Q_l(\omega/3)\hat{\Phi}(\omega/3), \quad l = 1, 2, \tag{3}$$

where $P(\omega) = \sum_{k \in \mathbb{Z}} P(k)e^{-ik\omega}$, $Q_l(\omega) = \sum_{k \in \mathbb{Z}} Q_l(k)e^{-ik\omega}$. P, Q_l are called matrix low-pass filter and high-pass filter in the application of signal precessing. The set $\{P, Q_l\}$ is called wavelet filter banks. We know that multi-scaling function and multi-wavelets are closely related to their corresponding filter banks. So, in the construction of multi-wavelets, we only consider the construction of multi-filter banks.

For column vector functions A and B with elements in $L^2(\mathbb{R})$, we define

$$\langle A, B \rangle = \int_{\mathbb{R}} A(x)B(x)^{\mathrm{T}} dx.$$

We call a vector of functions $\Phi(x) = (\phi_1(x), \phi_2(x), \ldots, \phi_r(x))^{\mathrm{T}}$ an orthonormal scaling vector if Φ satisfies (1), $\phi_j \in L_2(\mathbb{R})$, and

$$\int \phi_j(x - k)\overline{\phi_i(x)} dx = \delta(j - i)\delta(k), \quad 1 \leq j,\ i \leq r,\ k \in \mathbb{Z},$$

where δ is the Kronecker delta.

For compactly supported orthogonal multi-scaling function $\Phi(x)$ and its corresponding orthogonal multi-wavelet $\Psi_l(x)$, we have the following orthogonal conditions.

$$\begin{gathered}
\langle \Phi(\cdot), \Psi_l(\cdot - n) \rangle = \langle \Psi_l(\cdot), \Phi(\cdot - n) \rangle = O_{r \times r}, \\
\langle \Psi_l(\cdot), \Psi_l(\cdot - n) \rangle = \delta_{0,n} I_r, \quad l = 1, 2, \\
\langle \Phi(\cdot), \Phi(\cdot - n) \rangle = \delta_{0,n} I_r,
\end{gathered}$$

where $O_{r \times r}$ and I_r denote the zero matrix and unit matrix, respectively. This set of equations is equivalent to the equation

$$\sum_{k \in \mathcal{Z}} P(k)P(k + 3i)^{\mathrm{T}} = 3\delta_{i,0} I_r, \tag{4}$$

$$\sum_{k \in \mathcal{Z}} Q_l(k)Q_l(k + 3i)^{\mathrm{T}} = 3\delta_{i,0} I_r, \tag{5}$$

$$\sum_{k\in\mathcal{Z}} P(k)Q_l(k+3i)^{\mathrm{T}} = O_r. \tag{6}$$

Also, in terms of multi-filter banks, the orthogonalities of Φ and Ψ_l are represented

$$\sum_{k=0}^{2} P\left(\frac{\omega+2k\pi}{3}\right) P^*\left(\frac{\omega+2k\pi}{3}\right) = I_r, \tag{7}$$

$$\sum_{k=0}^{2} P\left(\frac{\omega+2k\pi}{3}\right) Q_l^*\left(\frac{\omega+2k\pi}{3}\right) = O_r, \tag{8}$$

$$\sum_{k=0}^{2} Q_l\left(\frac{\omega+2k\pi}{3}\right) Q_l^*\left(\frac{\omega+2k\pi}{3}\right) = \delta_{l-n} I_r,\ n<3, \tag{9}$$

where the superscript $*$ denotes the complex conjugate transpose. A sequence satisfying (4) or (7) is called a matrix conjugate quadrature filter (CQF). For more details about orthogonal multi-scaling function, we can refer to [12].

In this paper, we will focus on a class of symmetric or antisymmetric orthogonal multi-wavelet systems with multiplicity r and four coefficients. The symmetry is one of the most important properties of multi-wavelets. Before proceeding further, we will introduce some symmetric and antisymmetric properties on multi-scaling function and multi-wavelets.

Let us consider the situation when all the generating functions are symmetric or anti-symmetric about the same point, there exists some odd integer n such that, for all $j = 1,\dots,r$,

$$\phi_j(x) = \pm\phi_j(n-x), \quad \psi_j(x) = \pm\psi_j(n-x).$$

In vector form, we can write this as

$$\Phi(x) = \Sigma\Phi(n-x), \quad \Psi(x) = \Lambda\Psi(n-x),$$

where Σ and Λ are some diagonal matrices with the diagonal entries equal to ± 1. Next, we will present the following lemmas of multi-scaling functions and multi-wavelets with symmetry.

Lemma 2.1. *Let $\Phi(x) = (\phi_1(x), \phi_2(x), \dots, \phi_r(x))^{\mathrm{T}}$ be a multi-scaling function with dilation 3, and satisfy (1), and P be a finite impulse response (FIR) matrix filter. If P satisfies*

$$SD_a(3\omega)P(-\omega)SD_a(-\omega) = P(\omega), \tag{10}$$

for some $a = (a_1,\dots,a_r) \in \mathbb{R}^r$, then ϕ_j is symmetric or antisymmetric about $\frac{a_j}{4}$, i.e.,

$$\phi_j\left(\frac{a_j}{2} - x\right) = s_j\phi_j(x), \quad 1 \le j \le r, \tag{11}$$

where S is the diagonal matrix having coefficients s_j on the diagonal, $s_j \in \{-1, 1\}$, and

$$D_a(\omega) = diag\left(e^{-ia_1\omega/2}, e^{-ia_2\omega/2}, \dots, e^{-ia_r\omega/2}\right). \tag{12}$$

Lemma 2.2. *Assumed Φ is 3-band compactly supported multi-scaling functions with $\Phi(0) \neq 0$, P is an FIR matrix filter satisfying (10) for some $a = (a_1,\dots,r) \in \mathbb{R}^r$. Let $\Psi = (\psi_1,\dots,\psi_r)^T$ be a vector-valued function defined*

$$\Psi(x) = 3\sum_{k\in\mathbb{Z}} g(k)\Phi(3x-k),$$

for some FIR matrix filter G. If G satisfies

$$TD_b(3\omega)G(-\omega)TD_a(-\omega) = G(\omega), \tag{13}$$

for some $b = (b_1,\dots,b_r) \in \mathbb{R}^r$, then

$$\psi_j\left(\frac{b_j}{2} - x\right) = t_j\psi_j(x), \quad 1 \le j \le r, \tag{14}$$

where T is the diagonal matrix having coefficients t_j on the diagonal, $t_j \in \{-1, 1\}$, and

$$D_b(\omega) = diag\left(e^{-ib_1\omega/2}, e^{-ib_2\omega/2}, \ldots, e^{-ib_r\omega/2}\right). \tag{15}$$

Lemma 2.3. *Let $\Phi(x) = (\phi_1(x), \phi_2(x), \ldots, \phi_r(x))^{\mathrm{T}}$ be a compactly supported orthogonal symmetric or anti-symmetric multi-scaling function. $\Psi(x)$ is symmetric or anti-symmetric multi-wavelet associated with $\Phi(x)$. If $H(z) = \sum_{k=0}^{3(\alpha+1)-1} P(k)z^{-k}$, $Q_l(z) = \sum_{k=0}^{3(\alpha+1)-1} Q_l(k)z^{-k}$, $l = 1, 2$, for some $\alpha \in \mathbb{Z}_+/\{0\}$, then*

$$P(z) = z^{-(3(\alpha+1)-1)} diag\left(S_0 z^3, s_0\right) P\left(z^{-1}\right) diag(S_0, s_0 z), \tag{16}$$

$$Q_l(z) = z^{-(3(\alpha+1)-1)} diag\left(S_1^l z^3, S_2^l\right) Q_l\left(z^{-1}\right) diag(S_0, s_0 z), \quad l = 1, 2, \tag{17}$$

where $s_0 = \pm 1$, S_0, S_1^l, S_2^l are diagonal matrices with diagonal entries 1 *or* -1.

For the proof of these lemmas and detailed theory, see [13].

3. **Main Results.** In this section, we will give a set of explicit formulas for the construction of orthogonal symmetric multi-wavelet functions with dilation factor 3. Analogous to the method in [2], we propose the explicit formulations for matrix sequence $\{Q_l(k)\}$ directly in terms of matrix sequence $\{P(k)\}$. The corresponding orthogonal symmetric multi-wavelet with dilation 3 can be obtained. In this paper, we only discuss the construction of multi-scaling function with 4-coefficient, i.e., the multi-scaling function $\Phi(x)$ satisfies the following scaling relation

$$\Phi(x) = P_0\Phi(3x) + P_1\Phi(3x-1) + P_2\Phi(3x-2) + P_3\Phi(3x-3). \tag{18}$$

For the construction of the orthogonal multi-wavelets with 4-coefficient and multiplicity r, we have the following theorem.

Theorem 3.1. *Suppose that $\Phi(x)$ is an orthogonal compactly supported multi-scaling function, and satisfy (18), defining a matrix H, which satisfies*

$$H^2 = \left[3I_r - \left(P_1P_1^{\mathrm{T}} + P_2P_2^{\mathrm{T}}\right)\right]^{-1}\left(P_1P_1^{\mathrm{T}} + P_2P_2^{\mathrm{T}}\right). \tag{19}$$

Let

$$q_0^s = H_sP_0, \quad q_1^s = -H_s^{-1}P_1, \quad q_2^s = -H_s^{-1}P_2, \quad q_3^s = H_sP_3, \quad (s = 1, 2)$$

where H_s $(s = 1, 2)$ are 2 essentially different symmetric matrices of H^2. Take $\{Q_l(k)\} = \{q_k^s\}$ $(s = 1, 2)$, defining

$$\Psi_l(x) = 3\sum_{k=0}^{3} Q_l(k)\Phi(3x-k),\ l = 1, 2. \quad \Psi(x) = \left[\left(\Psi_1(x)^{\mathrm{T}}, \Psi_2(x)^{\mathrm{T}}\right)\right]^{\mathrm{T}}, \tag{20}$$

then $\Psi(x)$ is compactly supported orthogonal multi-wavelet with dilation 3 associated with $\Phi(x)$, and satisfies the following scaling matrix equation

$$\Psi(x) = \sum_{k=0}^{3}\left[(Q_k^1)^{\mathrm{T}}, (Q_k^2)^{\mathrm{T}}\right]^{\mathrm{T}}\Phi(3x-k). \tag{21}$$

Proof: In order to prove this theorem, we only need to verify the following four equations

$$P_0(q_3^s)^{\mathrm{T}} = O, \tag{22}$$
$$P_0(q_0^s)^{\mathrm{T}} + P_1(q_1^s)^{\mathrm{T}} + \cdots + P_3(q_3^s)^{\mathrm{T}} = O, \tag{23}$$
$$q_0^l(q_3^s)^{\mathrm{T}} = O, \quad (l, s = 1, 2) \tag{24}$$
$$q_0^s(q_0^s)^{\mathrm{T}} + q_1^s(q_1^s)^{\mathrm{T}} + \cdots + q_3^s(q_3^s)^{\mathrm{T}} = 3I_r. \tag{25}$$

According to [6], we know that H^2 is a symmetric positive definite matrix, and Equations (22) and (24) can be proved directly by using (5) and (6). We only prove (23) and (25). In order to verify (23), we have that

$$\begin{aligned}\sum_{j=0}^{3} P_j\left(q_j^s\right)^{\mathrm{T}} &= P_0P_0^{\mathrm{T}}(H_s)^{\mathrm{T}} + P_3P_3^{\mathrm{T}}(H_s)^{\mathrm{T}} - P_1P_1^{\mathrm{T}}\left(H_s^{-1}\right)^{\mathrm{T}} - P_2P_2^{\mathrm{T}}\left(H_s^{-1}\right)^{\mathrm{T}} \\ &= \left[\left(P_1P_1^{\mathrm{T}} + P_2P_2^{\mathrm{T}}\right) - \left(P_1P_1^{\mathrm{T}} + P_2P_2^{\mathrm{T}}\right)\right]\left(H_s^{-1}\right)^{\mathrm{T}} = 0.\end{aligned}$$

Next, to verify (25), we have that

$$\begin{aligned}\sum_{j=0}^{3} q_j^s\left(q_j^s\right)^{\mathrm{T}} &= H_sP_0P_0^{\mathrm{T}}H_s^{\mathrm{T}} + H_s^{-1}P_1P_1^{\mathrm{T}}\left(H_s^{-1}\right)^{\mathrm{T}} + H_s^{-1}P_2P_2^{\mathrm{T}}\left(H_s^{-1}\right)^{\mathrm{T}} + H_sP_3P_3^{\mathrm{T}}H_s^{\mathrm{T}} \\ &= H_s^{-1}\left\{H_s^2\left[3I_r - \left(P_1P_1^{\mathrm{T}} + P_2P_2^{\mathrm{T}}\right)\right](H_s^{\mathrm{T}})^2 + \left(P_1P_1^{\mathrm{T}} + P_2P_2^{\mathrm{T}}\right)\right\}(H_s^{-1})^{\mathrm{T}}.\end{aligned}$$

Since H^2 is a symmetric positive definite matrix, we have

$$\begin{aligned}\sum_{j=0}^{3} q_j^s\left(q_j^s\right)^{\mathrm{T}} &= H_s^{-1}\left[H_s^2\left(P_1P_1^{\mathrm{T}} + P_2P_2^{\mathrm{T}}\right) + \left(P_1P_1^{\mathrm{T}} + P_2P_2^{\mathrm{T}}\right)\right]\left(H_s^{-1}\right)^{\mathrm{T}} \\ &= H_s\left[\left(P_1P_1^{\mathrm{T}} + P_2P_2^{\mathrm{T}}\right) + H_s^{-2}\left(P_1P_1^{\mathrm{T}} + P_2P_2^{\mathrm{T}}\right)\right]\left(H_s^{-1}\right)^{\mathrm{T}} \\ &= H_s\left\{\left(P_1P_1^{\mathrm{T}} + P_2P_2^{\mathrm{T}}\right) + 3I_r - \left(P_1P_1^{\mathrm{T}} + P_2P_2^{\mathrm{T}}\right)\right\}\left(H_s^{-1}\right)^{\mathrm{T}} = 3I_r.\end{aligned}$$

This completes the proof of Theorem 3.1. □

Next, we will discuss the symmetry of orthogonal multi-wavelet defined in Theorem 3.1.

Theorem 3.2. *Let the orthogonal symmetric multi-scaling functions $\Phi(x)$ satisfy symmetric condition (10). If H_s is a diagonal matrix, which is defined as in (19), then we have that $\Psi_l(x)$ defined as in (20) satisfies $\Psi_l(x) = (-1)^{l-1}\Psi_l(3-x)$.*

Proof: Since $\Phi(x)$ satisfies symmetric condition (10), by Lemma 2.3, we have

$$P(z) = z^{-(3(\alpha+1)-1)}diag\left(S_0z^3, s_0\right)P\left(z^{-1}\right)diag(S_0, s_0z).$$

In addition, since H_s is a diagonal matrix, and

$$q_0^s = H_sP_0, \quad q_1^s = -H_s^{-1}P_1, \quad q_2^s = -H_s^{-1}P_2, \quad q_3^s = H_sP_3, \quad (s = 1, 2)$$

So, we can see that $\Psi_s(x)$ is the symmetric orthogonal multi-wavelet. □

According to Theorem 3.1, we can see that symmetric-antisymmetric orthonormal multi-scaling functions lead to symmetric-antisymmetric orthonormal multi-wavelets. This result can also refer to [3]. From the above discussion, for a given orthogonal multi-scaling function $\Phi(x)$ with four coefficients, Theorem 3.1 and Theorem 3.2 provide a set of explicit formulas for the construction of orthogonal multi-wavelet, as long as H^2 is a symmetric positive definite matrix. In the next section, the examples will be given to demonstrate that our proposed approach is efficient.

4. **Numerical Example.** In this section, we give an example for the construction of orthogonal multi-wavelet with dilation 3 from their corresponding multi-scaling function matrix sequences to illustrate the results in this paper.

Example. The multi-scaling function matrix CQF sequences P_0, P_1, P_2, P_3 were given in the following, which satisfy (18) with multiplicity 2, as follows:

$$P_0=\begin{pmatrix}\dfrac{10-3\sqrt{10}}{40} & \dfrac{5\sqrt{6}-2\sqrt{15}}{40}\\ \dfrac{5\sqrt{6}-3\sqrt{15}}{40} & \dfrac{5-3\sqrt{10}}{40}\end{pmatrix},\quad P_1=\begin{pmatrix}\dfrac{30+3\sqrt{10}}{40} & -\dfrac{5\sqrt{6}-2\sqrt{15}}{40}\\ \dfrac{5\sqrt{6}+7\sqrt{15}}{40} & \dfrac{15-3\sqrt{10}}{40}\end{pmatrix},$$

$$P_2=\begin{pmatrix}\dfrac{10-3\sqrt{10}}{40} & -\dfrac{5\sqrt{6}-2\sqrt{15}}{40}\\ -\dfrac{5\sqrt{6}-3\sqrt{15}}{40} & \dfrac{5-3\sqrt{10}}{40}\end{pmatrix},\quad P_3=\begin{pmatrix}\dfrac{30+3\sqrt{10}}{40} & \dfrac{5\sqrt{6}-2\sqrt{15}}{40}\\ -\dfrac{5\sqrt{6}+7\sqrt{15}}{40} & \dfrac{15-3\sqrt{10}}{40}\end{pmatrix}.$$

The matrix symbol $P(z)$ of matrix sequence $\{P_k\}$ $(k=0,1,2,3)$ can be given by the following

$$P(z)=\frac{1}{3}\begin{bmatrix}P_{0,0} & P_{0,1}\\ P_{1,0} & P_{1,1}\end{bmatrix},$$

where

$$\begin{aligned}
P_{0,0} &= \frac{10-3\sqrt{10}}{40}+\frac{30+3\sqrt{10}}{40}z+\frac{10-3\sqrt{10}}{40}z^2+\frac{30+3\sqrt{10}}{40}z^3,\\
P_{0,1} &= \frac{5\sqrt{6}-2\sqrt{15}}{40}-\frac{5\sqrt{6}-2\sqrt{15}}{40}z-\frac{5\sqrt{6}-2\sqrt{15}}{40}z^2+\frac{5\sqrt{6}-2\sqrt{15}}{40}z^3,\\
P_{1,0} &= \frac{5\sqrt{6}-3\sqrt{15}}{40}+\frac{5\sqrt{6}+7\sqrt{15}}{40}z-\frac{5\sqrt{6}-3\sqrt{15}}{40}z^2-\frac{5\sqrt{6}+7\sqrt{15}}{40}z^3,\\
P_{1,1} &= \frac{5-3\sqrt{10}}{40}+\frac{15-3\sqrt{10}}{40}z+\frac{5-3\sqrt{10}}{40}z^2+\frac{15-3\sqrt{10}}{40}z^3.
\end{aligned}$$

In other words, we have

$$\begin{cases}
\phi_1(x)=\dfrac{10-3\sqrt{10}}{40}\phi_1(3x)+\dfrac{30+3\sqrt{10}}{40}\phi_1(3x-1)+\dfrac{10-3\sqrt{10}}{40}\phi_1(3x-2)\\
\qquad +\dfrac{30+3\sqrt{10}}{40}\phi_1(3x-3)+\dfrac{5\sqrt{6}-2\sqrt{15}}{40}\phi_2(3x)-\dfrac{5\sqrt{6}-2\sqrt{15}}{40}\phi_2(3x-1)\\
\qquad -\dfrac{5\sqrt{6}-2\sqrt{15}}{40}\phi_2(3x-2)+\dfrac{5\sqrt{6}-2\sqrt{15}}{40}\phi_2(3x-3),\\
\phi_2(x)=\dfrac{5\sqrt{6}-3\sqrt{15}}{40}\phi_1(3x)+\dfrac{5\sqrt{6}+7\sqrt{15}}{40}\phi_1(3x-1)-\dfrac{5\sqrt{6}-3\sqrt{15}}{40}\phi_1(3x-2)\\
\qquad -\dfrac{5\sqrt{6}+7\sqrt{15}}{40}\phi_1(3x-3)+\dfrac{5-3\sqrt{10}}{40}\phi_2(3x)+\dfrac{15-3\sqrt{10}}{40}\phi_2(3x-1)\\
\qquad +\dfrac{5-3\sqrt{10}}{40}\phi_2(3x-2)+\dfrac{15-3\sqrt{10}}{40}\phi_2(3x-3).
\end{cases}$$

By Theorem 3.1, we see that the matrix sequence $\{q_0^s,q_1^s,q_2^s,q_3^s\}$ is given by

$$q_0^1=\begin{pmatrix}\dfrac{-5+3\sqrt{10}}{40} & \dfrac{5\sqrt{6}-3\sqrt{15}}{40}\\ \dfrac{5\sqrt{6}-2\sqrt{15}}{40} & \dfrac{-10+3\sqrt{10}}{40}\end{pmatrix},\ q_1^1=\begin{pmatrix}\dfrac{15-3\sqrt{10}}{40} & -\dfrac{5\sqrt{6}+7\sqrt{15}}{40}\\ \dfrac{5\sqrt{6}-2\sqrt{15}}{40} & \dfrac{30+3\sqrt{10}}{40}\end{pmatrix},$$

$$q_2^1 = \begin{pmatrix} \dfrac{-5+3\sqrt{10}}{40} & -\dfrac{5\sqrt{6}-3\sqrt{15}}{40} \\ -\dfrac{5\sqrt{6}-2\sqrt{15}}{40} & \dfrac{-10+3\sqrt{10}}{40} \end{pmatrix},\ q_3^1 = \begin{pmatrix} \dfrac{15-3\sqrt{10}}{40} & \dfrac{5\sqrt{6}+7\sqrt{15}}{40} \\ \dfrac{-5\sqrt{6}+2\sqrt{15}}{40} & \dfrac{30+3\sqrt{10}}{40} \end{pmatrix},$$

$$q_0^2 = \begin{pmatrix} \dfrac{-5+3\sqrt{10}}{40} & \dfrac{5\sqrt{6}-3\sqrt{15}}{40} \\ -\dfrac{5\sqrt{6}-2\sqrt{15}}{40} & -\dfrac{-10+3\sqrt{10}}{40} \end{pmatrix},\ q_1^2 = \begin{pmatrix} \dfrac{15-3\sqrt{10}}{40} & -\dfrac{5\sqrt{6}+7\sqrt{15}}{40} \\ -\dfrac{5\sqrt{6}-2\sqrt{15}}{40} & -\dfrac{30+3\sqrt{10}}{40} \end{pmatrix},$$

$$q_2^2 = \begin{pmatrix} \dfrac{-5+3\sqrt{10}}{40} & -\dfrac{5\sqrt{6}-3\sqrt{15}}{40} \\ \dfrac{5\sqrt{6}-2\sqrt{15}}{40} & -\dfrac{-10+3\sqrt{10}}{40} \end{pmatrix},\ q_3^2 = \begin{pmatrix} \dfrac{15-3\sqrt{10}}{40} & \dfrac{5\sqrt{6}+7\sqrt{15}}{40} \\ -\dfrac{-5\sqrt{6}+2\sqrt{15}}{40} & -\dfrac{30+3\sqrt{10}}{40} \end{pmatrix}.$$

The matrix symbols $Q^1(z)$, $Q^2(z)$ of matrix sequence $\{q_k^s\}$ $(k = 0, 1, 2, 3)$ can be given by the following

$$Q^1(z) = \frac{1}{3}\sum_{k=0}^{3} q_k^1 z^k = \frac{1}{3}\begin{bmatrix} Q_{0,0}^1 & Q_{0,1}^1 \\ Q_{1,0}^1 & Q_{1,1}^1 \end{bmatrix},\ Q^2(z) = \frac{1}{3}\sum_{k=0}^{3} q_k^2 z^k = \frac{1}{3}\begin{bmatrix} Q_{0,0}^2 & Q_{0,1}^2 \\ Q_{1,0}^2 & Q_{1,1}^2 \end{bmatrix},$$

where

$$Q_{0,0}^1 = \frac{-5+3\sqrt{10}}{40} + \frac{15-3\sqrt{10}}{40}z + \frac{-5+3\sqrt{10}}{40}z^2 + \frac{15-3\sqrt{10}}{40}z^3,$$

$$Q_{0,1}^1 = \frac{5\sqrt{6}-3\sqrt{15}}{40} - \frac{5\sqrt{6}+7\sqrt{15}}{40}z - \frac{5\sqrt{6}-3\sqrt{15}}{40}z^2 + \frac{5\sqrt{6}+7\sqrt{15}}{40}z^3,$$

$$Q_{1,0}^1 = \frac{5\sqrt{6}-2\sqrt{15}}{40} + \frac{5\sqrt{6}+2\sqrt{15}}{40}z - \frac{5\sqrt{6}-2\sqrt{15}}{40}z^2 + \frac{-5\sqrt{6}+2\sqrt{15}}{40}z^3,$$

$$Q_{1,1}^1 = \frac{-10+3\sqrt{10}}{40} + \frac{30+3\sqrt{10}}{40}z + \frac{-10+3\sqrt{10}}{40}z^2 + \frac{30+3\sqrt{10}}{40}z^3,$$

$$Q_{0,0}^2 = \frac{-5+3\sqrt{10}}{40} + \frac{15-3\sqrt{10}}{40}z + \frac{-5+3\sqrt{10}}{40}z^2 + \frac{15-3\sqrt{10}}{40}z^3,$$

$$Q_{0,1}^2 = \frac{5\sqrt{6}-3\sqrt{15}}{40} - \frac{5\sqrt{6}+7\sqrt{15}}{40}z - \frac{5\sqrt{6}-3\sqrt{15}}{40}z^2 + \frac{5\sqrt{6}+7\sqrt{15}}{40}z^3$$

$$Q_{1,0}^2 = -\frac{5\sqrt{6}-2\sqrt{15}}{40} - \frac{5\sqrt{6}+2\sqrt{15}}{40}z + \frac{5\sqrt{6}-2\sqrt{15}}{40}z^2 - \frac{-5\sqrt{6}+2\sqrt{15}}{40}z^3,$$

$$Q_{1,1}^2 = -\frac{-10+3\sqrt{10}}{40} - \frac{30+3\sqrt{10}}{40}z - \frac{-10+3\sqrt{10}}{40}z^2 - \frac{30+3\sqrt{10}}{40}z^3.$$

Then its corresponding orthogonal multi-wavelets $\psi(x)$ are given by

$$\begin{cases} \psi_1(x) = \dfrac{-5+3\sqrt{10}}{40}\phi_1(3x) + \dfrac{15-3\sqrt{10}}{40}\phi_1(3x-1) + \dfrac{-5+3\sqrt{10}}{40}\phi_1(3x-2) \\ \qquad + \dfrac{15-3\sqrt{10}}{40}\phi_1(3x-3) + \dfrac{5\sqrt{6}-3\sqrt{15}}{40}\phi_2(3x) - \dfrac{5\sqrt{6}+7\sqrt{15}}{40}\phi_2(3x-1) \\ \qquad - \dfrac{5\sqrt{6}-3\sqrt{15}}{40}\phi_2(3x-2) + \dfrac{5\sqrt{6}+7\sqrt{15}}{40}\phi_2(3x-3), \\ \psi_2(x) = \dfrac{5\sqrt{6}-2\sqrt{15}}{40}\phi_1(3x) + \dfrac{5\sqrt{6}+2\sqrt{15}}{40}\phi_1(3x-1) - \dfrac{5\sqrt{6}-2\sqrt{15}}{40}\phi_1(3x-2) \\ \qquad + \dfrac{-5\sqrt{6}+2\sqrt{15}}{40}\phi_1(3x-3) + \dfrac{-10+3\sqrt{10}}{40}\phi_2(3x) + \dfrac{30+3\sqrt{10}}{40}\phi_2(3x-1) \\ \qquad + \dfrac{-10+3\sqrt{10}}{40}\phi_2(3x-2) + \dfrac{30+3\sqrt{10}}{40}\phi_2(3x-3), \\ \psi_3(x) = \dfrac{-5+3\sqrt{10}}{40}\phi_1(3x) + \dfrac{15-3\sqrt{10}}{40}\phi_1(3x-1) + \dfrac{-5+3\sqrt{10}}{40}\phi_1(3x-2) \\ \qquad + \dfrac{15-3\sqrt{10}}{40}\phi_1(3x-3) + \dfrac{5\sqrt{6}-3\sqrt{15}}{40}\phi_2(3x) - \dfrac{5\sqrt{6}+7\sqrt{15}}{40}\phi_2(3x-1) \\ \qquad - \dfrac{5\sqrt{6}-3\sqrt{15}}{40}\phi_2(3x-2) + \dfrac{5\sqrt{6}+7\sqrt{15}}{40}\phi_2(3x-3), \\ \psi_4(x) = -\dfrac{5\sqrt{6}-2\sqrt{15}}{40}\phi_1(3x) - \dfrac{5\sqrt{6}+2\sqrt{15}}{40}\phi_1(3x-1) + \dfrac{5\sqrt{6}-2\sqrt{15}}{40}\phi_1(3x-2) \\ \qquad - \dfrac{-5\sqrt{6}+2\sqrt{15}}{40}\phi_1(3x-3) - \dfrac{-10+3\sqrt{10}}{40}\phi_2(3x) - \dfrac{30+3\sqrt{10}}{40}\phi_2(3x-1) \\ \qquad - \dfrac{-10+3\sqrt{10}}{40}\phi_2(3x-2) - \dfrac{30+3\sqrt{10}}{40}\phi_2(3x-3). \end{cases}$$

Figure 1 gives the graphs of scaling functions and the corresponding multi-wavelets with support $[0,3]$. These wavelets have the higher order of approximation than the Daubechies scaling function and wavelet with the same support.

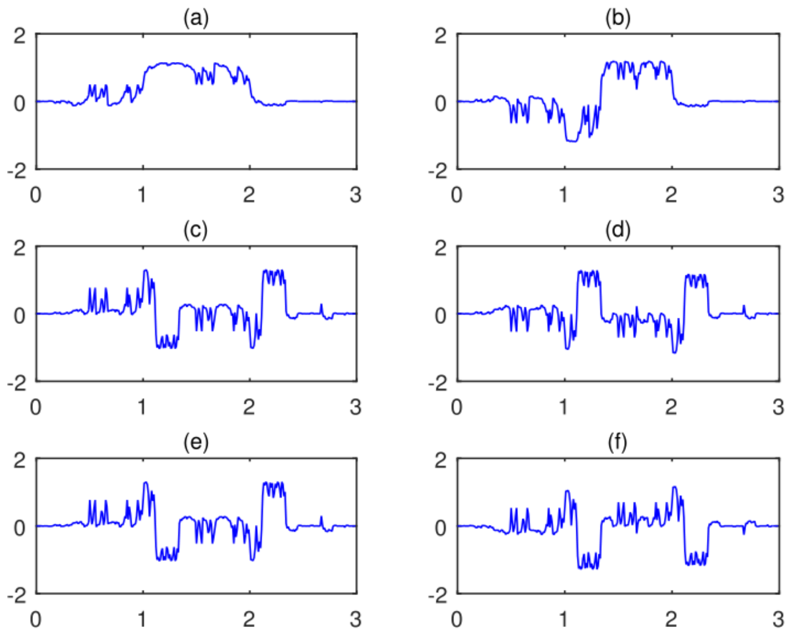

FIGURE 1. (a) ϕ_1; (b) ϕ_2; (c) ψ_1; (d) ψ_2;(e) ψ_3; (f) ψ_4

5. **Conclusions.** The wavelets play important roles in the fields of signal processing, image processing and communication systems. For vector-valued signal processing, the multi-wavelets have greater freedom and flexibility. It enables a finer frequency partitioning and can provide a more compact representation of signals. In this paper, we propose a method for the construction of compactly supported orthogonal symmetric or anti-symmetric multi-wavelets with dilation 3. We first obtain a set of explicit formulations for the matrix sequences of the multi-wavelet from its corresponding matrix sequences of scaling vector function. Next, the compactly supported orthogonal symmetric multi-wavelets with dilation 3 can be constructed via their corresponding matrix sequences. Finally, an example we give shows that our proposed approach is operable.

It is of interesting in both theory and application to have a family of compactly supported symmetric orthonormal wavelets. In the future, we will focus on the construction of compactly supported orthogonal symmetric or anti-symmetric multi-wavelets and complex wavelets with any general dilation factor.

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