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I-CONVERGENT DIFFERENCE SEQUENCE SPACES DEFINED BY COMPACT OPERATOR AND SEQUENCE OF MODULI

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Abstract. *The idea of difference sequence sets $X(\triangle) = \{x = (x_k) : \triangle x \in X\}$ with $X = \ell_\infty$, c and c_0 was introduced by Kizmaz. In this article, we introduce the sequence spaces $S_0^I(T, F, \triangle)$ and $S_\infty^I(T, F, \triangle)$ for the sequence of moduli $F = (f_k)$ and Compact Linear Operator T and study some inclusion relations that arise on said spaces.*
Keywords: Difference sequence spaces, I-convergence, Compact operator, Ideal, Filter

1. **Introduction.** Let $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ be the set of all Natural, Real and Complex numbers respectively. We write

$$\omega = \{x = (x_k) : x_k \in \mathbb{R} \text{ or } \mathbb{C}\}$$

the space of all real or complex sequences. Let ℓ_∞, c and c_0 be the linear spaces of bounded, convergent and null sequences respectively, normed by

$$\| x \|_\infty = \sup_k | x_k |, \quad \text{where } k \in \mathbb{N}$$

The idea of difference sequence spaces was introduced by Kizmaz. In 1981, Kizmaz [10] defined the sequence spaces

$$\ell_\infty(\triangle) = \{x = (x_k) \in \omega : (\triangle x_k) \in \ell_\infty\}$$
$$c(\triangle) = \{x = (x_k) \in \omega : (\triangle x_k) \in c\}$$
$$c_0(\triangle) = \{x = (x_k) \in \omega : (\triangle x_k) \in c_0\}$$

where $\triangle x = (\triangle x_k)_{k=1}^\infty = (x_k - x_{k+1})_{k=1}^\infty$ and $\triangle^0 x = (x_k)$. These are Banach spaces endowed with the norm

$$\| x \|_\triangle = | x_1 | + \| \triangle x \|_\infty$$

Let U be the set of all sequences u such that $u_k \neq 0$ $(k = 1, 2, \ldots)$. Then Malkowsky [7] defined the following sequence spaces

$$\ell_\infty(u; \triangle) = \{x = (x_k) \in \omega : (u_k(x_k - x_{k+1}))_{k=1}^\infty \in \ell_\infty\}$$
$$c(u; \triangle) = \{x = (x_k) \in \omega : (u_k(x_k - x_{k+1}))_{k=1}^\infty \in c\}$$
$$c_0(u; \triangle) = \{x = (x_k) \in \omega : (u_k(x_k - x_{k+1}))_{k=1}^\infty \in c_0\}$$

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where $u \in U$. The idea of modulus function was structured in 1953 by Nakano [11] which was defined as follows.

Definition 1.1. *A function $f : [0, \infty) \to [0, \infty)$ is called a modulus if*
(a) $f(t) = 0$ *if and only if* $t = 0$
(b) $f(t + u) \leq f(t) + f(u)$ *for all* $t, u \geq 0$
(c) f *is increasing, and*
(d) f *is continuous from the right at zero.*

Ruckle [25] used the idea of a modulus function to construct some spaces of complex sequences. Let X be a sequence space, defining the sequence space $X(f)$ as

$$X(f) = \{x = (x_k) : (f(\mid x_k \mid)) \in X\}$$

for a modulus f. Later on some sequence spaces, defined by a modulus function or sequence of moduli, were introduced and studied by Et [16], Bektaş and Çolak [2, 3] and many others. Kolk [5] gave an extension of $X(f)$ by considering a sequence of moduli $F = (f_k)$ that is

$$X(F) = \{x = (x_k) : (f_k(\mid x_k \mid)) \in X\}$$

After that Gaur and Mursaleen [1] defined the following sequence spaces

$$\ell_\infty(F, \triangle) = \{x = (x_k) : \triangle x \in \ell_\infty(F)\}$$

$$c_0(F, \triangle) = \{x = (x_k) : \triangle x \in c_0(F)\}$$

for a sequence of moduli $F = (f_k)$ and gave the necessary and sufficient condition for the inclusion relations between $X(\triangle)$ and $Y(F, \triangle)$, where $X, Y = \ell_\infty$ or c_0. Subsequently difference sequence spaces have been studied by Khan [21, 22], Khan and Lohani [23] and many others.

The notion of the statistical convergence was introduced by Fast [9]. Later on it was studied by Fridy [12, 13] from the sequence point of view and linked it with the summability theory. The concept of I-convergence is a generalization of the statistical convergence and also a generalization of ordinary convergence, and it is based on the notion of the ideal I of subsets of the set $\mathbb{N}$ of positive integers. It was studied at the initial stage by Kostyrko et al. [18]. Later on it was studied by Salat [19]. Salat et al. [20], Demirci [14], Khan and Ebadullah [24], Dems [15] and many authors.

Our aim in the present paper is to generalize the concept of ideal convergence in difference sequence spaces with the help of sequence of moduli $F = (f_k)$ and compact linear operator T and investigate some inclusion relations.

2. **Definitions and Preliminaries.** Before presenting our main results, we gave some well known definitions and results that are required to prove our main result.

Definition 2.1. [18] *Let* $\mathbb{N}$ *be the set of natural numbers. A family of subsets* I *of the power set of* $\mathbb{N}$ *is called an ideal in* $\mathbb{N}$ *if*
(a) $\phi \in I$,
(b) $A, B \in I \Rightarrow A \cup B \in I$,
(c) *for each* $A \in I$ *and* $B \subseteq A$, *we have* $B \in I$.

Remark 2.1. *An ideal* I *is said to be non-trivial if* $I \neq 2^{\mathbb{N}}$.

Definition 2.2. [18] *A non-empty set* $\mathcal{F} \in 2^{\mathbb{N}}$ *is said to be filter in* $\mathbb{N}$ *if*
(a) $\phi \neq \mathcal{F}$,
(b) *for* $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$,
(c) *for each* $A \in \mathcal{F}$ *with* $A \subseteq B$, *we have* $B \in \mathcal{F}$.

Remark 2.2. *For each ideal* I, *there is a filter* $\mathcal{F}(I)$ *associated with* I *defined as:*

$$\mathcal{F}(I) = \{M \subseteq \mathbb{N} : \mathbb{N} - M \in I\}.$$

Definition 2.3. *A non-trivial ideal I is said to be admissible if $\{\{n\} : n \in \mathbb{N}\} \subset I$.*

Definition 2.4. *A non-trivial ideal I is maximal if there does not exist any non-trivial ideal J such that $I \subseteq J$.*

Definition 2.5. *A sequence $(x_k) \in \omega$ is said to be I-convergent to a number L if for every $\epsilon > 0$, the set $\{k \in N : \mid x_k - L \mid \geq \epsilon\} \in I$. In this case we write $I - \lim x_k = L$.*

Definition 2.6. *A sequence $(x_k) \in \omega$ is said to be I-null if $L = 0$. In this case we write $I - \lim x_k = 0$.*

Definition 2.7. *A sequence $x_k \in \omega$ is said to be I-Cauchy if for each $\epsilon > 0$ there exists a number $m = m(\epsilon)$ such that $\{k \in N : \mid x_n - x_m \mid \geq \epsilon\} \in I$.*

Definition 2.8. [20] *A sequence $(x_k) \in \omega$ is said to be I-bounded if there exists $M > 0$ such that $\{k \in N : \mid x_k \mid > M\} \in I$.*

Definition 2.9. [4] *Let X and Y be two normed linear spaces and $T : D(T) \to Y$ be a linear operator, where $D(T) \subset X$. Then the operator T is said to be bounded if there exists a positive real $k > 0$ such that*

$$\parallel Tx \parallel \leq k \parallel x \parallel, \text{ for all } x \in D(T)$$

The set of all bounded linear operators $\mathcal{B}(X, Y)$ is a normed linear space normed by

$$\parallel T \parallel = \sup_{x \in X, \|x\|=1} \parallel Tx \parallel$$

and $\mathcal{B}(X, Y)$ is a Banach space if Y is Banach space.

Definition 2.10. [4] *Let X and Y be two normed linear spaces. An operator $T : X \to Y$ is said to be a compactor linear operator (or completely continuous linear operator), if*

(a) *T is linear and*
(b) *If for every bounded subset N of X, the image $T(N)$ is relatively compact, i.e., the closure $\overline{T(N)}$ is compact.*

Lemma 2.1. [6] *The condition* $\sup_k f_k(t) < \infty$, $t > 0$ *holds if and only if there is a point $t_0 > 0$ such that* $\sup_k f_k(t_0) < \infty$.

Lemma 2.2. [6] *The condition* $\inf_k f_k(t) > 0$ *holds if and only if there is a point $t_0 > 0$ such that* $\inf_k f_k(t_0) > 0$.

Lemma 2.3. [20] *Let $K \in \mathcal{F}(I)$ and $M \subseteq \mathbb{N}$. If $M \notin I$, then $M \bigcap K \notin I$.*

Lemma 2.4. [18] *If $I \subset 2^{\mathbb{N}}$ and $M \subseteq \mathbb{N}$. If $M \notin I$, then $M \bigcap K \notin I$.*

Lemma 2.5. [4] *Let $\mathcal{B}$ be a subset of a metric space X. Then*

(a) *If $\mathcal{B}$ is relatively compact, $\mathcal{B}$ is totally bounded.*
(b) *If $\mathcal{B}$ is totally bounded and X is complete, $\mathcal{B}$ is relatively compact.*

The set of all compact linear operators $\mathcal{C}(X, Y)$ is closed subspace of $\mathcal{B}(X, Y)$ which is a Banach space if Y is Banach space.

Following Başar and Altay [8] and Şengönül [17], we introduce the sequence spaces S and S_0 with the help of compact operator T on the real space $\mathbb{R}$ as follows.

$$S = \{x = (x_k) \in \ell_\infty : T(x) \in c\}$$

and

$$S_0 = \{x = (x_k) \in \ell_\infty : T(x) \in c_0\}$$

3. **Main Results.** This section is devoted to our main results. Here, we give the characterization of the underneath sequence spaces:

$$S_0^I(T,F,\triangle) = \{x=(x_k) \in \ell_\infty : I - \lim f_k(\mid T(\triangle x_k) \mid) = 0\} \in I$$

$$S_\infty^I(T,F,\triangle) = \{x=(x_k) \in \ell_\infty : \sup_k f_k(\mid T(\triangle x_k) \mid) < \infty\} \in I$$

Theorem 3.1. *For a sequence $F=(f_k)$ of moduli, the following statements are equivalent:*

(a) $S_\infty^I(\triangle) \subseteq S_\infty^I(T,F,\triangle)$
(b) $S_0^I(\triangle) \subseteq S_\infty^I(T,F,\triangle)$
(c) $\sup_k f_k(t) < \infty$, $t > 0$.

Proof: (a) $\Rightarrow$ (b) is obvious.
(b) $\Rightarrow$ (c): Let $S_0^I(\triangle) \subseteq S_\infty^I(T,F,\triangle)$. Suppose that (c) is not true.
Then by Lemma 2.1

$$\sup_k f_k(t) = \infty \text{ for all } t > 0$$

and therefore there is a sequence (k_i) of positive integers such that

$$f_{k_i}\left(\frac{1}{i}\right) > i \text{ for } i = 1,2,3,\ldots \tag{1}$$

Now, define the sequence $x=(x_k)$ as

$$x_k := \begin{cases} \dfrac{1}{i} & \text{if } k = k_i,\ i = 1,2,3,\ldots, \\ 0 & \text{otherwise} \end{cases}$$

Then $x \in S_0^I(\triangle)$ by (1) but $x \notin S_\infty^I(T,F,\triangle)$ which contradicts $S_0^I(\triangle) \subseteq S_\infty^I(T,F,\triangle)$. Hence (c) must hold.
(c) $\Rightarrow$ (a): Let (c) be satisfied and $x \in S_\infty^I(\triangle)$.
If we suppose that $x \notin S_\infty^I(T,F,\triangle)$, then

$$\sup_k f_k(\mid T(\triangle x_k) \mid) = \infty \text{ for } \triangle x \in S_\infty^I(\triangle)$$

If we take $t = \mid T(\triangle x_k) \mid$, then $\sup_k f_k(t) = \infty$ which contradicts (c).
Hence $S_\infty^I(\triangle) \subseteq S_\infty^I(T,F,\triangle)$. □

Theorem 3.2. *If $F=(f_k)$ is a sequence of moduli, then the following statements are equivalent:*

(a) $S_0^I(T,F,\triangle) \subseteq S_0^I(\triangle)$
(b) $S_0^I(T,F,\triangle) \subseteq S_\infty^I(\triangle)$
(c) $\inf_k f_k(t) > 0$, $t > 0$

Proof: (a) $\Rightarrow$ (b) is obvious.
(b) $\Rightarrow$ (c): Let $S_0^I(T,F,\triangle) \subseteq S_\infty^I(\triangle)$.
Suppose that (c) does not hold. Then by lemma [6],

$$\inf_k f_k(t) = 0,\ t > 0 \tag{2}$$

and therefore there is a sequence (k_i) of positive integers such that

$$f_{k_i}(i^2) < \frac{1}{i} \text{ for } i = 1,2,3,\ldots$$

Define the sequence $x=(x_k)$ by

$$x_k := \begin{cases} i^2 & \text{if } k = k_i,\ i = 1,2,3,\ldots, \\ 0 & \text{otherwise} \end{cases}$$

By (2), $x \in S_0^I(T, F, \triangle)$ but $x \notin S_\infty^I(\triangle)$ which contradicts the fact that $S_0^I(T, F, \triangle) \subseteq S_\infty^I(\triangle)$. Hence (c) must hold.

(c) $\Rightarrow$ (a): Let (c) hold and $x \in S_0^I(T, F, \triangle)$ that is,

$$I - \lim_k f_k(| T(\triangle x_k) |) = 0$$

Suppose that $x \notin S_0^I(\triangle)$. Then for some $\epsilon_0 > 0$ and positive integer k_0 we have

$$| T(\triangle x_k) | \geq \epsilon_0 \text{ for } k \geq k_0$$

Therefore, $f_k(\epsilon_0) \leq f_k(| T(\triangle x_k) |)$ for $k \geq k_0$ and consequently $\lim_k f_k(\epsilon_0) = 0$ which contradicts (c).

Thus, $S_0^I(T, F, \triangle) \subseteq S_0^I(\triangle)$. □

Theorem 3.3. *The inclusion* $S_\infty^I(T, F, \triangle) \subseteq S_0^I(\triangle)$ *holds if and only if*

$$\lim_k f_k(t) = \infty, \text{ for } t > 0 \tag{3}$$

Proof: Let $S_\infty^I(T, F, \triangle) \subseteq S_0^I(\triangle)$ such that $\lim_k f_k(t) = \infty$, for $t > 0$ does not hold. Then there is a number $t_0 > 0$ and a sequence (k_i) of positive integers such that

$$f_{k_i}(t_0) \leq M < \infty \tag{4}$$

Define the sequence $x = (x_k)$ as follows

$$x_k := \begin{cases} t_0 & \text{if } k = k_i,\ i = 1, 2, 3, \ldots, \\ 0 & \text{otherwise} \end{cases}$$

Thus, $x \in S_\infty^I(T, F, \triangle)$ by (4) but $x \notin S_0^I(\triangle)$.

Therefore, (3) must hold for $S_\infty^I(T, F, \triangle) \subseteq S_0^I(\triangle)$.

Conversely, let (3) hold. If $x \in S_\infty^I(T, F, \triangle)$ then $f_k(| T(\triangle x_k) |) \leq M < \infty$ for $k = 1, 2, 3, \ldots$.

Suppose that $x \notin S_0^I(\triangle)$. Then for some number $\epsilon_0 > 0$ and positive integer k_0 we have $| T(\triangle x_k) | \geq \epsilon_0$ for $k \geq k_0$.

Therefore, $f_k(\epsilon_0) \leq f_k(| T(\triangle x_k) |) \leq M$ for $k \geq k_0$ which contradict (3).

Hence $x \in S_0^I(\triangle)$. □

Theorem 3.4. *The inclusion* $S_\infty^I(\triangle) \subseteq S_0^I(T, F, \triangle)$ *holds if and only if*

$$\lim_k f_k(t) = 0, \text{ for } t > 0 \tag{5}$$

Proof: Suppose that $S_\infty^I(\triangle) \subseteq S_0^I(T, F, \triangle)$ but (5) does not hold. Then

$$\lim_k f_k(t_0) = l \neq 0, \text{ for some } t_0 > 0 \tag{6}$$

Define the sequence $x = (x_k)$ by

$$x_k := t_0 \sum_{v \doteq 0}^{k-1} (-1) \binom{k - v}{k - v}$$

for $k = 1, 2, 3, \ldots$.

Then $x \notin S_0^I(T, F, \triangle)$, by (6). Hence (5) must hold.

Conversely, suppose that (5) holds and let $x \in S_\infty^I(\triangle)$. Then $| T(\triangle x_k) | \leq M < \infty$ for $k = 1, 2, 3, \ldots$.

Therefore, $f_k(| T(\triangle x_k) |) \leq f_k(M)$ for $k = 1, 2, 3, \ldots$ and $\lim_k f_k(| T(\triangle x_k) |) \leq \lim_k f_k(M) = 0$, by (5).

Hence $x \in S_0^I(T, F, \triangle)$. □

4. **Conclusions.** In the paresent paper, we investigate new difference sequence spaces defined by compact operator and sequence of moduli via I_λ-convergence. Also, we study inclusion relation and contribute some results on these said spaces. The results will help the researchers for future work in different point of view. They may discuss topological and algebraic properties in the said spaces also evaluate these sequence spaces using different operators and functions such as bounded linear operator, and orlicz function.

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