ICIC Express Letters
Volume **13**, Number **11**, November **2019**

ICIC International ©2019 ISSN 1881-803X
pp. 1073–1077

PREDICTION INTERVAL OF THE FUTURE OBSERVATIONS OF THE TWO-PARAMETER EXPONENTIAL DISTRIBUTION UNDER MULTIPLY TYPE II CENSORING

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Received May 2019; accepted August 2019

Abstract. *Wu utilized the general weighted moments estimator (GWMEs) of the scale parameter of the one-parameter exponential distribution to construct the prediction intervals of the future observations under multiply type II censoring. Since two-parameter exponential distribution has better application for fitting data than one-parameter exponential distribution, Wu proposed the general weighted moments estimator (GWMEs) of the scale parameter for two-parameter exponential distribution and claimed that the proposed estimator outperforms the other 14 estimators including 12 weighted moments estimators proposed by Wu and Yang and approximate maximum likelihood estimator (AMLE) by Balakrishnan and the best linear unbiased estimator (BLUE) by Balasubramanian and Balakrishnan in terms of the exact mean squared errors (MSEs) in most cases. For two-parameter exponential distribution, we use the GWMEs to construct the pivotal quantities for the use of the prediction intervals of future observation. At last, one real life example is given to demonstrate the prediction intervals based on the GWMEs.*
Keywords: Type II multiply censored sample, Exponential distribution, General weighted moments estimator, Prediction interval

1. **Introduction.** In most literature of reliability, the exponential distribution is widely used as a model of lifetime data. There are many applications of exponential distribution in the analysis of reliability and the life test experiments. See for example, Johnson et al. [4]. The failure time Y follows a two-parameter exponential distribution if the probability density function (p.d.f.) of Y is given by $f(y) = \frac{1}{\theta}\exp\left(-\frac{y-\mu}{\theta}\right)$, $y \geq 0$, $\mu > 0$, $\theta > 0$, where μ is the location parameter and θ is the scale parameter. The location parameter of two-parameter exponential distributions are so-called threshold values or "guaranteed time" parameters in reliability and engineering. In dose-response experiments, this distribution is generally used to model the effective duration of a drug, where the location parameter μ is regarded as the guaranteed effective duration and the scale parameter θ is referred as the mean effective duration in addition to μ.

In life testing experiments, the experimenters may not be able to obtain the lifetimes of all items that are put on test due to the artificial mistakes or for implementing some purposes of experimental designs. Suppose that n items are put on the life test and the first r, middle l and the last s are unobserved or missing, this type of censoring is called the type II multiply censoring. Wu and Yu [10] proposed the simultaneous confidence intervals for all distances from the extreme populations for two-parameter exponential populations based on the multiply type II censored samples. Wu [9] proposed the prediction interval for the future observation for one-parameter exponential distribution based on type II multiply censored sample. The two-parameter exponential distribution has better application for

DOI: 10.24507/icicel.13.11.1073

fitting data than one-parameter exponential distribution (See Maurya et al. [6] for the application of two-parameter exponential distribution). For two-parameter exponential distribution, Wu [8] proposed some general WMEs by assigning a single weight to each observation instead of considering only two weights in Wu and Yang [7] under multiply type II censoring. The simulation comparison results show that the GWMEs outperform the 12 weighted moments estimators proposed by Wu and Yang [7] and approximate maximum likelihood estimator (AMLE) by Balakrishnan [2] and the best linear unbiased estimator (BLUE) by Balasubramanian and Balakrishnan [3] in terms of the exact mean squared errors (MSEs) in most cases for exponential distribution. Since GWMEs perform better than other 14 methods, we utilized the GWMEs proposed in Wu [8] to construct a pivotal quantity and use it to build the prediction interval of future observation. The structure of this research is organized as follows. In Section 2, the general WME for the two-parameter exponential distribution proposed in Wu [8] is defined. In Section 3, we proposed the prediction intervals of future observations based on our proposed pivotal quantities. The percentiles of the proposed pivotal quantities are listed for use in Table 1. One real life example to illustrate the proposed intervals is given in Section 4. At last, the conclusion is discussed in Section 5.

2. **The General Weighted Moments Estimation of the Scale Parameter of the Two-Parameter Exponential Distribution.** Suppose that the lifetimes Y follows a two-parameter exponential distribution with p.d.f. given by $f(y) = \frac{1}{\theta}\exp\left(-\frac{y-\mu}{\theta}\right)$, $y \geq 0$, $\mu > 0$, $\theta > 0$, where μ is the location parameter and θ is the scale parameter. Let $Y_{(r+1)} < \cdots < Y_{(r+k)} < Y_{(r+k+l+1)} < \cdots < Y_{(n-s)}$ be the available type II multiply censored sample from the above distribution.

The general WME to estimate the scale parameter θ is defined as follows.

The general WME $GWME^*$ is constructed by taking the weighted sum of the subtraction of the $r+1$ ordered observation from each observation in order to remove the influence of the unknown location parameter. There are $n-r-l-s-1$ weights $W^*_{r+2}, \ldots, W^*_{r+k}, \ldots, W^*_{r+k+l+1}, \ldots, W^*_{n-s}$ assigned for $GWME^*$ and the general WME is defined as $GWME^* = \tilde{\theta}^* = W^*_{r+2}Y^*_{(r+2)} + \cdots + W^*_{r+k}Y^*_{(r+k)} + W^*_{r+k+l+1}Y^*_{(r+k+l+1)} + \cdots + W^*_{n-s}Y^*_{(n-s)} == \underset{\sim}{W}^*\,\underset{\sim}{Y}^*$, where $\underset{\sim}{W}^* = \left[W^*_{r+2}, \ldots, W^*_{r+k}, W^*_{r+k+l+1}, \ldots, W^*_{n-s}\right]^T$, and $\underset{\sim}{Y}^* = \left(Y^*_{(r+2)}, \ldots, Y^*_{(n-s)}\right)$, where $Y^*_{(i)} = Y_{(i)} - Y_{(r+1)}$, $i = r+2, \ldots, r+k, r+k+l+1, \ldots, n-s$.

Know that the mean of $Y^*_{(i)}/\theta$ is $a^*_i = E\left(Y^*_{(i)}/\theta\right) = \sum_{j=r+2}^{i} \frac{1}{n-j+1}$ and the covariance of $Y^*_{(i)}/\theta$ and $Y^*_{(j)}/\theta$ is $b^*_{i,j} = Cov\left(Y^*_{(i)}/\theta, Y^*_{(j)}/\theta\right) = \sum_{j=r+2}^{i} \frac{1}{(n-j+1)^2}$, $i \leq j$, $(i,j) \in \{r+2, \ldots, r+k, r+k+l+1, \ldots, n-s\}$.

The weights $\underset{\sim}{W}^* = \left[W^*_{r+2}, \ldots, W^*_{r+k}, W^*_{r+k+l+1}, \ldots, W^*_{n-s}\right]^T$ are determined so that the MSE of the proposed general WME is minimized. From Wu [8], the optimal weights are $\underset{\sim}{W}^{*T} = A^{*-1}\,\underset{\sim}{a}^*$, where

$$A^* = \begin{bmatrix} b^*_{r+2,r+2} + a^{*2}_{r+2} & b^*_{r+2,r+3} + a^*_{r+2}a^*_{r+3} & \cdots & b^*_{r+2,n-s} + a^*_{r+2}a^*_{n-s} \\ b^*_{r+2,r+3} + a^*_{r+2}a^*_{r+3} & b^*_{r+3,r+3} + a^{*2}_{r+3} & \cdots & b^*_{r+3,n-s} + a^*_{r+3}a^*_{n-s} \\ \vdots & \ddots & \vdots & \vdots \\ b^*_{r+2,n-s} + a^*_{r+2}a^*_{n-s} & \cdots & \cdots & b^*_{n-s,n-s} + a^{*2}_{n-s} \end{bmatrix}$$

and $a^* = \left(a^*_{r+2}, \ldots, a^*_{r+k}, a^*_{r+k+l+1}, \ldots, a^*_{n-s}\right)$ which is the mean vector of the random vector $\underset{\sim}{Y}^*/\theta$. The general WME (GWME) with minimum MSE is obtained as

$$\tilde{\theta}^* = \underset{\sim}{W}^{*T} \underset{\sim}{Y}^* = A^{*-1} \underset{\sim}{a}^* \underset{\sim}{Y}^* \tag{1}$$

The minimum MSE of GWME is

$$\text{MSE}\left(\tilde{\theta}^*\right) = \left(\underset{\sim}{W}^{*T} B^* \underset{\sim}{W}^* + \left(\underset{\sim}{W}^{*T} a^* - 1\right)^2\right)\theta^2 \tag{2}$$

where $B^* = [b^*_{i,j}]_{i=r+2,\ldots,r+k,r+k+l+1,\ldots,n-s,j=r+2,\ldots,r+k,r+k+l+1,\ldots,n-s}$ is the covariance matrix of the random vector $\underset{\sim}{Y}^* / \theta$.

3. **Prediction Intervals of Future Observation.** In order to predict the future observation, the pivotal quantity is considered as $U = (Y_{(j)} - Y_{(n-s)})/\tilde{\theta}^*$, $n - s < j \le n$ based on the *GWME* $\tilde{\theta}^*$ defined in (1). Since $\frac{Y_{(1)}}{\theta}, \ldots, \frac{Y_{(n)}}{\theta}$ are the n order statistics from a standard exponential distribution and $\frac{\tilde{\theta}^*}{\theta} = \frac{\underset{\sim}{W}^{*T}\underset{\sim}{Y}^*}{\theta}$ is a linear combination of n order statistics from a standard exponential distribution, the distribution of pivotal quantity $U = \left(\frac{Y_{(j)}}{\theta} - \frac{Y_{(n-s)}}{\theta}\right) \Big/ \frac{\tilde{\theta}^*}{\theta}$ is independent of θ, $n - s < j \le n$. Let $U(\delta; n, j, r, k, l, s)$ be the δ percentile of the distribution of U satisfying $P(U \le U(\delta; n, j, r, k, l, s)) = \delta$.

Make use of the pivotal quantity, the prediction interval of future observations $Y_{(j)}$, $n - s < j \le n$ is proposed in the following theorem.

Theorem 3.1. *For type II multiply censored sample* $Y_{(r+1)} < \cdots < Y_{(r+k)} < Y_{(r+k+l+1)} < \cdots < Y_{(n-s)}$, *the prediction interval of* $Y_{(j)}$, $n-s < j \le n$ *is* $\Big(Y_{(n-s)} + U\left(\frac{\alpha}{2}; n, j, r, k, l, s\right)\tilde{\theta}^*$, $Y_{(n-s)} + U\left(1 - \frac{\alpha}{2}; n, j, r, k, l, s\right)\tilde{\theta}^*\Big)$.

Proof: Observe that $1-\alpha = P\left(U\left(\frac{\alpha}{2}; n, j, r, k, l, s\right) \le \left(\frac{Y_{(j)} - Y_{(n-s)}}{\tilde{\theta}^*}\right) \le U\left(1 - \frac{\alpha}{2}; n, j, r, k, l, s\right)\hat{\theta}\right) = P\left(Y_{(n-s)} + U\left(\frac{\alpha}{2}; n, j, r, k, l, s\right)\tilde{\theta}^* \le Y_{(j)} \le Y_{(n-s)} + U\left(1 - \frac{\alpha}{2}; n, j, r, k, l, s\right)\tilde{\theta}^*\right)$.
□

Since the exact distribution of U is too hard to derive algebraically, the δ percentile of the distribution of U is obtained based on Monte Carlo simulation. Moreover, all the simulations were run with the aid of AbSoft Fortran Inclusive of IMSL [1]. In the simulation, 100,000 replicates are used to compute the percentiles of U for each combination of n, r, k, l, s, j, where $j = n-s+1, \ldots, n$. Due to the limitation of the number of pages, only part of the percentiles of U are given in Table 1, for $\delta = 0.005$, 0.010, 0.025, 0.050, 0.100, 0.900, 0.950, 0.975, 0.990, 0.995 under $n = 12$, 24 (see Table 1). Any specific percentile $U(\delta; n, j, r, k, l, s)$ for any censoring scheme (n, r, k, l, s) for the jth future observation, $j = n - s + 1, \ldots, n$, can be obtained by the software program provided by the author.

4. **Example.** We use the example of times to breakdown of an insulating fluid between electrodes recorded at five different voltages (Nelson [6]) in this section to demonstrate the prediction interval of future observations. Such a distribution of time to breakdown is usually assumed to be exponentially distributed in engineering theory. We choose 35 kV, and the multiple type II censored data with $n = 12$, $r = 2$, $k = 3$, $l = 1$ and $s = 5$ is $-, -, 41, 87, 93, -, 116, -, -, -, -, -$. The weights are 0.23568, 0.12544, 0.19776, 0.8058 and the estimated scale parameter is $\tilde{\theta}^* = W^*_{r+2}Y^*_{(r+2)} + \cdots + W^*_{r+k}Y^*_{(r+k)} + W^*_{r+k+l+1}Y^*_{(r+k+l+1)} + \cdots + W^*_{n-s}Y^*_{(n-s)} = W^*_4\left(Y^*_{(4)} - Y^*_{(3)}\right) + W^*_5\left(Y^*_{(5)} - Y^*_{(3)}\right) + 7^*_5\left(Y^*_{(7)} - Y^*_{(3)}\right) = 0.20047 * 46 + 0.31604 * 52 + 1.28774 * 75 = 122.2362$.

Using Theorem 3.1, the 90% and 95% prediction intervals for $Y_{(8)}$, $Y_{(9)}$, $Y_{(10)}$, $Y_{(11)}$, $Y_{(12)}$ are obtained in Table 2.

TABLE 1. The δ percentile of the pivotal quantity $U = \left(Y_{(j)} - Y_{(n-s)}\right)/\tilde{\theta}^*$ and $P(U \le U(\delta; n, j, r, k, l, s)) = \delta$

						δ									
n	r	k	l	s	j	0.005	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990	0.995
12	2	5	2	2	11	0.0028	0.0056	0.0143	0.0293	0.0605	1.5646	2.1479	2.7893	3.7520	4.5520
12	2	5	2	2	12	0.0803	0.1147	0.1872	0.2767	0.4212	4.1812	5.4994	6.9563	9.0850	10.8650
12	3	5	2	1	12	0.0057	0.0115	0.0290	0.0591	0.1216	3.1415	4.3191	5.6124	7.5610	9.1770
12	3	5	1	2	11	0.0030	0.0059	0.0149	0.0302	0.0623	1.6416	2.2760	2.9804	4.0670	4.9960
12	3	5	1	2	12	0.0797	0.1154	0.1902	0.2827	0.4294	4.4029	5.8598	7.4686	9.8840	11.9630
12	2	5	3	1	12	0.0057	0.0114	0.0286	0.0580	0.1202	3.0407	4.1567	5.3696	7.1270	8.6030
12	2	5	1	3	10	0.0020	0.0039	0.0099	0.0201	0.0415	1.0917	1.5103	1.9866	2.7080	3.3360
12	2	5	1	3	11	0.0454	0.0656	0.1084	0.1612	0.2449	2.4216	3.1951	4.0591	5.3710	6.4850
12	2	5	1	3	12	0.1943	0.2530	0.3670	0.4948	0.6869	5.1021	6.6523	8.3571	10.9250	13.0820
12	1	5	3	2	11	0.0029	0.0058	0.0144	0.0290	0.0599	1.5118	2.0629	2.6612	3.5390	4.2870
12	1	5	3	2	12	0.0779	0.1127	0.1854	0.2748	0.4171	4.0199	5.2445	6.5640	8.5090	10.1340
12	1	5	2	3	10	0.0019	0.0039	0.0097	0.0197	0.0406	1.0421	1.4301	1.8563	2.4800	3.0230
12	1	5	2	3	11	0.0457	0.0655	0.1075	0.1591	0.2421	2.2986	2.9997	3.7700	4.8970	5.8640
12	1	5	2	3	12	0.1931	0.2515	0.3635	0.4894	0.6793	4.8364	6.2364	7.7662	10.0360	11.9130
12	4	5	1	1	12	0.0060	0.0118	0.0296	0.0602	0.1242	3.2786	4.5590	5.9841	8.1510	9.9870
12	1	5	4	1	12	0.0056	0.0110	0.0281	0.0574	0.1189	2.9757	4.0464	5.2159	6.8860	8.2520
24	2	5	2	2	23	0.0027	0.0054	0.0135	0.0273	0.0559	1.2880	1.7097	2.1454	2.7430	3.2260
24	2	5	2	2	24	0.0745	0.1082	0.1778	0.2620	0.3952	3.3630	4.2547	5.1688	6.4130	7.3970
24	3	5	2	1	24	0.0053	0.0108	0.0269	0.0541	0.1116	2.5742	3.4139	4.2896	5.4990	6.4650
24	3	5	1	2	23	0.0027	0.0053	0.0134	0.0272	0.0556	1.2981	1.7206	2.1574	2.7690	3.2450
24	3	5	1	2	24	0.0759	0.1089	0.1787	0.2629	0.3975	3.3913	4.2958	5.2205	6.5190	7.5110
24	2	5	3	1	24	0.0052	0.0106	0.0267	0.0541	0.1110	2.5640	3.3982	4.2695	5.4600	6.3660
24	2	5	1	3	22	0.0018	0.0036	0.0089	0.0181	0.0373	0.8640	1.1475	1.4400	1.8410	2.1620
24	2	5	1	3	23	0.0433	0.0628	0.1025	0.1509	0.2269	1.8650	2.3396	2.8290	3.4900	4.0090
24	2	5	1	3	24	0.1898	0.2464	0.3525	0.4718	0.6456	3.8774	4.8165	5.7671	7.0730	8.1010
24	1	5	3	2	23	0.0027	0.0053	0.0133	0.0269	0.0554	1.2819	1.6964	2.1241	2.7170	3.1860
24	1	5	3	2	24	0.0753	0.1090	0.1780	0.2624	0.3952	3.3498	4.2332	5.1404	6.3730	7.3140
24	1	5	2	3	22	0.0018	0.0035	0.0089	0.0181	0.0370	0.8582	1.1384	1.4285	1.8280	2.1390
24	1	5	2	3	23	0.0433	0.0625	0.1021	0.1504	0.2260	1.8524	2.3264	2.8057	3.4620	3.9670
24	1	5	2	3	24	0.1898	0.2457	0.3523	0.4698	0.6428	3.8506	4.7665	5.7014	6.9740	7.9950
24	4	5	1	1	24	0.0054	0.0106	0.0269	0.0543	0.1116	2.5943	3.4390	4.3196	5.5450	6.4840
24	1	5	4	1	24	0.0053	0.0106	0.0265	0.0538	0.1104	2.5461	3.3722	4.2239	5.3880	6.2740

5. **Conclusion.** Since the two-parameter exponential distribution can be better fitting data than one-parameter exponential distribution, we utilized the GWMEs to construct a pivotal quantity so that we can build a prediction interval for future observation for two-parameter exponential distribution based on the pivotal quantity in Theorem 3.1. For application use, we tabulated the percentiles of proposed pivotal quantity by Monte-Carlo method in Table 1. At last, we give one real life example to illustrate the proposed prediction intervals. In the future, this research can be extended to other location scale family, for example Pareto distributions.

Acknowledgement. The author's research was supported by Ministry of Science and Technology MOST 103-2118-M-032 -004 -MY2, MOST 105-2118-M-032 -005 -MY2, MOST 107-2118-M-032-006 and MOST 108-2118-M-032-001- in Taiwan.

TABLE 2. 90% and 95% prediction interval for future observation $Y_{(j)}$, $j = 8, \ldots, 12$

90%		
Future observation	$U(0.05; 12, j, 2, 3, 1, 5)$, $U(0.95; 12, j, 2, 3, 1, 5)$	Prediction interval
$Y_{(8)}$	0.0129, 1.1162	$(117.5768, 252.4400)$
$Y_{(9)}$	0.0923, 2.1646	$(127.2824, 380.5925)$
$Y_{(10)}$	0.2321, 3.5346	$(144.3710, 548.0561)$
$Y_{(11)}$	0.4503, 5.5985	$(171.0430, 800.3394)$
$Y_{(12)}$	0.8401, 9.9343	$(218.6906, 1330.3311)$
95%		
Future observation	$U(0.025; 12, j, 2, 3, 1, 5)$, $U(0.975; 12, j, 2, 3, 1, 5)$	Prediction interval
$Y_{(8)}$	0.0063, 1.5150	$(116.7701, 301.1878)$
$Y_{(9)}$	0.0622, 2.8567	$(123.6031, 465.1922)$
$Y_{(10)}$	0.1714, 4.6035	$(136.9513, 678.7143)$
$Y_{(11)}$	0.3475, 7.2547	$(158.4771, 1002.7870)$
$Y_{(12)}$	0.6603, 12.8745	$(196.7126, 1689.7300)$

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