ICIC Express Letters
Volume **13**, Number **5**, May **2019**

ICIC International ©2019 ISSN 1881-803X
pp. 401–408

HIGH-ORDER MOMENT UKF FOR MARKOV JUMP NONLINEAR SYSTEMS

Ziheng Zhou, Haiying Wan, Xiaoli Luan and Fei Liu

Key Laboratory of Advanced Process Control for Light Industry (Ministry of Education)
Institute of Automation
Jiangnan University
No. 1800, Lihu Avenue, Wuxi 214122, P. R. China
xlluan@jiangnan.edu.cn

Received October 2018; accepted January 2019

Abstract. *This paper reports on the high-order moment state estimation of the Markov jump nonlinear systems based on an unscented Kalman filter (UKF). Considering that once the state and the external disturbances deviate from the Gaussian distribution, it is difficult to obtain a good estimation of the actual state by only focusing on the mean and mean square error. Therefore, a high-order moment component form of the original systems is obtained from the state accumulation based on the cumulant generating function, which makes it possible to fuse the high-order moment information with the state estimation. Then, to improve the estimation accuracy, a novel high-order moment recursive state estimator is proposed to handle the nonlinear filtering issues via a UKF. Finally, a numerical example is provided to illustrate the effectiveness of the proposed filter.*

Keywords: Markov jump systems, High-order moment, Unscented Kalman filter, Cumulant generating function

1. **Introduction.** As a fundamental problem in the areas of signal processing and control, the state estimation of a dynamic system has attracted much attention during the last several decades. Specially, considering the high-accuracy and high-quality requirements of modern control systems, it is necessary to develop nonlinear filtering techniques to meet the growing demand. The extended Kalman filter (EKF), which approximates the nonlinear systems by its first-order linearization and applies a Kalman filter to it, is the first extension of the Kalman filter to solve the nonlinear filtering issues [1]. However, due to its limitations, including the low accuracy, poor convergence, and the difficulty of computing the Jacobian matrix, the EKF has serious limitation for many applications [2,3]. To overcome these disadvantages, [4] proposed an unscented Kalman filter (UKF), which has better accuracy than the EKF and simultaneously avoids the cumbersome calculation of the Jacobian matrix. Subsequently, the UKF has been developed rapidly and applied widely in practical applications, such as gravity matching navigation [5], multi-sensor fusion [6], and friction coefficient estimation [7].

However, the majority of the aforementioned nonlinear filtering results are only efficient in single-mode systems whose dynamics depend only on the time evolution. In fact, multi-mode nonlinear systems with interacting continuous-time and discrete-event dynamics are more common and useful in practical processes, such as Markov jump nonlinear systems (MJNLSs). Due to the nonlinearity of the state evolution and the randomness in the mode jumping process, it is difficult to estimate the state of the MJNLSs. To this end, some attempts had been made. For example, [8] proposed a consensus-based distributed UKF design method for discrete-time MJNLSs. [9] designed an interacting multiple sensor

DOI: 10.24507/icicel.13.05.401

UKF via considering the collaborative tracking procedure as an MJNLS. Furthermore, the stability analysis of a discrete-time UKF for nonlinear stochastic systems with Markov packet dropouts was addressed in [10].

Although certain progress has been obtained in the state estimation of MJNLSs, these studies are mainly based on the assumption that the state and external disturbances of the systems are subject to a Gaussian distribution. In this case, it is enough to estimate the system dynamics based only on the mean or mean square error. However, in nonlinear systems, the general Gaussian distribution (GGD) is a common phenomenon and it is difficult to estimate the state accurately based only on the mean and mean square error [11]. For example, unavoidable errors will result if high-order moment information is not incorporated such as the skewness in economy systems [12]. Therefore, it is necessary to take the high-order moment characteristics into consideration during the filter design. In our previous work [13], the cumulant generating function (CGF) was used to analyze the high-order moment stability of Markov jump systems and this method provides a new approach to estimate the system state based on high-order moment characteristics.

Motivated by the aforementioned analysis, this study attempts to design a high-order moment UKF to estimate the state of MJNLSs based on the CGF. The main contributions of this paper are three-fold. First, the original MJNLS is transformed into a deterministic higher order component expression based on the CGF so that the high-order moment information can be used in the filter design. Then, the high-order moment errors are taken into account to deal with the case of the GGD. Finally, the strong nonlinearity of the systems has been considered in the state recursive estimation and a UKF is designed.

The rest of the paper is organized as follows. Section 2 formulates the preliminaries and transforms the original MJNLS into a deterministic higher order component expression based on the CGF. Section 3 presents the UKF algorithm for the transformed system. Section 4 illustrates the proposed filtering technique using a numerical example. The conclusions are summarized and the future research directions are pointed out in Section 5.

Notations: I_n is the n-dimensional unit matrix. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $\|A\|$ denotes the Euclidean norm of matrix A. A^{T} is the transposition of matrix A. $E\{\cdot\}$ represents the statistical expectation of the stochastic process or vector. $A \otimes B$ is the Kronecker product of A and B. $A^{\otimes p}$ is p times Kronecker product of the A.

2. **Problem Statement and Preliminaries.**

Consider the following discrete-time MJNLS:

$$\begin{cases} x(k+1) = f_{\theta_k}(x(k)) + w(k) \\ y(k) = g_{\theta_k}(x(k)) + v(k) \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^{n_x}$ is the state vector with a known initial value, $y(k) \in \mathbb{R}^{n_y}$ is the measured output, and $w(k) \in \mathbb{R}^{n_w}$ and $v(k) \in \mathbb{R}^{n_v}$ are the system noise and measurement noise. θ_k represents a Markov process taking values in the finite set $M = \{1, 2, \ldots, m\}$ and the items of the transition probability matrix Π are given by:

$$\pi_{ij} = P_r(\theta_k = j | \theta_{k-1} = i)$$

where π_{ij} denotes the transition probability from mode i at time $k-1$ to mode j at time k and satisfies $\pi_{ij} > 0$, $\sum_{j=1}^{m} \pi_{ij} = 1$, $\forall i, j \in M$.

According to the UKF technique of MJNLS (1) in [14], the following assumption is given for the process noise $w(k)$ and measurement noise $v(k)$.

Assumption 2.1. ([14]) *For all $k \geq 0$, $m \geq 0$,*

(a) $E\{w(k)\} = 0$, $E\{w(k)w^{\mathrm{T}}(m)\} = H\delta(k-m)$, *where* $H > 0$.

(b) $E\{v(k)\} = 0$, $E\{v(k)v^{\mathrm{T}}(m)\} = Q\delta(k-m)$, *where* $Q > 0$.

(c) $E\{w(k)v^{\mathrm{T}}(m)\} = 0$.

where $\delta(\cdot)$ *is the Dirac function, which implies* $\delta\{k\} = \begin{cases} 1 & k = 0 \\ 0 & others \end{cases}$. *With this assumption proposed, the UKF algorithm is provided in the next section.*

The difficulty in the recursive estimation of MJNLSs is the stochasticity during mode-jumping. One of the most widely accepted ideas to handle this is weighting based on the transition probability matrix. Motivated by this idea, we transform the MJNLS (1) into a deterministic high-order moment component form with the same norm of the original states based on the CGF and design a UKF for the transformed system. To establish the recursive p-moment state estimation, first, a derandomization technique is introduced.

Firstly, the indicator function $\mathbf{1}_{\mathbb{A}}$ is defined:

$$\mathbf{1}_{\mathbb{A}}(\omega) = \begin{cases} 1 & \text{if } \omega \in \mathbb{A} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

and then define:

$$q_i(k) = E\left\{\|x(k)\| \, \mathbf{1}_{\{\theta_k = i\}}(\theta_k)\right\} \tag{3}$$

Combining (3) and (1), we have:

$$q_j(k+1) = \sum_{i=1}^{m} f_i\left(x(k)\right) \mathbf{1}_{\theta_k = i}(\theta_k) \mathbf{1}_{\theta_{k+1} = j}(\theta_{k+1}) + w(k) = \sum_{i=1}^{m} \pi_{ij} f_i\left(q_i(k)\right) + w(k) \tag{4}$$

Definition 2.1. ([13]) *For a random variable* $z \in R^p$ *with the distribution density function* $p(z)$*, the moment generating function (MGF) is defined by* $\Phi_z(\varpi) = \int_{R^p} e^{\varpi^{\mathrm{T}} z} p(z) dz$ *and the CGF is defined by* $\Psi_z(\varpi) = \log \Phi_z(\varpi)$.

If it is analytical for the MGF $\Phi_z(\varpi)$ and CGF $\Psi_z(\varpi)$ defined in Definition 2.1, then it can be expanded into a Taylor's series in the neighborhood of $\varpi = 0$ as:

$$\Phi_z(\varpi) = \sum_{p=0}^{\infty} m(p,n)^{\mathrm{T}} \frac{\varpi^{\otimes p}}{p!} \tag{5}$$

$$\Psi_z(\varpi) = \sum_{p=0}^{\infty} c(p,n)^{\mathrm{T}} \frac{\varpi^{\otimes p}}{p!} \tag{6}$$

where $m(p,n)$ is the p-order moment vector with dimension $n^p \times 1$, $p = \{1, 2, \ldots, l\}$ given by:

$$m(p,n) = \int\limits_{R^n} z^{\otimes p} p(z) dz \tag{7}$$

$c(p,n)$ is the p-order cumulant, which can be calculated by:

$$c(p,n) = m(p,n) - \sum_{l=1}^{p-1} \begin{pmatrix} p-1 \\ l \end{pmatrix} Q_l c(p-l, n) \otimes m(l,n) \tag{8}$$

where Q_l is a specific commutation matrix with an appropriate dimension.

According to (5) and (6), by taking the CGF on both sides of (4) and expanding it to a Taylor's series in the neighborhood of $\varpi = 0$, the left side of (4) is described as:

$$\Psi_{q_j(k+1)} = \sum_{p=0}^{\infty} c_{q_j(k+1)}(p,n)^{\mathrm{T}} \frac{\varpi^{\otimes p}}{p!} \tag{9}$$

and the right side of (4) is formulated as:

$$\Psi_{\sum\limits_{i=1}^{m} \pi_{ij} f_i(q_i(k))} = \sum_{i=1}^{m} \left\{ \sum_{p=0}^{\infty} c_{\pi_{ij} f_i(q_i(k))}(p,n)^{\mathrm{T}} \frac{\varpi^{\otimes p}}{p!} \right\} = \sum_{p=0}^{\infty} \left\{ \sum_{i=1}^{m} c_{\pi_{ij} f_i(q_i(k))}(p,n)^{\mathrm{T}} \frac{\varpi^{\otimes p}}{p!} \right\} \tag{10}$$

Then it yields:

$$c_{q_j(k+1)}(p,n)=\sum_{i=1}^{m}c_{\pi_{ij}f_i(q_i(k))}(p,n) \tag{11}$$

With the cumulant property $c_{Tz}(p)=T^{\otimes p}C_z(p)$ where T is the transformation matrix, it is easy to find that:

$$c_{q_j(k+1)}(p,n)=\sum_{i=1}^{m}\pi_{ij}f_i^{\otimes p}\left(c_{q_i(k)}(p,n)\right) \tag{12}$$

We define $s_j(k,p)=c_{q_j(k)}(p,n)$, the state vector $S(k,p)=\left\{s_1^{\mathrm{T}}(k,p),\dots,s_m^{\mathrm{T}}(k,p)\right\}^{\mathrm{T}}$, the noise vector $w_{mp}(k)=\left\{w^{\mathrm{T}}(k),\dots,w^{\mathrm{T}}(k)\right\}^{\mathrm{T}}$ with an appropriate dimension, and then (12) can be rewritten as:

$$S(k+1,p)=F_{tp}\left(S(k,p)\right)+w_{mp}(k) \tag{13}$$

where $F_{tp}=\left(\Pi^{\mathrm{T}}\otimes I_{n_x^p}\right)\cdot diag\left\{f_1^{\otimes p},f_2^{\otimes p},\dots,f_m^{\otimes p}\right\}\in R^{\left(n_x^p\times m\right)\times\left(n_x^p\times m\right)}$.

From these definitions, the MJNLS (1) is transformed into a deterministic form (13) with the p-order moment component based on the CGF. Therefore, the recursive estimation of $S(k,p)$ is equivalent to the high-order moment state estimation of $x(k)$ in system (1).

Remark 2.1. *When $p=1$ and $p=2$, the high-order moment state estimation is reduced to the mean error estimation and the mean square error estimation, respectively, which completely describes the Gaussian distribution parameters. When it comes to higher moment orders, the high-order dynamic characteristics of the system, such as the skewness error and the kurtosis in the generalized Gaussian distribution, are estimated when $p=3$ and $p=4$, respectively.*

3. **Main Results.** The UKF algorithm based on the high-order moment is provided in this section. Consider the transformed nonlinear deterministic system:

$$\begin{cases} S(k+1,p)=F_{tp}\left(S(k,p)\right)+w_{mp}(k) \\ y_{tp}(k)=G_{tp}\left(S(k,p)\right)+v_{mp}(k) \end{cases} \tag{14}$$

where $y_{tp}(k)=\left\{y^{\mathrm{T}}(k),\dots,y^{\mathrm{T}}(k)\right\}^{\mathrm{T}}$ and $v_{mp}(k)=\left\{v^{\mathrm{T}}(k),\dots,v^{\mathrm{T}}(k)\right\}^{\mathrm{T}}$ with appropriate dimensions; w_{mp}, v_{mp} are the uncorrelated white Gaussian noise with covariance H and Q as described in Assumption 2.1.

Sigma points are:

$$\mathcal{S}^i=\begin{cases} \hat{S}^i(k,p) & i=0 \\ \hat{S}^i(k,p)+\left(\sqrt{(L+\lambda)P^i(k)}\right)_n & i=1,\dots,L \\ \hat{S}^i(k,p)-\left(\sqrt{(L+\lambda)P^i(k)}\right)_n & i=L+1,\dots,2L \end{cases} \tag{15}$$

where λ is a scaling factor; $\left(\sqrt{(L+\lambda)P^i(k)}\right)_n$ denotes the nth row or column of $\sqrt{(L+\lambda)P^i(k)}$.

Then the high-order moment form of the state evolution of the sigma points is:

$$\begin{cases} S^i(k+1,p)=F\left(S^i(k,p)\right) \\ y_{tp}^i(k)=G_{tp}\left(S^i(k,p)\right) \end{cases} \tag{16}$$

and the estimation values $\hat{S}^-(k,p)$, $\hat{y}_{tp}^-(k)$ and the output covariance $P_y^-(k)$ can be calculated as:

$$\begin{cases} \hat{S}^-(k,p) = \sum_{i=0}^{2L} W_i^m S^i(k,p) \\ \hat{y}_{tp}^-(k) = \sum_{i=0}^{2L} W_i^m y^i(k) \end{cases} \tag{17}$$

$$P_y^-(k) = \sum_{i=0}^{2L} W_i^c \left(y_{tp}^i(k) - \hat{y}_{tp}(k)\right)\left(y_{tp}^i(k) - \hat{y}_{tp}(k)\right)^{\mathrm{T}} \tag{18}$$

where the weights are $W_i^m = \frac{1}{2(L+\lambda)}$ and $W_i^c = \frac{1}{2(L+\lambda)}$.

The implementation of the high-order moment UKF algorithm is as follows.

1) Initialization

$$\hat{S}(0,p) = E\left(S(0,p)\right)$$

$$P(0) = E\left[\left(S(0,p) - \hat{S}(0,p)\right)\left(S(0,p) - \hat{S}(0,p)\right)^{\mathrm{T}}\right]$$

where $S(0,p)$ is the initial state with the high-order moment and $P(0)$ is the initial covariance between the true state and the estimated state.

2) Sigma point setting

For $i = 1, \ldots, 2L$, set the sigma points as:

$$\mathcal{S}^i(k) = \left\{\hat{S}^0(k,p), \hat{S}^1(k,p) + \left(\sqrt{(L+\lambda)P^i(k)}\right), \ldots, \hat{S}^{L+1}(k,p) - \left(\sqrt{(L+\lambda)P^i(k)}\right), \ldots\right\}$$

3) Time updating

$$S^i(k,p) = F_{tp}\left(S^i(k-1,p)\right) \tag{19}$$

$$\hat{S}^-(k,p) = \sum_{i=0}^{2L} W_i^m S^i(k,p) \tag{20}$$

$$P^-(k) = \sum_{i=0}^{2L} W_i^c \left(y_{tp}^i(k) - \hat{y}_{tp}(k)\right)\left(y_{tp}^i(k) - \hat{y}_{tp}(k)\right)^{\mathrm{T}} + H \tag{21}$$

$$y_{tp}^i(k) = G_{tp}\left(S^i(k,p)\right) \tag{22}$$

$$\hat{y}_{tp}^-(k) = \sum_{i=0}^{2L} W_i^m y_{tp}^i(k) \tag{23}$$

4) Measurement updating

$$P_y^-(k) = \sum_{i=0}^{2L} W_i^c \left(y_{tp}^i(k) - \hat{y}_{tp}(k)\right)\left(y_{tp}^i(k) - \hat{y}_{tp}(k)\right)^{\mathrm{T}} \tag{24}$$

$$P_{xy}(k) = \sum_{i=0}^{2L} W_i^c \left(S^i(k,p) - \hat{S}^-(k,p)\right)\left(S^i(k,p) - \hat{S}^-(k,p)\right)^{\mathrm{T}} + Q \tag{25}$$

Then the UKF compensation gain is:

$$K(k) = P_{xy}(k) P_y^-(k) \tag{26}$$

The estimation for the high-order moment component form of the state is given as:

$$\hat{S}(k,p) = \hat{S}^-(k,p) + K(k)\left(y_{tp}(k) - \hat{y}_{tp}^-(k)\right) \tag{27}$$

The covariance is updated as:

$$P(k) = P^-(k) - K(k) P_y(k) K(k)^{\mathrm{T}} \tag{28}$$

According to the aforementioned algorithm, the optimal recursive estimation $\hat{S}(k,p)$ can be obtained. The gain of the UKF $K(k)$ is updated based on the variance of the system error $w(k)$ and the measurement error $v(k)$. In the last step of the algorithm, the error variance $P(k)$ is updated and decreased; due to the fact that the additional information is used, the compensation gain $K(k)$ becomes more accurate than the former. In general, the proposed method is a solution for the high-order moment state estimation of MJNLSs.

From the definitions of the CGF and $S(k,p)$, the high-order moment component form of the augmented state $S(k,p)$ has the same norm as $(q(k))^{\otimes p}$. It is obvious that the transformed state $q(k)$ has the same norm as $x(k)$ when considering (3). Consequently, the high-order moment component form of the augmented state $S(k,p)$ has the same norm as $(x(k))^{\otimes p}$. The proposed estimation algorithm has two parameters: the original state $x(k)$ and the moment p. When $p = 1$ and $p = 2$, the filter is simplified to the recursive mean and mean square state error estimation, respectively, which is similar to the existing results of the recursive MJLS state estimation. In high-order moment cases, more information on the state distribution is used. Under the assumption that the state does not obey a Gaussian distribution, the high-order moment state estimation is indispensable and shows the variation of the Gaussian distribution.

4. **Simulations.** In this section, we present a numerical example to verify the effectiveness of the proposed algorithm.

Consider a two-mode MJNLS (1) as follows:

$$f_1(x(k)) = x(k) - \frac{12}{x(k)} + 2\log(x(k)) + \sin(x(k)),\ f_2(x(k)) = x(k) + \frac{2}{x^2(k)} + 20\cos(k).$$

The covariance of system noise is $H = 10$, the covariance of measurement noise is $Q = 8$, the parameter of the UKF is $L = 4$, and the initial value of the system is: $x_0 = [-1 \quad 1]^{\mathrm{T}}$. Then the true trajectory and the trajectory calculated by the high-order moment UKF for $p = 1$ based on the proposed algorithm are depicted in Figure 1.

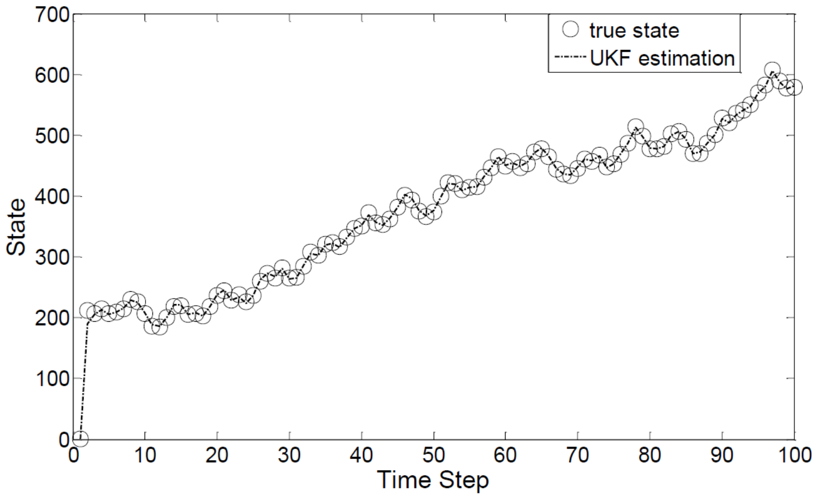

FIGURE 1. The estimated and true value of system for the moment $p = 1$

Figure 1 shows that the trajectory of the estimation tracks the true values when $p = 1$. Moreover, the mean value estimation ($p = 1$) of the state is just a special case of the high-order moment. Consequently, the high-order moment UKF is achievable as the performance for the moment $p = 1$ is effective.

To demonstrate the effect of the moment order p on the filtering performance, the root mean square errors (RMSEs) of different moment orders are illustrated in Figure 2.

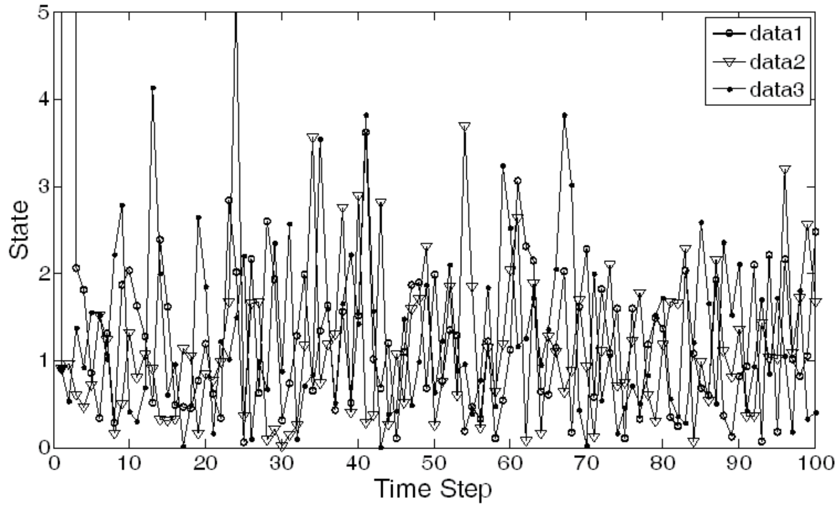

Figure 2. The RMSEs of different moment orders for the MJNLS

Table 1. The average of the RMSEs of different moment orders for the MJNLS

Moment p	RMSE
$p = 1$	2.0346
$p = 2$	2.1908
$p = 3$	2.5122

Figure 2 shows the RMSEs of different moments p for the MJNLS. It is evident that the RMSEs are higher when the order of the moment p is higher. The reason is that for a higher order moment, the RMSEs are also accumulated and the high-order moment component form of the state estimation is imprecise. Compared with the existing results of the state estimation, the estimation of the state in a high-order moment component form provides additional information hidden in the GGD of the state. Consequently, the accuracy is worse for the higher-order moment cases. The same tendency is also shown in Table 1.

5. **Conclusions.** In this study, we proposed a high-order moment UKF algorithm for MJNLSs that can be applied to solving nonlinear GGD dynamic state space problems. Based on the CGF, a deterministic system with a high-order moment component form transformed from the original system is obtained. In this manner, the high-order moment information is available in the state estimation. Therefore, a UKF based on the high-order moment is designed and takes account of the nonlinearity and high-order moment of the GGD. Due to the accumulated error from each moment, the higher the order of the moment is, the worse the performance of the recursive estimation is. A numerical example is provided to illustrate the effectiveness of the proposed technique. Moreover, the transition probability of MJNLS in this paper is completely known, the more general situation which transition probability partly unknown will be considered in further work.

Acknowledgment. This work is partially supported by the National Natural Science Foundation of China (61722306, 61833007, and 61473137).

REFERENCES

[1] H. Cox, On the estimation of state variables and parameters for noisy dynamic systems, *IEEE Trans. Automatic Control*, vol.9, no.1, pp.5-12, 1964.
[2] J. Llinas and D. L. Hall, *Handbook of Multisensor Data Fusion*, CRC Press, Boca Raton, FL, 2008.

[3] W. Zhang and R. Sun, Research on performance comparison of EKF and UKF and their application, *Journal of Nanjing University of Science and Technology*, vol.39, no.5, pp.614-618, 2015.

[4] S. Julier, J. Uhlmann and H. F. Durrant-Whyte, A new method for the nonlinear transformation of means and covariances in filters and estimators, *IEEE Trans. Automatic Control*, vol.45, no.3, pp.477-482, 2000.

[5] M. Liu and G. Chang, Gravity matching aided inertial navigation technique based on marginal robust unscented Kalman filter, *Mathematical Problems in Engineering*, vol.2015, 2015.

[6] W. Li, G. Wei, F. Han et al., Weighted average consensus-based unscented Kalman filtering, *IEEE Trans. Cybernetics*, vol.46, no.2, pp.558-567, 2016.

[7] Y. Zhao, B. Liang and S. Iwnicki, Friction coefficient estimation using an unscented Kalman filter, *Vehicle System Dynamics*, vol.52, no.sup1, pp.220-234, 2014.

[8] W. Li and Y. Jia, Consensus-based distributed multiple model UKF for jump Markov nonlinear systems, *IEEE Trans. Automatic Control*, vol.57, no.1, pp.227-233, 2012.

[9] Z. Liu and J. Wang, Interacting multiple sensor unscented Kalman filter, *Proc. of the 10th World Congress on Intelligent Control and Automation*, Bejing, China, pp.4409-4413, 2012.

[10] L. Li and Y. Xia, Unscented Kalman filter over unreliable communication networks with Markovian packet dropouts, *IEEE Trans. Automatic Control*, vol.58, no.12, pp.3224-3230, 2013.

[11] H. Zhang, G. Feng and C. Han, Linear estimation for random delay systems, *Systems & Control Letters*, vol.60, no.7, pp.450-459, 2011.

[12] V. Chow, W. Jiang and J. V. Li, *Does VIX Truly Measure Return Volatility?*, Social Science Electronic Publishing, 2014.

[13] X. Luan, B. Huang and F. Liu, Higher order moment stability region for Markov jump systems based on cumulant generating function, *Automatica*, vol.93, pp.389-396, 2018.

[14] H. Zhou, H. Huang, H. Zhao et al., Adaptive unscented Kalman filter for target tracking in the presence of nonlinear systems involving model mismatches, *Remote Sensing*, vol.9, no.7, pp.657-677, 2017.