ICIC Express Letters
Volume **13**, Number **7**, July **2019**

ICIC International ©2019 ISSN 1881-803X
pp. 547–555

NUMERICAL ANALYSIS OF FRACTIONAL-ORDER VARIABLE SECTION CANTILEVER BEAMS

Yujing Feng[1], Lechun Liu[1] and Yiming Chen[1,2,3]

[1]School of Science
Yanshan University
No. 438, West Hebei Avenue, Qinhuangdao 066004, P. R. China
695985374@qq.com

[2]LE STUDIUM Loire Valley Institute for Advanced Studies
1 rue Dupanloup 45000 Orléans, France

[3]LAME Laboratory
INSA Centre Val de Loire
3 rue de la Chocolaterie CS 23410-41034 Blois Cedex, France

Received December 2018; accepted February 2019

Abstract. *A new method for solving the bending vibration equation of viscoelastic cantilever beam with variable cross section is presented in this paper. The fractional order governing equation is established based on the constitutive relation of viscoelastic material, the theory of variable cross-section beam and the motion equation of beam. Numerical results for viscoelastic beams with two different cross sections are given, and the validity of the algorithm is verified. The displacements of variable cross-section beams composed of two kinds of viscoelastic materials under different uniform loads, different harmonic loads and different taper ratios of cross-sections are compared. Therefore, the bending vibration of viscoelastic cantilever beam with variable cross-section is analyzed theoretically.*

Keywords: Constitutive relation of fractional viscoelastic material, Viscoelastic cantilever beam, Variable cross-section beam, Bernstein polynomial, Operator matrix, Numerical solution

1. Introduction.

The beam whose section changes along the axis is called variable cross-section beam. This kind of beam has great applications in material mechanics [1], mechanical engineering [2] and civil engineering [3]. With the research of advanced viscoelastic materials in recent years, its application value is getting higher and higher.

In many papers, straight beams with a rectangular [4] or circular [5] cross-section are usually selected as the research content. In recent years, the establishment of models for variable cross-section beams [6, 7] and numerical solution algorithms [8] have become more and more mature.

With the rapid development of fractional order, most of the constitutive equations of viscoelastic materials adopt fractional order equation [9, 10]. Combined with the theoretical knowledge of beams and the constitutive relation of viscoelastic materials, the fractional order governing equations of various beams are derived. Numerical simulations are carried out in time domain by various numerical methods. The main analysis methods include variational iteration method [11], multi-scale method [12], finite element method [13] and so on.

DOI: 10.24507/icicel.13.07.547

In the field of numerical computation, the polynomial algorithm for numerical solution of fractional equations is relatively mature. Legendre polynomials [14], Chebyshev polynomials [15], Bernoulli wavelets [16] and Bernstein polynomials [17] are often used as the basis functions for numerical simulation.

This article will numerically analyze the bending vibration of a viscoelastic variable section cantilever beam. By using the fractional constitutive equation of viscoelastic material and the Hamiltonian principle, the bending vibration equation of the variable cross-section cantilever beam is established. Using the Bernstein polynomial to solve the problem, through the study of numerical results, find the material, section, and the impact of the load on the displacement. The results show that the larger the uniform load is, the larger the displacement is; the larger the frequency of harmonic load is, the larger the displacement is; the larger the taper ratio of variable cross-section beam is, the larger the displacement is. When the same beam composed of polybutadiene and butadiene B252 receives the same load, the displacement of butyl B252 is smaller than that of polybutadiene, and the corresponding damping is larger.

This paper is organized as follows. In Section 2 we introduce the definition of fractional order, the viscoelastic constitutive relation and the concepts of Bernstein polynomials. In Section 3 the fractional bending vibration equation of a variable cross-section cantilever beam is derived. In Section 4 the algorithm for solving equations by using Bernstein polynomials is given. In Section 5 several numerical examples of beams with variable cross-section are given. Finally, Section 6 concludes the paper.

2. **Preliminary Knowledge.**

The fractional differential theory, viscoelastic constitutive equation and Bernstein polynomials are introduced below, which provides a theoretical basis for this study.

2.1. **Basic knowledge of fractional calculus.**

In order to better study the performance of materials, scholars use fractional calculus to establish constitutive equation. At present, the definition of fractional calculus is not unified, and we mainly introduce the definition of Caputo fractional calculus.

Definition 2.1. *Fractional Caputo differential definition*

$$ {}_a^C D_t^\alpha f(t) = \begin{cases} \dfrac{d^m f(t)}{dt^m}, & \alpha = m \in N^+ \\ \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_a^t \dfrac{f^{(m)}(T)}{(t-T)^{\alpha-m+1}} dT, & 0 \le m-1 < \alpha < m \end{cases} \tag{1}$$

where m is the smallest positive integer greater than α.

For Caputos derivative we have

$$ {}_a^C D_t^\alpha C = 0 \tag{2}$$

where C is a constant.

$$ {}_a^C D_t^\alpha x^m = \begin{cases} 0, & m = 0 \\ \dfrac{\Gamma(m+1)}{\Gamma(m+1-\alpha)} x^{m-\alpha}, & m = 1, 2, \dots \end{cases} \tag{3}$$

where $0 < \alpha \le 1$.

2.2. **Fractional viscoelastic constitutive equation.**

For viscoelastic analysis, mechanical model can be used to study. Most mechanical models are composed of a combination of spring and sticky pot. Viscoelasticity is a material that is a polymer between an ideal elasticity and an ideal viscous material with a stress-strain between the two. This paper adopts Kelvin-Voigt model of the form

$$ \sigma(x,t) = E_0 \varepsilon(x,t) + E_1 \frac{d^\alpha \varepsilon(x,t)}{dt^\alpha} \tag{4}$$

2.3. **The definition of Bernstein polynomials basis.** Bernstein polynomials of degree are defined by

$$B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i} \qquad (i = 1, 2, \ldots, n) \tag{5}$$

By using the binomial expansion of $(1-x)^{n-i}$, Equation (5) can be expressed as

$$B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i} = \sum_{k=0}^{n-i} \binom{n}{i} \binom{n-i}{k} x^{i+k} \tag{6}$$

Now, we define

$$\Phi(x) = [B_{0,n}(x), B_{1,n}(x), \ldots, B_{n,n}(x)]^T \tag{7}$$

Equation (7) is expressed in the form of a matrix

$$\Phi(x) = AT_n(x) \tag{8}$$

where

$$A = \begin{bmatrix} (-1)^0 \binom{n}{0} & (-1)^1 \binom{n}{0} \binom{n-0}{1} & \cdots & (-1)^1 \binom{n}{0} \binom{n-0}{n-0} \\ 0 & (-1)^0 \binom{n}{1} \binom{n-1}{0} & \cdots & (-1)^{n-1} \binom{n}{1} \binom{n-1}{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (-1)^0 \binom{n}{n} \end{bmatrix}$$

$$T_n(x) = \begin{bmatrix} 1 \\ x \\ \vdots \\ x^n \end{bmatrix}$$

A is called the coefficient matrix of Bernstein polynomial. Because the coefficient matrix is reversible, we can get

$$T_n(x) = A^{-1}\Phi(x) \tag{9}$$

3. **The Establishment of Bending Vibration Equation.** As shown in Figure 1, x is the geometric axis, y is the vertical axis, and z is the horizontal direction. The cross section is circular or rectangular type, and the sectional area decreases along the x-axis direction. The length of the beam is l. The beam deflection is $u(x,t)$, the beam length is ρ, the cross-sectional area is $A(x)$, the transverse load is $f(x,t)$, and the bending moment is $M(x,t)$.

The dynamic equations of beams with variable cross-section are as follows.

$$\rho A(x) \frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial^2 M(x,t)}{\partial x^2} = f(x,t) \tag{10}$$

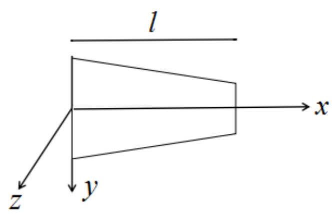


FIGURE 1. Variable section cantilever beam diagram

This paper adopts Kelvin-Voigt model of the form

$$\sigma(x,t) = E_0\varepsilon(x,t) + E_1 D_t^\alpha \varepsilon(x,t) \tag{11}$$

where $\varepsilon(x,t)$ is the axial strain. $\sigma(x,t)$ is the normal cross-section stress. E_0, E_1, α are the parameters of the viscoelastic material. For viscoelastic materials, we selected polybutadiene and butyl B252. The material is polybutadiene, parameter $\alpha = 0.528$, $E_0 = 8.14 \times 10^5$ MPa, $E_1 = 7.31 \times 10^4$ MPa. The material is butyl B252, the order is $\alpha = 0.519$, $E_0 = 1.05 \times 10^6$ MPa, $E_1 = 2.44 \times 10^5$ MPa [18]. D_t^α uses Caputo type fractional differential definition.

The relationship between bending moment and stress is written as

$$M(x,t) = -\int_A z\sigma(x,t)dz \tag{12}$$

For small deformations, the strain-displacement relationship is given by

$$\varepsilon(x,t) = z\frac{\partial^2 u(x,t)}{\partial x^2} \tag{13}$$

According to Equation (13), the beam bending moment is equal to

$$M(x,t) = E_0 I(x)\frac{\partial^2 u(x,t)}{\partial x^2} + E_1 I(x) D_t^\alpha \frac{\partial^2 u(x,t)}{\partial x^2} \tag{14}$$

where $I(x)$ is the moment of inertia of the section given by $\int_{A(x)} z^2 dA(x)$.

Thus, after the substitution of Equation (14) in Equation (10), the bending vibration equation of viscoelastic beam with variable cross-section is obtained

$$\rho A(x)\frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(E_0 I(x)\frac{\partial^2 u(x,t)}{\partial x^2} + E_1 I(x) D_t^\alpha \frac{\partial^2 u(x,t)}{\partial x^2}\right) = f(x,t) \tag{15}$$

A beam with a variable cross-section is a cantilever beam, and the boundary condition and initial condition are

$$u(0,t) = \frac{\partial u(0,t)}{\partial x} = 0$$
$$\frac{\partial^2 u(l,t)}{\partial x^2} = \frac{\partial^3 u(l,t)}{\partial x^3} = 0$$
$$u(x,0) = \frac{\partial u(x,0)}{\partial t} = 0$$

4. Numerical Method of Bernstein Polynomial.

4.1. Function approximation.

We can approximate function $u(x,t) \in L^2([0,1]\times[0,1])$ by an expansion of the form

$$u(x,t) \cong \sum_{i=0}^{n}\sum_{j=0}^{n} u_{i,j} B_{i,n}(x) B_{j,n}(t) = \Phi^T(x) U \Phi(t) \tag{16}$$

where U is the coefficient matrix to be solved

$$U = \begin{bmatrix} u_{00} & u_{01} & \cdots & u_{0n} \\ u_{10} & u_{11} & \cdots & u_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n0} & u_{n1} & \cdots & u_{nn} \end{bmatrix} \tag{17}$$

The U can be calculated by the following formula

$$U = Q^{-1}(\Phi(x), (\Phi(t), u(x,t)))Q^{-1} \tag{18}$$

4.2. **Numerical algorithm.** The first derivative of $\Phi(x)$ can be obtained

$$\Phi'(x) = D\Phi(x) \tag{19}$$

where D is an order matrix, which is called the first order differential operator matrix of the Bernstein polynomial.

Now deriving Equation (8), we can get

$$\Phi'(x) = A \begin{bmatrix} 0 \\ 1 \\ \vdots \\ nx^{n-1} \end{bmatrix} \tag{20}$$

Define the $(n+1)\times n$ matrix $V_{(n+1)\times n}$ and vector $T_n^*(x)$ as

$$V_{(n+1)\times n} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n \end{bmatrix}, \quad T_n^*(x) = \begin{bmatrix} 1 \\ x \\ \vdots \\ x^{n-1} \end{bmatrix} \tag{21}$$

$T_n^*(x)$ is expanded in the form of $\Phi(x)$, and we can get

$$T_n^*(x) = B^*\Phi(x) \tag{22}$$

where

$$B^* = \begin{bmatrix} A_{[1]}^{-1} & A_{[2]}^{-1} & \cdots & A_{[n]}^{-1} \end{bmatrix}^T$$

$A_{[k]}^{-1}$ is kth row of A^{-1}, $k = 1, 2, \ldots, n$, and we have

$$\Phi'(x) = AV_{(n+1)\times n}B^*\Phi(x) \tag{23}$$

Converting $\frac{\partial u(x,t)}{\partial t}$ into a matrix form, we obtain

$$\frac{\partial u(x,t)}{\partial t} \cong \frac{d\left(\Phi^T(x)U\Phi(t)\right)}{dx} = \Phi^T(x)U\frac{d(\Phi(t))}{dx} = \Phi^T(x)\,UD\Phi(x) \tag{24}$$

where

$$D = AV_{(n+1)\times n}B^* \tag{25}$$

By deducing Equation (8) again, we can get

$$\Phi''(t) = D^2\Phi(t) \tag{26}$$

then

$$\frac{\partial^2 u(x,t)}{\partial t^2} \cong \frac{\partial^2\left(\Phi^T(x)U\Phi(t)\right)}{\partial t^2} = \Phi^T(x)(D)^2U\Phi(t) = \Phi^T(x)\big(AV_{(n+1)\times n}B^*\big)^2U\Phi(t) \tag{27}$$

Similarly,

$$\begin{aligned}
\frac{\partial^2}{\partial x^2}\left(E_0I(x)\frac{\partial^2 u(x,t)}{\partial x^2}\right) &\cong \frac{\partial^2}{\partial x^2}\left(E_0I(x)\frac{\partial^2(\Phi^T(x)U\Phi(t))}{\partial x^2}\right) \\
&= \frac{\partial^2}{\partial x^2}\left(E_0I(x)\Phi^T(x)\left(D^T\right)^2\Phi(t)\right) \\
&= E_0\Big(I''(x)\Phi^T(x)\left(D^T\right)^2U\Phi(t) + 2I'(x)\Phi^T(x)\left(D^T\right)^3U\Phi(t) \\
&\quad + I(x)\Phi^T(x)\left(D^T\right)^4U\Phi(t)\Big) \\
&= E_0\bigg(I''(x)\Phi^T(x)\Big(\big(AV_{(n+1)\times n}B^*\big)^T\Big)^2U\Phi(t) \\
&\quad + 2I'(x)\Phi^T(x)\Big(\big(AV_{(n+1)\times n}B^*\big)^T\Big)^3U\Phi(t)
\end{aligned}$$

$$+ I(x)\Phi^T(x)\Big(\big((AV_{(n+1)\times n}B^*)^T\big)^4 U\Phi(t)\Big) \tag{28}$$

Now $D_t^\alpha u(x,t)$ is expressed as a matrix; combined with Equation (8), we have

$$\begin{aligned}
D_t^\alpha u(x,t) &\cong D_t^\alpha \Phi^T(x)U\Phi(t)\\
&= \Phi^T(x)\,UD_t^\alpha\Phi(t)\\
&= \Phi^T(x)\,UAD_t^\alpha T_n(t)\\
&= \Phi^T(x)\,UAD_t^\alpha \begin{bmatrix} 1\\ t\\ \vdots\\ t^n\end{bmatrix}\\
&= \Phi^T(x)\,UA\begin{bmatrix} 0 & 0 & \cdots & 0\\ 0 & \frac{\Gamma(2)}{\Gamma(2-\alpha)}t^{-\alpha} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}t^{-\alpha}\end{bmatrix}\begin{bmatrix} 1\\ t\\ \vdots\\ t^n\end{bmatrix}\\
&= \Phi^T(x)\,UAMA^{-1}\Phi(t)
\end{aligned} \tag{29}$$

where

$$M = \begin{bmatrix} 0 & 0 & \cdots & 0\\ 0 & \frac{\Gamma(2)}{\Gamma(2-\alpha)}t^{-\alpha} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}t^{-\alpha}\end{bmatrix}$$

Then

$$\begin{aligned}
\frac{\partial^2}{\partial x^2}\left(E_1 I(x) D_t^\alpha \frac{\partial^2 u(x,t)}{\partial x^2}\right) \cong E_1\Big(& I''(x)\Phi^T(x)\Big(\big(AV_{(n+1)\times n}B^*\big)^T\Big)^2 UAMA^{-1}\Phi(t)\\
&+ 2I'(x)\Phi^T(x)\Big(\big(AV_{(n+1)\times n}B^*\big)^T\Big)^3 UAMA^{-1}\Phi(t)\\
&+ I(x)\Phi^T(x)\Big(\big(AV_{(n+1)\times n}B^*\big)^T\Big)^4 UAMA^{-1}\Phi(t)\Big)
\end{aligned} \tag{30}$$

Finally, insert Equation (27), Equation (28) and Equation (30) into Equation (15). We have

$$\begin{aligned}
&\rho A(x)\Phi^T(x)\big(AV_{(n+1)\times n}B^*\big)^2\Phi(t) + E_0\Big(I''(x)\Phi^T(x)\Big(\big(AV_{(n+1)\times n}B^*\big)^T\Big)^2 U\Phi(t)\\
&+ 2I'(x)\Phi^T(x)\Big(\big(AV_{(n+1)\times n}B^*\big)^T\Big)^3 U\Phi(t) + I(x)\Phi^T(x)\Big(\big(AV_{(n+1)\times n}B^*\big)^T\Big)^4 U\Phi(t)\Big)\\
&+ E_1\Big(I''(x)\Phi^T(x)\Big(\big(AV_{(n+1)\times n}B^*\big)^T\Big)^2 UAMA^{-1}\Phi(t) + 2I'(x)\Phi^T(x)\Big(\big(AV_{(n+1)\times n}B^*\big)^T\Big)^3\\
&UAMA^{-1}\Phi(t) + I(x)\Phi^T(x)\Big(\big(AV_{(n+1)\times n}B^*\big)^T\Big)^4 UAMA^{-1}\Phi(t)\Big) = f(x,t)
\end{aligned} \tag{31}$$

Using the collocation method, the discrete variable (x,t) is (x_i,t_i), and Equation (31) is transformed into a linear system of equations. By using Matlab and least square method, the solution coefficient $u_{i,j}$, $(i=0,1,\dots,n;\ j=0,1,\dots,n)$ can be obtained, and thus the numerical solution of the original problem can be obtained.

5. **Numerical Examples.** The solid line in the graph below indicates that the material used is polybutadiene, and the dashed line indicates that the material used is butyl B252.

For cantilever beam parameters $l = 1$ m, $A_0 = 10$ cm^2, $I_0 = 20.833$ cm^4, $\rho = 8000$ kg/m^3, in this paper, two kinds of variable section viscoelastic beams are studied. The first type is a circular section beam with parameters of

$$A(x) = A_0(1 - cx/l)^2, \quad I(x) = I_0(1 - cx/l)^4$$

where $0 \le c < 1$ is the taper ratio of the variable section beam.

The second type is a rectangular beam with parameters of

$$A(x) = A_0(1 - cx/l), \quad I(x) = I_0(1 - cx/l)^3$$

5.1. **Round cross-section beam.** When the cantilever beams are subjected to loads of 10Heaviside(t), 30Heaviside(t) and 50Heaviside(t), the solution of displacement when $c = 0.5$, $t = 0.5455$ can be obtained as Figure 2(a). When the cantilever beams are subjected to loads of $\sin\left(\frac{\pi}{2}t\right)$, $\sin(\pi t)$ and $\sin\left(\frac{3\pi}{2}t\right)$, the solution of displacement when $c = 0.5$, $t = 0.5455$ can be obtained as Figure 2(b). When the load is 10Heaviside(t), $t = 0.5455$, the taper ratio is 0.25, 0.5, and 0.75 when the displacement solution is as Figure 2(c).

5.2. **Rectangular cross-section beam.** When the cantilever beams are subjected to loads of 10Heaviside(t), 30Heaviside(t) and 50Heaviside(t), the solution of displacement when $c = 0.5$, $t = 0.5455$ can be obtained as Figure 3(a). When the cantilever beams are subjected to loads of $\sin\left(\frac{\pi}{2}t\right)$, $\sin(\pi t)$ and $\sin\left(\frac{3\pi}{2}t\right)$, the solution of displacement when

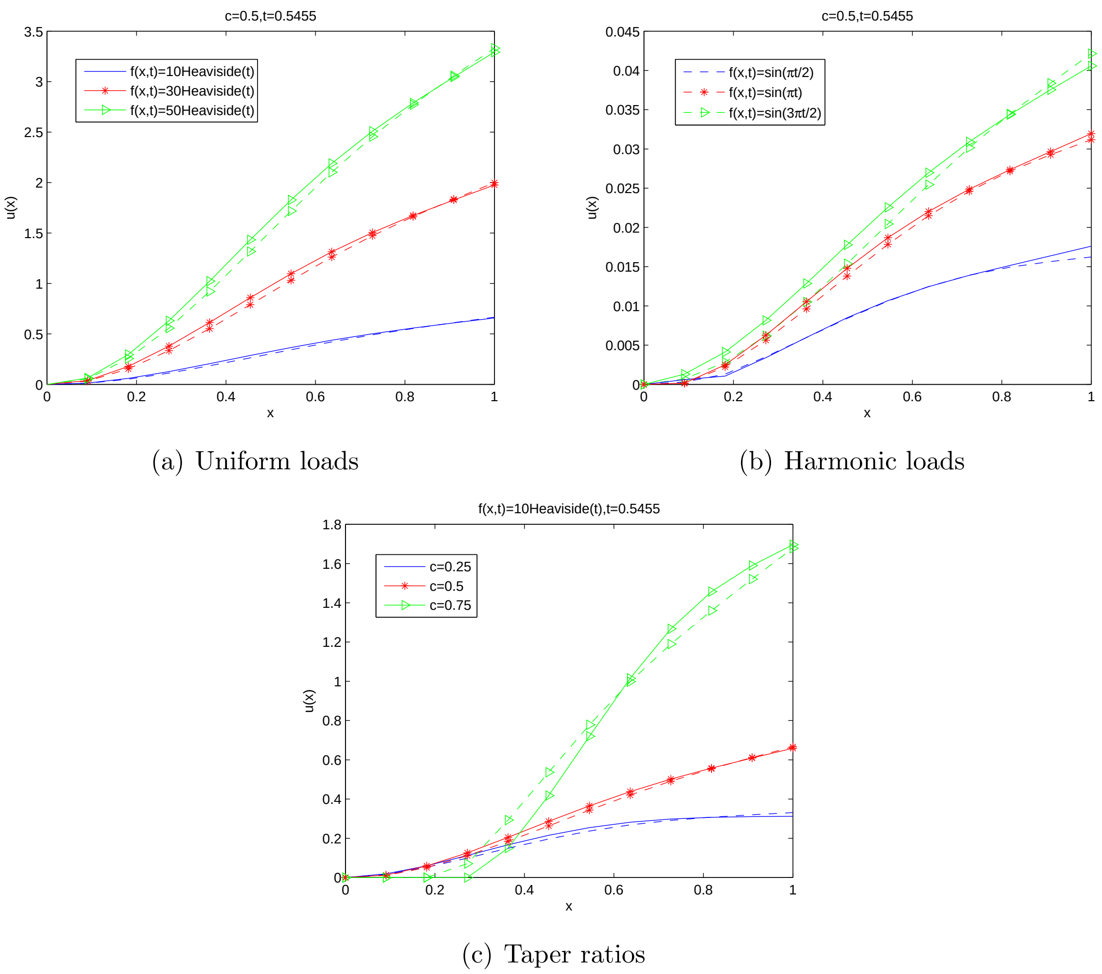
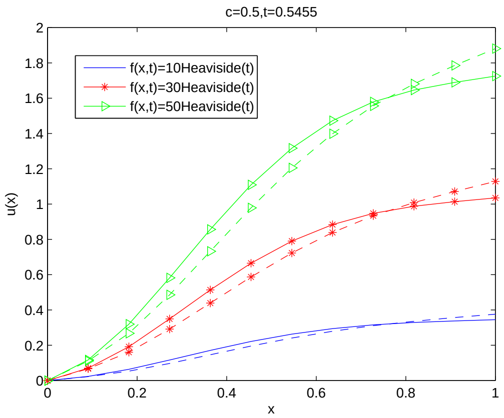
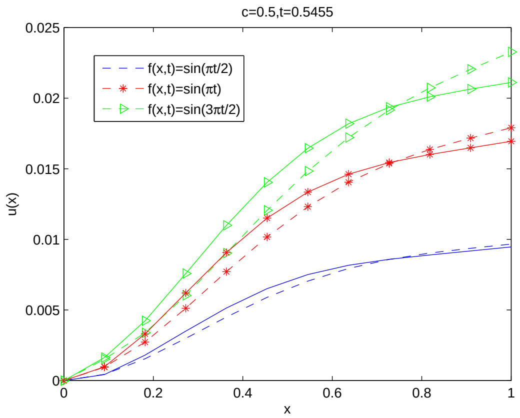


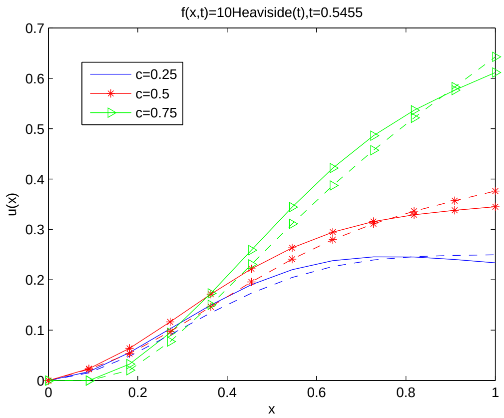
Figure 2. Displacement solutions of round cross-section beam



(a) Uniform loads



(b) Harmonic loads



(c) Taper ratios

FIGURE 3. Displacement solutions of rectangular cross-section beam

$c = 0.5$, $t = 0.5455$ can be obtained as Figure 3(b). When the load is 10Heaviside(t), $t = 0.5455$, the taper ratio is 0.25, 0.5, and 0.75 when the displacement solution is as Figure 3(c).

It can be seen from Figure 2 that the displacement of viscoelastic cantilever beam increases with the increase of uniform load. It can be seen from Figure 3 that the displacement of the viscoelastic cantilever beam increases with the increase of the harmonic load frequency. It can be seen from Figure 3 that the displacement of viscoelastic cantilever beams increases with increasing taper ratio of variable cross-section. From Figure 2 and Figure 3, it can be seen that the beam formed by polybutadiene material beam is larger than that of butyl B252, and the corresponding damping is relatively small.

6. **Conclusion.** The fractional order governing equations are established according to the constitutive relations of viscoelastic materials, the theory of variable cross-section beams and the equations of motion of beams. In this paper, a new numerical algorithm for solving the fractional order equation of bending vibration of a variable cross-section cantilever beam is established in the time domain. The numerical examples of two kinds of variable cross-section beams are given, and the displacement solutions of beams composed of two viscoelastic materials under different loads are given, which verifies the effectiveness of the algorithm.

The effects of uniform load, harmonic load, variable cross section taper ratio and viscoelastic material on displacement are analyzed. Through image observation, it can be found that with the increase of uniform load, the displacement of beams increases. As the frequency of harmonic load increases, the displacement of beams increases. With the increase of the taper ratio of the beam with variable cross-section, the displacement of the beam becomes larger. The displacement of a variable cross section beam composed of polybutadiene and butadiene B252 is compared. When the same load and the same cross section, the displacement of butyl B252 is less than the displacement of polybutadiene, and the corresponding damping is also larger.

In this paper, only the displacement of cantilever beam is considered. In future work, rotation angle is also taken into account. Then the fractional order equations are constructed. By solving the equation numerically, the properties of the material can be analyzed.

Acknowledgements. This work is supported by the Natural Science Foundation of Hebei Province (A2017203100) in China and the LE STUDIUM RESEARCH PROFESSORSHIP award of Centre-Val de Loire region in France.

REFERENCES

[1] M. G. Gnay and T. Timarci, Static analysis of thin-walled laminated composite closed-section beams with variable stiffness, *Composite Structures*, vol.182, 2017.
[2] M. A. Carruth and J. M. Allwood, The development of a hot rolling process for variable cross-section I-beams, *Journal of Materials Processing Technology*, vol.212, no.8, pp.1640-1653, 2012.
[3] S. J. E. Dias and J. A. O. Barros, NSM shear strengthening technique with CFRP laminates applied in high T cross section RC beams, *Composites Part B: Engineering*, vol.114, pp.256-267, 2017.
[4] I. S. Choi, G. W. Jang, S. Choi et al., Higher order analysis of thin-walled beams with axially varying quadrilateral cross sections, *Computers & Structures*, vol.179, pp.127-139, 2017.
[5] S. M. Abdelghany, K. M. Ewis, A. A. Mahmoud et al., Vibration of a circular beam with variable cross sections using differential transformation method, *Beni-Suef University Journal of Basic and Applied Sciences*, vol.4, no.3, pp.185-191, 2015.
[6] A. J. Hull, D. Perez and D. L. Cox, An analytical model of a curved beam with a T shaped cross section, *Journal of Sound Vibration*, vol.416, pp.29-54, 2018.
[7] R. Friedrich, R. Lammering and M. Rösner, On the modeling of flexure hinge mechanisms with finite beam elements of variable cross section, *Precision Engineering*, vol.38, no.4, pp.915-920, 2014.
[8] V. S. Zhernakov, V. P. Pavlov and V. M. Kudoyarova, Spline-method for numerical calculation of natural-vibration frequency of beam with variable cross-section, *Procedia Engineering*, vol.206, pp.710-715, 2017.
[9] M. D. Paola, R. Heuer and A. Pirrotta, Fractional visco-elastic Euler-Bernoulli beam, *International Journal of Solids & Structures*, vol.50, nos.22-23, pp.3505-3510, 2013.
[10] S. M. S. Bahraini, M. Farid and E. Ghavanloo, Large deflection of viscoelastic beams using fractional derivative model, *Journal of Mechanical Science & Technology*, vol.27, no.4, pp.1063-1170, 2013.
[11] O. Martin, A modified variational iteration method for the analysis of viscoelastic beams, *Applied Mathematical Modelling*, vol.40, nos.17-18, pp.7988-7995, 2016.
[12] X. Q. He, M. Rafiee, S. Mareishi et al., Large amplitude vibration of fractionally damped viscoelastic CNTs/fiber/polymer multiscale composite beams, *Composite Structures*, vol.131, pp.1111-1123, 2015.
[13] C. Chazal and R. M. Pitti, Integral approach for time dependent materials using finite element method, *Journal of Theoretical & Applied Mechanics*, vol.4, no.4, pp.1029-1048, 2011.
[14] Z. Meng, M. Yi, J. Huang et al., Numerical solutions of nonlinear fractional differential equations by alternative Legendre polynomials, *Applied Mathematics & Computation*, vol.336, pp.454-464, 2018.
[15] J. Xie, Z. Yao, H. Gui et al., A two-dimensional Chebyshev wavelets approach for solving the Fokker-Planck equations of time and space fractional derivatives type with variable coefficients, *Applied Mathematics & Computation*, vol.332, pp.197-208, 2018.
[16] J. Wang, T. Z. Xu, Y. Q. Wei et al., Numerical solutions for systems of fractional order differential equations with Bernoulli wavelets, *International Journal of Computer Mathematics*, pp.1-21, 2018.
[17] Y. Chen, L. Liu, B. Li et al., Numerical solution for the variable order linear cable equation with Bernstein polynomials, *Applied Mathematics & Computation*, vol.238, no.7, pp.329-341, 2014.
[18] R. L. Bagley and P. J. Torvik, A theoretical basis for the application of fractional calculus to viscoelasticity, *Journal of Rheology*, vol.27, no.3, 1983.