ICIC Express Letters
Volume **13**, Number **7**, July **2019**
ICIC International ©2019 ISSN 1881-803X
pp. **645–652**

CLOSED-FORM FORMULAS FOR CONTINUOUS/DISCRETE-TIME PIDAJ CONTROLLER'S PARAMETERS

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Received December 2018; accepted March 2019

Abstract. *This paper focuses on establishing the closed-form formulas to design a proportional-integral-derivative-acceleration-jerk (PIDAJ) controller for fourth-order plants. After plant modeling, the analog PIDAJ controller in continuous-time domain can be designed by utilizing the proposed formulas in a vector-matrix form. Based on bilinear transform, the digital PIDAJ controller in discrete-time domain can then be achieved. The use of the proposed formulas to determine the PIDAJ controller's parameters for controlling the air-fuel ratio of an engine is described as an application example to confirm their feasibility. MATLAB simulation results confirm the consistency between step responses of the example system controlled by the designed controllers in continuous-time domain and discrete-time domain.*

Keywords: PIDAJ controller, Continuous-time, Controller design, Discrete-time, Fourth-order plant, Bilinear transform

1. **Introduction.** Because of its acceptable performances, a proportional-integral-derivative (PID) controller is broadly utilized in continuous process controls. According to the type and order, most type-0 plants in industrial process sector consist of either one first-order lag plus dead time or three to five first-order lag components [1]. Generally, the PID controller is accurately employed in second-order plants to achieve optimum solutions. Unfortunately, the transient response characteristics are frequently dissatisfied in case of controlling both third-order plants and higher-order plants by utilizing the PID controller for the reason that the number of zeros in the PID controller is less than the order of the plant [2]. The PID-based control of higher-order plants provides zero state-state error, but the transient responses cannot meet desired specifications. In order to minimize this limitation, a proportional-integral-derivative-acceleration (PIDA) controller implemented by adding one zero to the PID controller has been proposed [3,4]. However, applying the PIDA controller to controlling fourth-order plants still produces unacceptable transient responses. Therefore, the aim of this paper is to propose a technique to design a new controller called 'proportional-integral-derivative-acceleration-jerk' (PIDAJ) controller that is suitable for controlling the fourth-order plant. The proposed technique is based on the closed-form formulas to provide ease of finding controller parameters not only in continuous-time (CT) systems but also in discrete-time (DT) systems, since using the closed-form formulas offers the straightforward solution by reducing the calculations required in controller design [5,6].

The rest of this paper is organized as follows. The proposed closed-form formulas for designing the PIDAJ controller are explained in Section 2. Section 3 demonstrates how to apply the proposed closed-form formulas for finding the parameters of the PIDAJ

DOI: 10.24507/icicel.13.07.645

controller for an engine air-fuel ratio control, and the MATLAB simulation results are also included in this section. The conclusions and possible future work are given in Section 4.

2. **Proposed Closed-Form Formulas for PIDAJ Controller.** Figure 1 shows a general architecture of the control system used to describe the controller design approach. The control system design involves finding the parameters of both the CT PIDAJ controller $K(s)$ and DT PIDAJ controller $K(z)$ for the fourth-order plant $G(s)$. These required parameters should provide the transient responses to agree with the desired specifications including settling time (t_s) and percent overshoot ($P.O.$).

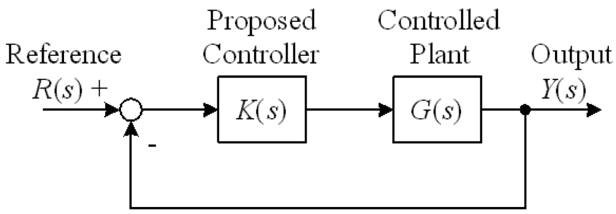


FIGURE 1. General structure of control system

Based on the procedures for digital controller design in [7], three possible ways to obtain the DT controller are discretization of analog controller, sampled-data design from analog model, and discrete-time based design from digital model. In this article, the formulas for finding simultaneously the proportional gain K_p, integral gain K_i, derivative gain K_d, acceleration gain K_a, and jerk gain K_j of the PIDAJ controllers in both CT and DT systems are explained. Once, the satisfied CT PIDAJ controller with satisfied transient response characteristic is obtained. Only using bilinear transform to discretize this CT PIDAJ controller, then the DT PIDAJ controller can be easily achieved.

2.1. **Continuous-time frameworks.** Let the transfer function of the plant $G(s)$ be assumed as

$$G(s) = \frac{b_1 s + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}, \tag{1}$$

where a_3, a_2, a_1, a_0 and b_1, b_0 are the numerator and denumerator coefficients of the given plant model, respectively. The transfer function of the PIDAJ controller can be stated as

$$K(s) = K\frac{(s+a)(s+b)(s+c)(s+d)}{s} = \frac{K_j s^4 + K_a s^3 + K_d s^2 + K_p s + K_i}{s}, \tag{2}$$

where a, b, c and d are the zeros of PIDAJ controller. Then, the transfer function of the closed-loop system in Figure 1 is

$$\frac{Y(s)}{R(s)} = \frac{K(s)G(s)}{1 + K(s)G(s)}. \tag{3}$$

Hence, the actual characteristic equation can be written as

$$\left\{ \begin{aligned} \frac{\underset{\text{actual}}{F(s)}}{(1+K_j b_1)} &= s^5 + \frac{(a_3 + K_a b_1 + K_j b_0)}{(1+K_j b_1)} s^4 + \cdots + \frac{(a_2 + K_d b_1 + K_a b_0)}{(1+K_j b_1)} s^3 \\ &+ \cdots + \frac{(a_1 + K_p b_1 + K_d b_0)}{(1+K_j b_1)} s^2 + \cdots + \frac{(a_0 + K_p b_0 + K_i b_1)}{(1+K_j b_1)} s \\ &+ \cdots + \frac{(K_i b_0)}{(1+K_j b_1)}. \end{aligned} \right. \tag{4}$$

The desired specifications of the control system that are designed are frequently stated in characteristics of transient and steady state responses, which are exhibited by the dominant closed-loop poles as follows:

$$P.O. = e^{\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)} \times 100\%,\ t_s = -\ln\left(0.02\sqrt{1-\zeta^2}\right)\Big/\zeta\omega_n. \tag{5}$$

From (5) in terms of $P.O.$, the damping ratio can be obtained as

$$\zeta = \sqrt{\left[\ln\left(\frac{P.O.}{100}\right)\right]^2 \Big/ \left\{\pi^2 + \left[\ln\left(\frac{P.O.}{100}\right)\right]^2\right\}}, \tag{6}$$

and the undamped natural frequency can be achieved from the given $t_s(\pm 2\%)$ as

$$\omega_n = -\ln\left(0.02\sqrt{1-\zeta^2}\right)\Big/\zeta t_s. \tag{7}$$

Hence, the locations of the dominant closed-loop poles are

$$q, \hat{q} = s_{d\pm} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}. \tag{8}$$

The objective of the proposed design is to find the K_p, K_i, K_d, K_a and K_j of the PIDAJ controller to provide that all closed-loop poles or the roots of characteristic equation in (3) are placed at the locations for obtaining desired output responses. These desired pole locations can be expressed by the designed characteristic equation as

$$\begin{cases} \underset{\text{design}}{F(s)} = \left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)(s+R)(s+r)(s+p) = 0 \\ \quad = (s+q)(s+\hat{q})(s+R)(s+r)(s+p) \\ \quad = s^5 + (p+r+R+2\zeta\omega_n)s^4 + \cdots + \left\{(r+R)p + Rr + 2\zeta\omega_n(p+r+R) + \omega_n^2\right\}s^3 \\ \quad\quad + \cdots + \left\{Rrp + 2\zeta\omega_n\left[(r+R)p + Rr\right] + \omega_n^2(p+r+R)\right\}s^2 \\ \quad\quad + \cdots + \left\{2\zeta\omega_n Rrp + \omega_n^2\left[(r+R)p + Rr\right]\right\}s + \omega_n^2 Rrp. \end{cases} \tag{9}$$

Based on the design region proposed in [3], the real pole $(S+R)$ can be chosen by neglecting the poles $(s+r)$ and $(s+p)$. Equating the coefficients with similar power series between (4) and (9) yields:

$$\begin{cases} \underline{s^4}| : (a_3 + K_a b_1 + K_j b_0)/(1 + K_j b_1) = (p + r + R + 2\zeta\omega_n), \\ \underline{s^3}| : (a_2 + K_d b_1 + K_a b_0)/(1 + K_j b_1) = \left\{(r+R)p + Rr + \cdots + 2\zeta\omega_n(p+r+R) + \omega_n^2\right\}, \\ \underline{s^2}| : (a_1 + K_p b_1 + K_d b_0)/(1 + K_j b_1) = \left\{Rrp + \cdots + 2\zeta\omega_n\left[(r+R)p + Rr\right] + \cdots + \omega_n^2(p+r+R)\right\}, \\ \underline{s^1}| : (a_0 + K_p b_0 + K_i b_1)/(1 + K_j b_1) = \left\{2\zeta\omega_n Rrp + \cdots + \omega_n^2\left[(r+R)p + Rr\right]\right\}, \\ \underline{s^0}| : (K_i b_0)/(1 + K_j b_1) = \omega_n^2 Rrp. \end{cases} \tag{10}$$

The simple linear system can then be obtained in a vector-matrix form $Ax = b$ as follows:

$$\begin{cases} A = \begin{bmatrix} 0 & 0 & 0 & b_1 & \{b_0 - b_1(p + r + R + 2\zeta\omega_n)\} \\ 0 & 0 & b_1 & b_0 & -b_1\left\{(r+R)p + Rr + \cdots + 2\zeta\omega_n(p+r+R) + \omega_n^2\right\} \\ b_1 & 0 & b_0 & 0 & -b_1\left\{Rrp + \cdots + 2\zeta\omega_n\left[(r+R)p + Rr\right] + \cdots + \omega_n^2(p+r+R)\right\} \\ b_0 & b_1 & 0 & 0 & -b_1\left\{2\zeta\omega_n Rrp + \cdots + \omega_n^2\left[(r+R)p + Rr\right]\right\} \\ 0 & b_0 & 0 & 0 & -b_1(\omega_n^2 Rrp) \end{bmatrix}, \\ x = \begin{bmatrix} K_p \\ K_i \\ K_d \\ K_a \\ K_j \end{bmatrix},\ b = \begin{bmatrix} -a_3 + (p + r + R + 2\zeta\omega_n) \\ -a_2 + \left\{(r+R)p + Rr + \cdots + 2\zeta\omega_n(p+r+R) + \omega_n^2\right\} \\ -a_1 + \left\{Rrp + 2\zeta\omega_n\left[(r+R)p + Rr\right] + \cdots + \omega_n^2(p+r+R)\right\} \\ -a_0 + \left\{2\zeta\omega_n Rrp + \omega_n^2\left[(r+R)p + Rr\right]\right\} \\ \omega_n^2 Rrp \end{bmatrix}. \end{cases} \tag{11}$$

From design specification criteria, r, R, q, $\hat{q}$ and p are the desired root locations. To find the K_p, K_i, K_d, K_a and K_j of the PIDAJ controller in (2), the formula is

$$x = \begin{bmatrix} K_p & K_i & K_d & K_a & K_j \end{bmatrix}^T = A^{-1}b. \tag{12}$$

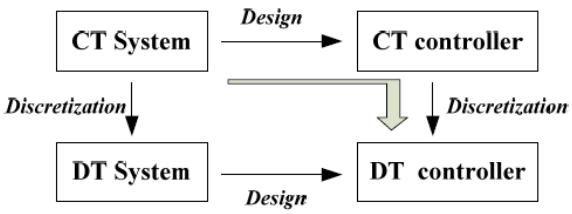


FIGURE 2. Discretization

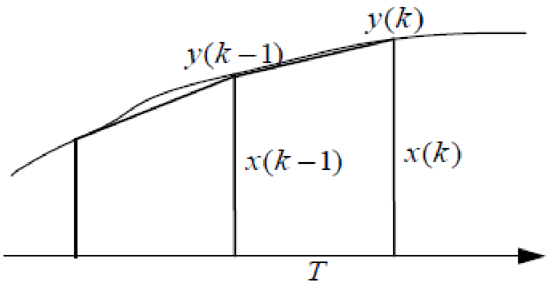


FIGURE 3. Trapezoidal approximation [9]

2.2. **Discrete-time frameworks.** In order to achieve the DT controller in z-domain, the CT controller is then discretized as depicted in Figure 2 [8].

One of possible methods for mapping from the s-plane to z-plane is the exact conversion between the Laplace and z-plane by using

$$z = e^{sT}, \tag{13}$$

where T is the sampling time. From Figure 3 [9], let $y(k-1)$ and $y(k)$ be the integrated areas under curve from the start time to the $(k-1)^{\text{th}}$ sample and to the k^{th} sample, respectively. Then, the total area under curve can be approximated as

$$y(k) = y(k-1) + \frac{T}{2}\left\{x(k) + x(k-1)\right\}. \tag{14}$$

Hence, the z transform of (14) is

$$\frac{Y(z)}{X(z)} = \frac{T}{2}\left(\frac{1+z^{-1}}{1-z^{-1}}\right);\ \left(\equiv \frac{1}{s}\right). \tag{15}$$

Finally, the bilinear transformation can be written as

$$s = \frac{2}{T}\left(\frac{z-1}{z+1}\right). \tag{16}$$

Substituting (16) into (3), the formula of the parameters of the DT PIDAJ controller can be given by

$$K(z) = \frac{\beta_4 z^4 + \beta_3 z^3 + \beta_2 z^2 + \beta_1 z + \beta_0}{(z-1)(z+1)^3} = \frac{K_z(z-a_z)(z-b_z)(z-c_z)(z-d_z)}{(z-1)(z+1)^3}, \tag{17}$$

where, the discrete-time coefficients vector is

$$\begin{bmatrix} \beta_4 \\ \beta_3 \\ \beta_2 \\ \beta_1 \\ \beta_0 \end{bmatrix} = \frac{1}{2T^3}\begin{bmatrix} 2T^3 & T^4 & 4T^2 & 8T & 16 \\ 4T^3 & 4T^4 & 0 & -16T & -64 \\ 0 & 6T^4 & -8T^2 & 0 & 96 \\ -4T^3 & 4T^4 & 0 & 16T & -64 \\ -2T^3 & T^4 & 4T^2 & -8T & 16 \end{bmatrix}\begin{bmatrix} K_p \\ K_i \\ K_d \\ K_a \\ K_j \end{bmatrix}. \tag{18}$$

3. **Example of Usage.** Reducing the exhaust pollution emissions in an automobile by using the feedback control of the air-to-fuel ratio has been a significant topic since 1980s [10]. An engine operation at or near a particular air-to-fuel ratio needs the regulation of the air and fuel flow rates into the manifold system. The fuel command and engine speed are considered as the input and output, respectively. The plant transfer function is given as

$$\begin{aligned} G(s) &= \frac{2.381}{(s+0.25)(s+4.762)(s+15.1515 \pm j15.1515)} \\ &= \frac{b_1 s + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}, \end{aligned} \tag{19}$$

where $b_1 = 0$, $b_0 = 2.381$, $a_3 = 35.315$, $a_2 = 612.205$, $a_1 = 2337.3$ and $a_0 = 564.6013$, respectively. In this case, the desired specifications are $P.O. \leq 5\%$ and $t_s(\pm 2\%) \leq 1$ sec.

3.1. **Continuous-time frameworks solution.** From the given $P.O.$, the damping ratio is

$$\zeta = \sqrt{\left[\ln\left(\frac{P.O.}{100}\right)\right]^2 \Big/ \left\{\pi^2 + \left[\ln\left(\frac{P.O.}{100}\right)\right]^2\right\}} = 0.6901, \tag{20}$$

and from the given $t_s(\pm 2\%)$, then the undamped natural frequency is

$$\omega_n = -\ln\left(0.02\sqrt{1-\zeta^2}\right) \Big/ \zeta t_s = 6.1373 \text{ rad/sec}. \tag{21}$$

Therefore, the dominant closed-loop poles are located at

$$q, \hat{q} = s_{d\pm} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -4.2354 \pm j4.4416. \tag{22}$$

Substituting, b_1, b_0, a_3, a_2, a_1 and a_0 from (19), $\zeta = 0.6901$, $\omega_n = 6.1373$ rad/sec, $p = 0.1$ and $r,\ R = 15.5 \pm j15.5$ into (11), yields:

$$\begin{cases} x = \begin{bmatrix} 0 & 0 & 0 & 0.42 & 0 \\ 0 & 0 & 0 & 0 & 0.42 \\ 0 & 0 & 0.42 & 0 & 0 \\ 0 & 0.42 & 0 & 0 & 0 \\ 0.42 & 0 & 0 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 4.2558 \\ 172.5027 \\ 2978.7 \\ 18076 \\ 1809.9 \end{bmatrix} \\ \quad = \begin{bmatrix} K_p & K_i & K_d & K_a & K_j \end{bmatrix}^T = \begin{bmatrix} 7591.7 & 760.131 & 1251 & 72.4497 & 1.7874 \end{bmatrix}^T. \end{cases} \tag{23}$$

From (2) and (23), the transfer function of the designed PIDAJ controller using the closed-form formula can be achieved as follows:

$$\left\{ K(s) = \frac{K(s+a)(s+b)(s+c)(s+d)}{s}, \quad \begin{array}{|c|c|} \hline K = 1.7874, & a, b = -14.4626 \pm j12.4015, \\ \hline c = -11.5068, & d = -0.1018. \\ \hline \end{array} \right. \tag{24}$$

3.2. **Discrete-time frameworks solution.** To obtain the DT PIDAJ controller in (17), firstly, the transform matrix in (18) is used to transform the vector of CT coefficients, which contains the PIDAJ controller's parameters, to the vector of DT coefficients as follows.

For the sampling time 1/100 sec/samples, the vector of DT coefficients can be stated as

$$\begin{bmatrix} \beta_4 & \beta_3 & \beta_2 & \beta_1 & \beta_0 \end{bmatrix}^T = \begin{bmatrix} 1.7455 \times 10^7 \\ -6.2978 \times 10^7 \\ 8.5295 \times 10^7 \\ -5.1416 \times 10^7 \\ 1.1644 \times 10^7 \end{bmatrix}. \tag{25}$$

If the CT PIDAJ controller is stated in the zero-pole-gain form, the DT PIDAJ controller can be obtained by the following formula:

$$\begin{cases} K(s) = \dfrac{K_s(s+a_s)(s+b_s)(s+c_s)(s+d_s)}{s}, \; K(z) = \dfrac{K_z(z-a_z)(z-b_z)(z-c_z)(z-d_z)}{(z-1)(z+1)^3}, \\ \begin{cases} K_z = \dfrac{K_s(a_sT+2)(b_sT+2)(c_sT+2)(d_sT+2)}{2T^3}, \\ a_z = \left(\dfrac{a_sT-2}{a_sT+2}\right), \; b_z = \left(\dfrac{b_sT-2}{b_sT+2}\right), \; c_z = \left(\dfrac{c_sT-2}{c_sT+2}\right), \; d_z = \left(\dfrac{d_sT-2}{d_sT+2}\right). \end{cases} \end{cases} \tag{26}$$

Then, the CT PIDAJ controller in (24) can be discretized by bilinear transformation as

$$\left\{ K(z) = \frac{K_z(z-a_z)(z-b_z)(z-c_z)(z-d_z)}{(z-1)(z+1)^3}, \; \boxed{\begin{array}{l|l} K_z = 1.7455 \times 10^7, & c_z = 0.8912, \\ \hline a_z, b_z = 0.8589 \pm j0.1075, & d_z = 0.999. \end{array}} \right. \tag{27}$$

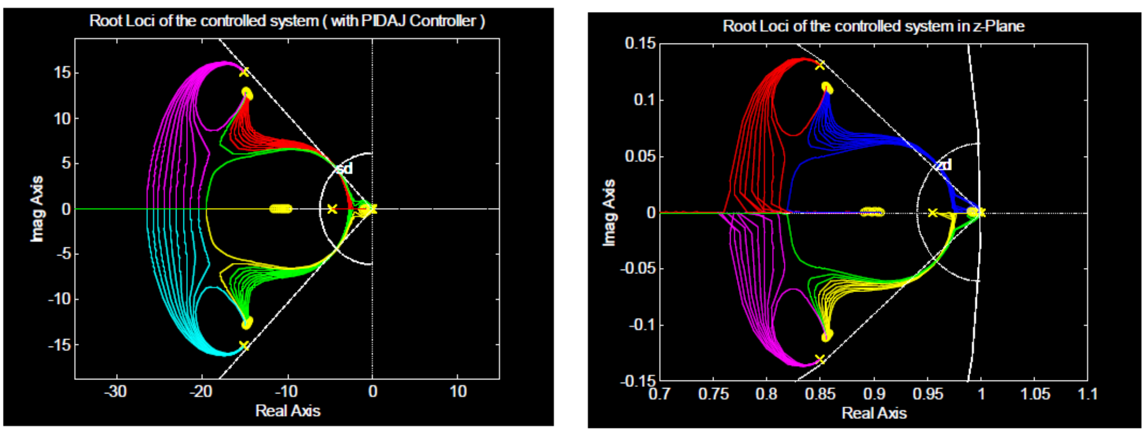
In the same way, the DT plant $G(z)$ can also be obtained as follows:

$$\begin{cases} G(s) = \dfrac{K_s}{(s+p_{1s})(s+p_{2s})(s+p_{3s})(s+p_{4s})}, \; G(z) = \dfrac{K_z(z+1)^4}{(z-p_{1z})(z-p_{2z})(z-p_{3z})(z-p_{4z})}, \\ \begin{cases} K_z = \dfrac{K_sT^4}{(p_{1s}T+2)(p_{2s}T+2)(p_{3s}T+2)(p_{4s}T+2)}, \\ p_{1z} = \left(\dfrac{p_{1s}T-2}{p_{1s}T+2}\right), \; p_{2z} = \left(\dfrac{p_{2s}T-2}{p_{2s}T+2}\right), \; p_{3z} = \left(\dfrac{p_{3s}T-2}{p_{3s}T+2}\right), \; p_{4z} = \left(\dfrac{p_{4s}T-2}{p_{4s}T+2}\right). \end{cases} \end{cases} \tag{28}$$

Then, the DT plant $G(z)$ can be written as

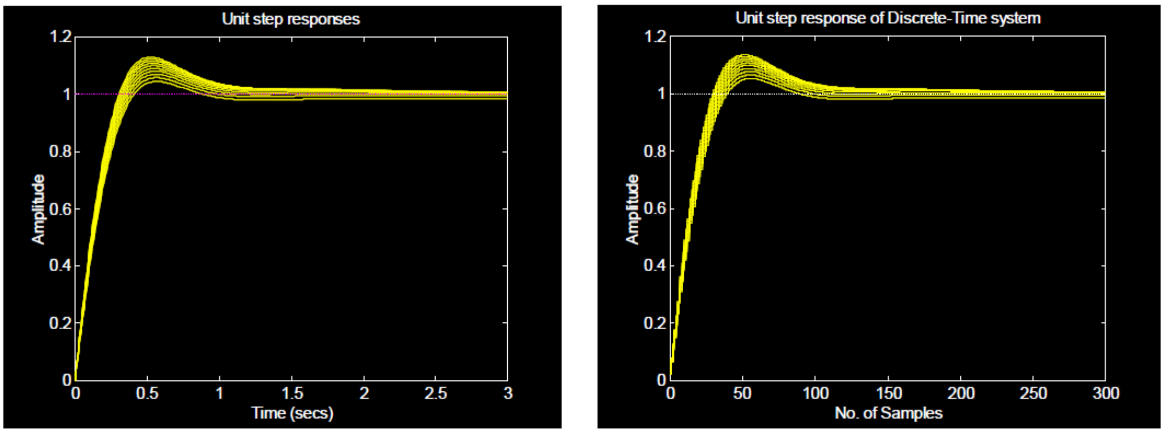
$$\left\{ G(z) = \frac{K_z(z+1)^4}{(z-p_{1z})(z-p_{2z})(z-p_{3z})(z-p_{4z})}, \; \boxed{\begin{array}{l|l} \multicolumn{2}{c}{K_z = 1.2482 \times 10^{-9},} \\ \hline p_{1z} = 0.9975, & p_{3z} = 0.85 + j0.1303, \\ \hline p_{2z} = 0.9535, & p_{4z} = 0.85 - j0.1303. \end{array}} \right. \tag{29}$$

3.3. **Simulation results.** Figure 4 shows the root loci in both s-plane and z-plane of the controlled system when p is varied from 0.1 to 1.0. The corresponding unit step responses of the root loci are displayed in Figure 5. Figure 6 shows the unit step response when $p = 1.0$, and the gain $K = K_j$ by increasing to 10 times. The simulation results show that the unit step responses of the controlled system in both continuous-time and discrete-time are similar.



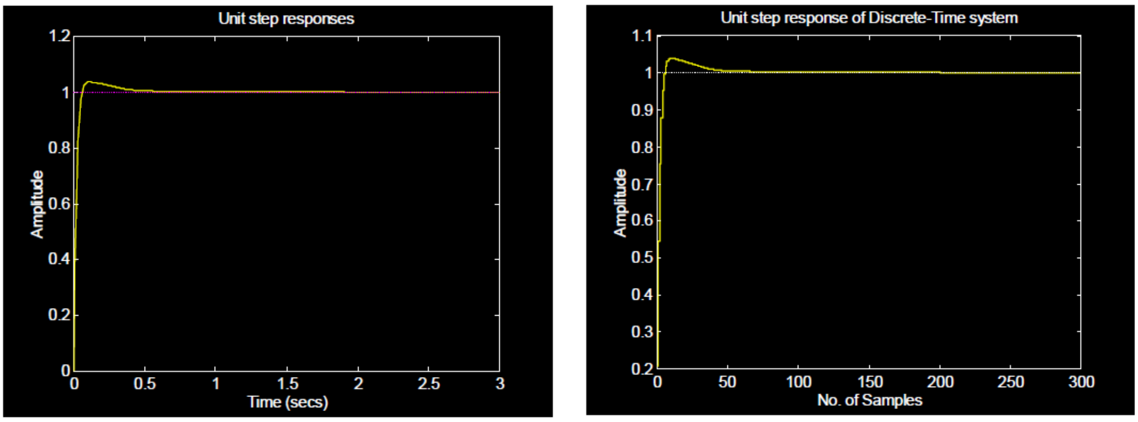
(a) s-plane (b) z-plane

FIGURE 4. Root loci of the controlled system for $p = 0.1$-1.0



(a) Continuous-time control (b) Discrete-time control

Figure 5. Unit step responses of the controlled system by varying p from 0.1 to 1.0



(a) Continuous-time control (b) Discrete-time control

Figure 6. Unit step responses of the controlled system for $p = 1.0$ and $K \times 10$

4. **Conclusions.** The closed-form formulas to find the parameters of the PIDAJ controller are derived and given in both CT and DT systems. Example usage of the proposed formulas confirms that the PIDAJ controller can be designed with non-complicated procedures. Furthermore, these formulas are also available for other design methods such as root locus technique. Implementation of the PIDAJ controller in real applications is the future work.

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