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EXTENDED KALMAN FILTERS FOR LINEAR FRACTIONAL-ORDER SYSTEMS WITH UNKNOWN FRACTIONAL-ORDER

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Abstract. *This study presents extended Kalman filters for continuous-time linear fractional-order systems with unknown fractional-order. Based on the Grünwald-Letnikov difference, the fractional-order state equation is discretized to obtain a difference equation. Then, the Sigmoid function is used to ensure the estimation of fractional-order in a suitable range. Because the fractional-order contained in linear fractional-order systems is unknown, the investigated system can be viewed as a nonlinear system, and the nonlinear function can be linearized using the first-order Taylor expression. Considering that the initial value in terms of a relatively small fractional-order can produce a significant impact on state estimation and fractional-order identification for the linear fractional-order system, the augmented vector method is adopted to achieve the initial value compensation. Finally, the effectiveness of the proposed algorithms is validated by simulation results.*

Keywords: Fractional-order systems, Extended Kalman filters, State estimation, Initial value compensation

1. **Introduction.** Because of the memorability, fractional-order calculus operators can better describe the dynamic characteristics of many physical systems such as viscoelasticity [1] and anomalous diffusivity [2]. The introduction of fractional-order operators can make the controller design more flexible. Therefore, fractional-order calculus is widely used in the field of control such as fractional-order sliding mode controls [3], fractional-order iterative learning controls [4] and fractional-order PID controls [5]. Meanwhile, the problems on modeling and control of fractional-order systems [6] have attracted extensive attention of many scholars in recent years.

For integer-order systems, the research methods of state estimation and parameter identification are summarized in the neural network method [7], Kalman filters [8, 9], nonlinear Kalman filters [10] and adaptive Kalman filters [11]. Similarly, the Kalman filter is an effective robust observer for the state estimation and parameter identification of fractional-order systems including input and output signals and is widely applied in industrial control systems, target radar tracking, multi-sensor fusion, positioning systems, communication and signal processing [12, 13]. For linear fractional-order systems, the fractional-order Kalman filters based on the fractional-order average derivative method were presented in [14] to enhance the accuracy of state estimation in terms of the uncorrelated and correlated noises. For discrete-time linear fractional-order systems, the

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fractional-order Kalman filter was investigated in [15] to obtain the estimation over networks with packet losses. Taking into account the considerable interest in parameter identification, the fractional-order Kalman filter was proposed in [16] to estimate state and parameter for a fractional-order system with colored measurement noise. Besides, the problems on estimations of state and parameters for continuous-time nonlinear fractional-order systems containing the uncorrelated and correlated process and measurement noises were dealt with cubature Kalman filters based on Grünwald-Letnikov (G-L) difference in [17], which used the third-degree spherical-radical rule, and the nonlinear functions in the state equation and output equation were performed by the cubature points. For stochastic discrete-time nonlinear fractional-order chaotic systems with the channel additive noise and process noise, the extended Kalman filter was presented in [18] to achieve the parameter estimation and perfect synchronization.

In this paper, it is a major issue to estimate the unknown fractional-order, and the effect of initial value on the estimations of state and fractional-order is explored using the extended Kalman filter for continuous-time linear fractional-order systems. Then, some main achievements of this study are summarized. 1) By using the Sigmoid function, the estimation of fractional-order is got in a suitable range. 2) Because the fractional-order contained in the linear fractional-order system is unknown, the investigated system can be viewed as a nonlinear system, and the nonlinear function can be linearized using the first-order Taylor expression. 3) For a relatively small unknown fractional-order, the augmented vector method is adopted to achieve the initial value compensation. 4) Extended Kalman filters based on G-L difference method are investigated to deal with problems on state estimation and fractional-order identification.

The rest of this paper is organized as follows. In Section 2, the Sigmoid function is introduced and the initial value compensation problem is put forward. In Section 3.1, the treatment on the unknown fractional-order is provided. In Section 3.2, the extended Kalman filter based on G-L difference for continuous-time linear fractional-order systems is presented to obtain the effective estimation of fractional-order. Then, the initial value compensation problem is explored for a relatively small fractional-order in Section 3.3. Then, the simulation results are given to validate the effectiveness of the proposed Kalman filters in Section 4. Finally, we sum up the full paper in Section 5.

2. **Problem Statement.** Considering the following continuous-time linear fractional-order system

$$ {}_0^C D_t^{\beta} x(t) = Ax(t) + Bu(t) + Gw(t), \tag{1} $$

$$ z(t) = Cx(t) + v(t), \tag{2} $$

where $\beta \in (0, 1)$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $G \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{q\times n}$, the symbols $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^p$ and $z(t) \in \mathbb{R}^q$ are the state, the input and output, $w(t) \in \mathbb{R}^m$ and $v(t) \in \mathbb{R}^q$ are process noise and measurement noise with the covariance matrices Q and R, respectively.

To get the suitable estimation of the fractional-order β in $(0, 1)$, the Sigmoid function $S(x)$ is provided by [19] as follows

$$ S(x) = \frac{1}{1 + \mathrm{e}^{-x}}. \tag{3} $$

In fact, the function can map all the independent variables in $(0, 1)$. Besides, it is continuous, smooth, increasing monotonically, differentiable everywhere in the domain $(-\infty, +\infty)$ and its derivative is $S'(x) = S(x)(1 - S(x))$.

Then, the following transformation for $\alpha \in (-\infty, +\infty)$ based on the Sigmoid function is introduced as

$$ \beta = S(\alpha), \tag{4} $$

where $\mathrm{d}\beta/\mathrm{d}\alpha = \beta(1 - \beta)$ and $\alpha = -\ln(\beta^{-1} - 1)$.

The vectors $x(t)$, $u(t)$, $w(t)$, $v(t)$ and $z(t)$ are replaced by $x(k)$, $u(k)$, $w(k)$, $v(k)$ and $z(k)$ at $t = kT$ for convenience. Then, the β-order derivative based on the G-L difference given in [20] with $x(k-j) = 0$ for $j > k$ is approximated by

$$ {}_{0}^{C}D_t^{\beta} x(k) \approx \Delta^{\beta} x(k) = \frac{1}{T^{\beta}} \sum_{j=0}^{k} (-1)^j \binom{\beta}{j} x(k-j). \tag{5}$$

Furthermore, the fractional-order state Equation (1) is discretized as follows

$$x(k) = \sum_{j=1}^{k} A_j x(k-j) + B_{\beta} u(k-1) + G_{\beta} w(k-1), \tag{6}$$

where

$$A_j = \begin{cases} T^{\beta} A + \beta I & j = 1 \\ (-1)^{j+1} c_j I & j > 1 \end{cases}, \ B_{\beta} = T^{\beta} B, \ G_{\beta} = T^{\beta} G \text{ and } c_j = \binom{\beta}{j}.$$

Then, the extended Kalman filters are proposed to deal with the problems on estimations of the state and fractional-order for continuous-time linear fractional-order systems. Besides, the initial value compensation problem is concerned for a relatively small fractional-order to improve the accuracy of state estimation and fractional-order identification.

3. **Main Results.**

3.1. **Treatment on unknown fractional-order.**

In this section, the problem on fractional-order identification can be dealt with. In order to get effective estimation of fractional-order, the parameter α is estimated. In fact, the unknown parameter $\alpha = S^{-1}(\beta)$ contained in the state Equation (1) can be viewed as a random vector $\alpha(k)$ for estimation of the kth iteration, which satisfies

$$\alpha(k) = \alpha(k-1) + w_{\alpha}(k-1), \tag{7}$$

where $w_{\alpha}(k)$ is white Gaussian noise satisfying the mathematical expectation 0 and Q_{α} with a small positive real number, the condition $\beta(k) = S(\alpha(k))$ can be established obviously. Besides, the noises $w(k)$, $w_{\alpha}(k)$ and $v(k)$ are uncorrelated mutually.

From Equation (7), Equation (6) can be represented by

$$x(k) = \sum_{j=1}^{k} A_j^{\beta}(k) x(k-j) + B_{\beta}(k-1) u(k-1) + G_{\beta}(k-1) w(k-1), \tag{8}$$

where

$$A_j^{\beta}(k) = \begin{cases} T^{\beta(k-1)} A + \beta(k-1) I & j = 1 \\ (-1)^{j+1} c_j(k) I & j > 1 \end{cases}, \ B_{\beta}(k-1) = T^{\beta(k-1)} B,$$

$$G_{\beta}(k-1) = T^{\beta(k-1)} G, \ c_j(k) = \binom{\beta(k)}{j}.$$

To deal with unknown parameter $\alpha(k)$, the augmented state vector $\kappa(k) = \left[x^{\mathrm{T}}(k), \alpha(k)\right]^{\mathrm{T}}$ and the augmented noise vector $\xi(k) = \left[w^{\mathrm{T}}(k), w_{\alpha}(k)\right]^{\mathrm{T}}$ with $E\left[\xi(k)\xi^{\mathrm{T}}(k)\right] = \mathcal{Q} = \begin{bmatrix} Q & \mathbf{0} \\ \mathbf{0} & Q_{\alpha} \end{bmatrix}$ are defined.

Then, we have

$$\kappa(k) = f(\kappa(k-1), u(k-1)) + \sum_{j=2}^{k} \mathcal{A}_j^{\beta}(k) \kappa(k-j) + \mathcal{G}_{\beta}(k-1) \xi(k-1), \tag{9}$$

where

$$f(\kappa(k-1),u(k-1)) = \begin{bmatrix} A_1^\beta(k)x(k-1)+B_\beta(k-1)u(k-1) \\ \alpha(k-1) \end{bmatrix}$$
$$= \begin{bmatrix} \left(T^{\beta(k-1)}A+\beta(k-1)I\right)x(k-1)+T^{\beta(k-1)}Bu(k-1) \\ \alpha(k-1) \end{bmatrix},$$
$$\mathcal{A}_j^\beta(k) = \begin{bmatrix} A_j^\beta(k) & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix},\ \mathcal{G}_\beta(k-1) = \begin{bmatrix} G_\beta(k-1) & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}.$$

For the unknown fractional-order contained in the linear fractional-order system (1), the nonlinear function $f(\kappa(k-1),u(k-1))$ on the right side of Equation (9) using the first-order Taylor expression at $\kappa(k-1)=\widehat{\kappa}(k-1|k-1)$ is approximated by

$$f(\kappa(k-1),u(k-1)) \approx f(\widehat{\kappa}(k-1|k-1),u(k-1))+M(k-1)(\kappa(k-1)-\widehat{\kappa}(k-1|k-1)), \quad (10)$$

where

$$M(k-1) = \left.\begin{bmatrix} T^\beta A+\beta I & N(k-1) \\ \mathbf{0} & 1 \end{bmatrix}\right|_{\beta=\widehat{\beta}(k-1|k-1)},$$
$$N(k-1) = \left(T^\beta(\ln T)A+I\right)\beta(1-\beta)\widehat{x}(k-1|k-1)+T^\beta(\ln T)\beta(1-\beta)Bu(k-1).$$

Meanwhile, the derivations of coefficients $c_j(k)$ and $\mathcal{G}_\beta(k-1)$ at the right of Equation (9) is difficult to obtain. Therefore, the approximations of $c_j(k)$ and $\mathcal{G}_\beta(k-1)$ are replaced by the estimations corresponding to the $(k-j)$th iteration to simplify the calculation. Then, we can get

$$\widetilde{c}_j(k) = \binom{\widehat{\beta}(k-j|k-j)}{j},\ \widetilde{\mathcal{G}}_\beta(k-1) = \begin{bmatrix} T^{\widehat{\beta}(k-1|k-1)}G & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}.$$

Furthermore, the matrix $\mathcal{A}_j^\beta(k)$ for $j>1$ can be represented by

$$\overline{\mathcal{A}}_j^\beta(k) = \begin{bmatrix} (-1)^{j+1}\widetilde{c}_j(k)I & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}.$$

Combining Equations (9) and (10), the augmented state equation can be provided by

$$\kappa(k) = \sum_{j=1}^{k}\widetilde{\mathcal{A}}_j^\beta(k)\kappa(k-j)+f\left(\widehat{\kappa}(k-1|k-1),u(k-1)\right)-M(k-1)\widehat{\kappa}(k-1|k-1)$$
$$+\widetilde{\mathcal{G}}_\beta(k-1)\xi(k-1), \quad (11)$$

where

$$\widetilde{\mathcal{A}}_j^\beta(k) = \begin{cases} M(k-1) & j=1 \\ \overline{\mathcal{A}}_j^\beta(k) & j>1 \end{cases}.$$

The output equation becomes $z(k)=\mathcal{C}\kappa(k)+v(k)$, where $\mathcal{C}=[C,0]$.

3.2. **Fractional-order Kalman filter based on G-L difference.** The estimation of $\kappa(k)$ is defined as $\widehat{\kappa}(k|k)=E[\kappa(k)|\sigma(k)]$, where the information of $z(0),z(1),\ldots,z(k)$, $u(0),u(1),\ldots,u(k)$ are contained in $\sigma(k)$. Meanwhile, the prediction of $\kappa(k)$ is defined as $\widehat{\kappa}(k|k-1)=E[\kappa(k)|\sigma(k-1)]$. Thus, $\widehat{\beta}(k|k)=S(\widehat{\alpha}(k|k))$ is achieved.

Then, the extended Kalman filter based on G-L difference is proposed to achieve state estimation for a continuous-time linear fractional-order system with unknown fractional-order.

Theorem 3.1. *For a continuous-time linear fractional-order system with unknown fractional-order described by Equations* (1) *and* (2)*, the extended Kalman filter based on G-L difference is designed as follows*

$$\widehat{\kappa}(k|k-1) = \sum_{j=1}^{k}\widetilde{\mathcal{A}}_j^\beta(k)\widehat{\kappa}(k-j|k-j)+f(\widehat{\kappa}(k-1|k-1),u(k-1))-M(k-1)\widehat{\kappa}(k-1|k-1),$$

$$\widehat{\kappa}(k|k) = \widehat{\kappa}(k|k-1) + K(k)(z(k) - \mathcal{C}\widehat{\kappa}(k|k-1)),$$
$$P(k|k-1) = \sum_{j=1}^{k} \widetilde{\mathcal{A}}_j^{\beta}(k)P(k-j|k-j)\left(\widetilde{\mathcal{A}}_j^{\beta}(k)\right)^{\mathrm{T}} + \widetilde{\mathcal{G}}_{\beta}(k-1)\mathcal{Q}\widetilde{\mathcal{G}}_{\beta}^{\mathrm{T}}(k-1),$$
$$K(k) = P(k|k-1)\mathcal{C}^{\mathrm{T}}(\mathcal{C}P(k|k-1)\mathcal{C}^{\mathrm{T}} + R)^{-1},$$
$$P(k|k) = (I - K(k)\mathcal{C})P(k|k-1).$$

Proof: The results are easy to get. Therefore, the proof process is omitted.

3.3. **Treatment on initial value compensation problem.** Because the initial value can produce a great influence on the state estimation and fractional-order identification for a relatively small fractional-order $\beta \in (0, 1)$, it is necessary that the initial value compensation problem is considered to obtain the more accurate estimations of the state and fractional-order.

Based on the relationship between Riemann-Liouville and Caputo [21], we can get

$${}_0^C D_t^{\beta} x(t) = {}_0^R D_t^{\beta} x(t) - \sum_{p=0}^{n-1} \frac{x^{(p)}(0)}{\Gamma(p-\beta+1)} t^{p-\beta}. \tag{12}$$

For $\beta \in (0, 1)$, we set $n = 1$, then we have

$${}_0^C D_t^{\beta} x(t) \approx \frac{1}{T^{\beta}} \sum_{j=0}^{k} (-1)^j c_j x(k-j) - \frac{H(k)}{T^{\beta}} x(0), \tag{13}$$

where $H(k) = k^{-\beta}/\Gamma(1-\beta)$.

From Equations (1) and (13), we obtain

$$\frac{1}{T^{\beta}} \sum_{j=0}^{k} (-1)^j c_j x(k-j) - \frac{H(k)}{T^{\beta}} x(0) = Ax(k-1) + Bu(k-1) + Gw(k-1). \tag{14}$$

Then, Equation (6) can be simplified as

$$x(k) = \sum_{j=1}^{k} A_j x(k-j) + B_{\beta} u(k-1) + G_{\beta} w(k-1) + H(k)x(0). \tag{15}$$

Considering that the fractional-order β is contained in the matrix $H(k)$, we replace the fractional-order with its estimation $\widehat{\beta}(k-1|k-1)$. Thus, the matrix $H(k)$ is redefined as $H_{\beta}(k)$. Meanwhile, we view the augmented vector $\overline{x}(k)$ as the initial state $x(0)$ for estimation of the kth iteration. It follows that

$$\overline{x}(k) = \overline{x}(k-1) + \overline{w}(k-1), \tag{16}$$

where $\overline{w}(k)$ is white Gaussian noise satisfying the mathematical expectation $\mathbf{0}$ and the covariance matrix Q_1 with a very small norm.

Therefore, Equation (15) can be rewritten by

$$x(k) = \sum_{j=1}^{k} A_j x(k-j) + B_{\beta} u(k-1) + G_{\beta} w(k-1) + H_{\beta}(k)\overline{x}(k-1). \tag{17}$$

In order to reduce the errors of state estimation and fractional-order identification produced by the initial value for the investigated fractional-order system, we define $\varsigma(k) = \left[x^{\mathrm{T}}(k), \overline{x}^{\mathrm{T}}(k)\right]^{\mathrm{T}}$ and $\zeta(k) = \left[w^{\mathrm{T}}(k), \overline{w}^{\mathrm{T}}(k)\right]^{\mathrm{T}}$ with $E\left[\zeta(k)\zeta^{\mathrm{T}}(k)\right] = Q_2 = \begin{bmatrix} Q & \mathbf{0} \\ \mathbf{0} & Q_1 \end{bmatrix}$ and $E\left[\zeta(k)v^{\mathrm{T}}(l)\right] = \mathbf{0}$.

According to Equations (7) and (8), the augmented state equation based on Equation (16) can be represented by

$$\varsigma(k)=\sum_{j=1}^{k}\mathbf{A}_j^{\beta}(k)\varsigma(k-j)+\mathbf{B}_{\beta}(k-1)u(k-1)+\mathbf{G}_{\beta}(k-1)\zeta(k-1), \tag{18}$$

where

$$\mathbf{A}_j^{\beta}(k)=\begin{cases}\begin{bmatrix}A_1^{\beta}(k) & H_{\beta}(k)\\ \mathbf{0} & I\end{bmatrix} & j=1\\ \begin{bmatrix}A_j^{\beta}(k) & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} & j>1\end{cases},\ \mathbf{B}_{\beta}(k-1)=\begin{bmatrix}B_{\beta}(k-1)\\ \mathbf{0}\end{bmatrix},$$

$$\mathbf{G}_{\beta}(k-1)=\begin{bmatrix}G_{\beta}(k-1) & \mathbf{0}\\ \mathbf{0} & I\end{bmatrix}.$$

Similarly, we can define the augmented vectors $\eta(k)=\left[\varsigma^{\mathrm{T}}(k),\alpha(k)\right]^{\mathrm{T}}=\left[x^{\mathrm{T}}(k),\overline{x}^{\mathrm{T}}(k),\alpha(k)\right]^{\mathrm{T}}$ and $\varepsilon(k)=[\zeta^{\mathrm{T}}(k),w_{\alpha}(k)]^{\mathrm{T}}=[w^{\mathrm{T}}(k),\overline{w}^{\mathrm{T}}(k),w_{\alpha}(k)]^{\mathrm{T}}$ with $E[\varepsilon(k)\varepsilon^{\mathrm{T}}(k)]=\mathcal{Q}=\begin{bmatrix}Q_2 & \mathbf{0}\\ \mathbf{0} & Q_{\alpha}\end{bmatrix}=\begin{bmatrix}Q & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & Q_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & Q_{\alpha}\end{bmatrix}$. Then, we have

$$\eta(k)=g(\eta(k-1),u(k-1))+\sum_{j=2}^{k}\overline{\mathbf{A}}_j^{\beta}(k)\eta(k-j)+\overline{\mathbf{G}}_{\beta}(k-1)\varepsilon(k-1), \tag{19}$$

where

$$\begin{aligned}&g(\eta(k-1),u(k-1))\\ =&\begin{bmatrix}A_1^{\beta}(k)x(k-1)+B_{\beta}(k-1)u(k-1)+H_{\beta}(k)\overline{x}(k-1)\\ \overline{x}(k-1)\\ \alpha(k-1)\end{bmatrix}\\ =&\begin{bmatrix}\left(T^{\beta(k-1)}A+\beta(k-1)I\right)x(k-1)+T^{\beta(k-1)}Bu(k-1)+H_{\beta}(k)\overline{x}(k-1)\\ \overline{x}(k-1)\\ \alpha(k-1)\end{bmatrix},\\ &\overline{\mathbf{A}}_j^{\beta}(k)=\begin{bmatrix}\mathbf{A}_j^{\beta}(k) & \mathbf{0}\\ \mathbf{0} & 0\end{bmatrix},\ \overline{\mathbf{G}}_{\beta}(k-1)=\begin{bmatrix}\mathbf{G}_{\beta}(k-1) & \mathbf{0}\\ \mathbf{0} & 1\end{bmatrix}.\end{aligned}$$

By using the first-order Taylor expression at $\eta(k-1)=\widehat{\eta}(k-1|k-1)$, the nonlinear function $g(\eta(k-1),u(k-1))$ on the right side of Equation (19) is written by

$$g(\eta(k-1),u(k-1))\approx g(\widehat{\eta}(k-1|k-1),u(k-1))+\mathcal{M}(k-1)(\eta(k-1)-\widehat{\eta}(k-1|k-1)), \tag{20}$$

where

$$\mathcal{M}(k-1)=\left.\begin{bmatrix}T^{\beta}A+\beta I & H_{\beta}(k) & \mathcal{N}(k-1)\\ \mathbf{0} & I & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & 1\end{bmatrix}\right|_{\beta=\widehat{\beta}(k-1|k-1)},$$

$$\begin{aligned}\mathcal{N}(k-1)=&\left(T^{\beta}(\ln T)A+I\right)\beta(1-\beta)\widehat{x}(k-1|k-1)+T^{\beta}(\ln T)\beta(1-\beta)Bu(k-1).\\ &+\frac{k^{-\beta}}{\Gamma(1-\beta)}\widehat{\overline{x}}(k-1|k-1).\end{aligned}$$

Being similar as Section 3.1, the matrices $\overline{\mathbf{A}}_j^{\beta}(k)$ for $j>1$ and $\overline{\mathbf{G}}_{\beta}(k-1)$ can be transformed as

$$\widetilde{\mathbf{A}}_j^{\beta}(k)=\begin{bmatrix}(-1)^{j+1}\widetilde{c}_j(k)I & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & 0\end{bmatrix},\ \overline{\mathcal{G}}_{\beta}(k-1)=\begin{bmatrix}T^{\widehat{\beta}(k-1|k-1)}G & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & I & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & 1\end{bmatrix}.$$

Then, the fractional-order state equation can be obtained by

$$\begin{aligned}\eta(k) = & \sum_{j=1}^{k} \overline{\mathcal{A}}_j^{\beta}(k)\eta(k-j) + g(\widehat{\eta}(k-1|k-1), u(k-1)) - \mathcal{M}(k-1)\widehat{\eta}(k-1|k-1) \\ & + \overline{\mathcal{G}}_{\beta}(k-1)\varepsilon(k-1), \end{aligned} \tag{21}$$

where

$$\overline{\mathcal{A}}_j^{\beta}(k) = \begin{cases} \mathcal{M}(k-1) & j = 1 \\ \widetilde{\mathbf{A}}_j^{\beta}(k) & j > 1 \end{cases}.$$

The output equation becomes $z(k) = \overline{\mathcal{C}}\eta(k) + v(k)$, where $\overline{\mathcal{C}} = [C, \mathbf{0}, 0]$.

Theorem 3.2. *For a continuous-time linear fractional-order system with unknown fractional-order described by Equations* (1) *and* (2)*, the extended Kalman filter based on G-L difference with initial value compensation is designed as follows*

$$\begin{aligned}
\widehat{\eta}(k|k-1) &= \sum_{j=1}^{k} \overline{\mathcal{A}}_j^{\beta}(k)\widehat{\eta}(k-j|k-j) + g\left(\widehat{\eta}(k-1|k-1), u(k-1)\right) \\
&\quad - \mathcal{M}(k-1)\widehat{\eta}(k-1|k-1), \\
\widehat{\eta}(k|k) &= \widehat{\eta}(k|k-1) + K(k)\left(z(k) - \overline{\mathcal{C}}\widehat{\eta}(k|k-1)\right), \\
P(k|k-1) &= \sum_{j=1}^{k} \overline{\mathcal{A}}_j^{\beta}(k)P(k-j|k-j)\left(\overline{\mathcal{A}}_j^{\beta}(k)\right)^{\mathrm{T}} + \overline{\mathcal{G}}_{\beta}(k-1)\overline{\mathcal{Q}}\,\overline{\mathcal{G}}_{\beta}^{\mathrm{T}}(k-1), \\
K(k) &= P(k|k-1)\overline{\mathcal{C}}^{\mathrm{T}}\left(\overline{\mathcal{C}}P(k|k-1)\overline{\mathcal{C}}^{\mathrm{T}} + R\right)^{-1}, \\
P(k|k) &= \left(I - K(k)\overline{\mathcal{C}}\right)P(k|k-1).
\end{aligned}$$

Proof: The proof is easy to calculate. Thus, the process is not given.

4. **Numerical Example.** If the algorithms proposed by Theorem 3.1 and Theorem 3.2 are applied to the practical engineering, the truncation discussed in [14] is necessary to be considered. Therefore, we set the length of truncation as 30, then the simulation result is given as follows.

Consider the following linear fractional-order system with unknown fractional-order as

$${}_0^C D_t^{\beta} x(t) = Ax(t) + Bu(t) + Gw(t), \tag{22}$$

$$z(t) = Cx(t) + v(t), \tag{23}$$

where $x(t) = [x_1(t), x_2(t)]^{\mathrm{T}}$, $A = \begin{bmatrix} -9 & 11 \\ 4 & -5 \end{bmatrix}$, $B = [1, 1]^{\mathrm{T}}$, $G = [1, 1]^{\mathrm{T}}$, $C = [1, 0.5]$, the covariance matrices of noises are $Q = 0.2$, $R = 1.2$, $Q_{\alpha} = 0.0001$ and $Q_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$.

The sampling period is $T = 0.6$s, the running time is 600s and the real value of fractional-order is set as $\beta = 0.2$. The initial conditions for the fractional-order Kalman filters are selected as $x(0) = [5, -5]^{\mathrm{T}}$, $w(0) = 0$, $\widehat{\kappa}(0|0) = [0, 0, 0]^{\mathrm{T}}$, $P(0|0) = I$ and $K(0) = [0, 0, 0]^{\mathrm{T}}$. The input signal $u(t)$ is set as a sine wave function that is $u(t) = 10\sin(0.1t)$ to obtain the sampling value of the input and measurement signals of state estimation. Then, the measurement $z(k)$ is drawn in Figure 1, the state vector $x(k)$, the unknown fractional-order $\beta(k)$ and its corresponding estimations using the extended Kalman filters proposed by Theorem 3.1 and Theorem 3.2 are shown in Figures 2-3, respectively.

From Figures 2-3, it is seen that the estimations of state $x(k)$ and fractional-order $\beta(k)$ are very close to the real values, which indicates that the two kinds of extended Kalman filters proposed by Theorem 3.1 and Theorem 3.2 can achieve the estimations of the state and fractional-order. Besides, the conclusion can be drawn that the effect of state and

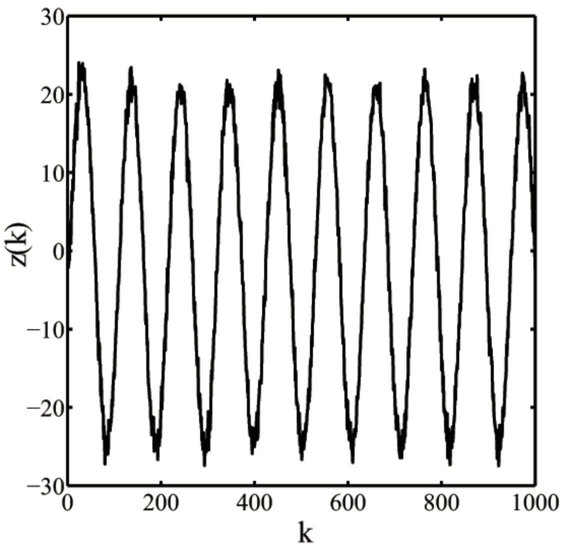
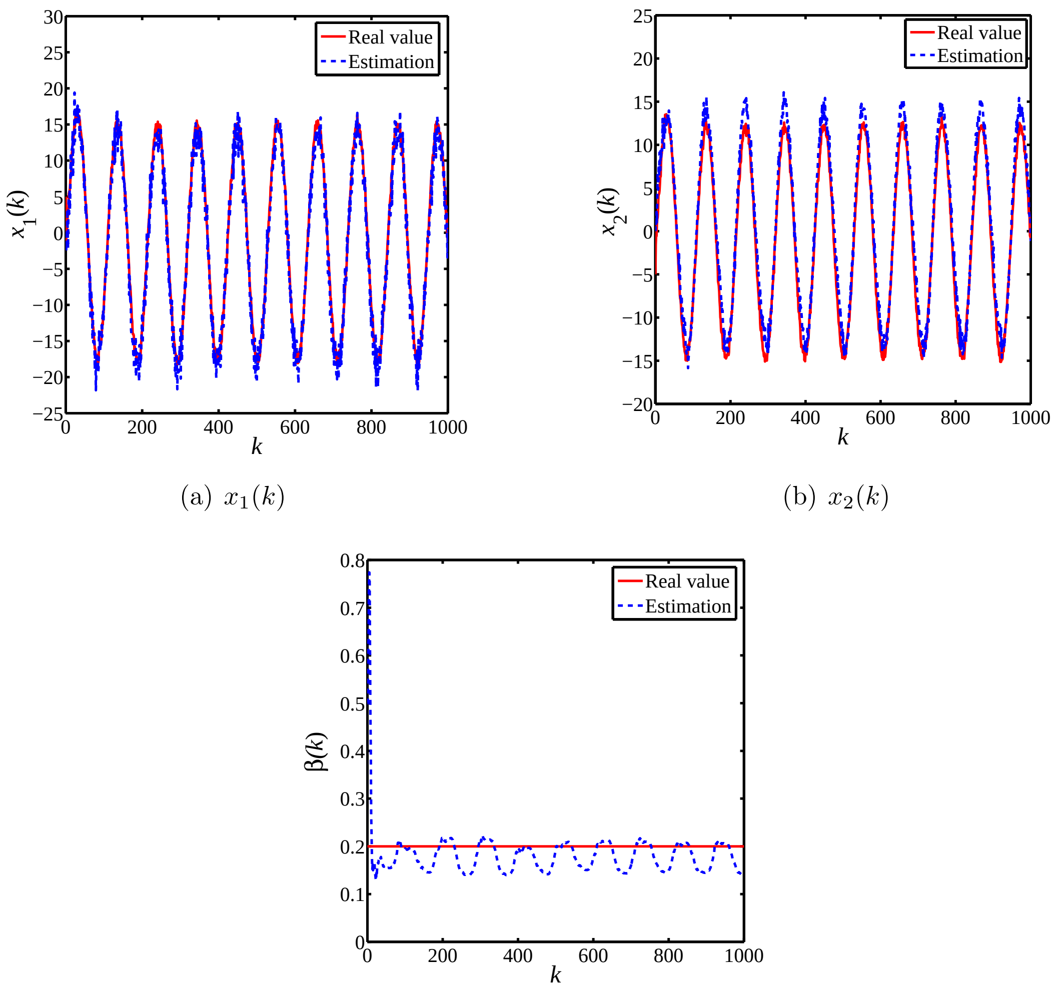


Figure 1. Measurement $z(k)$

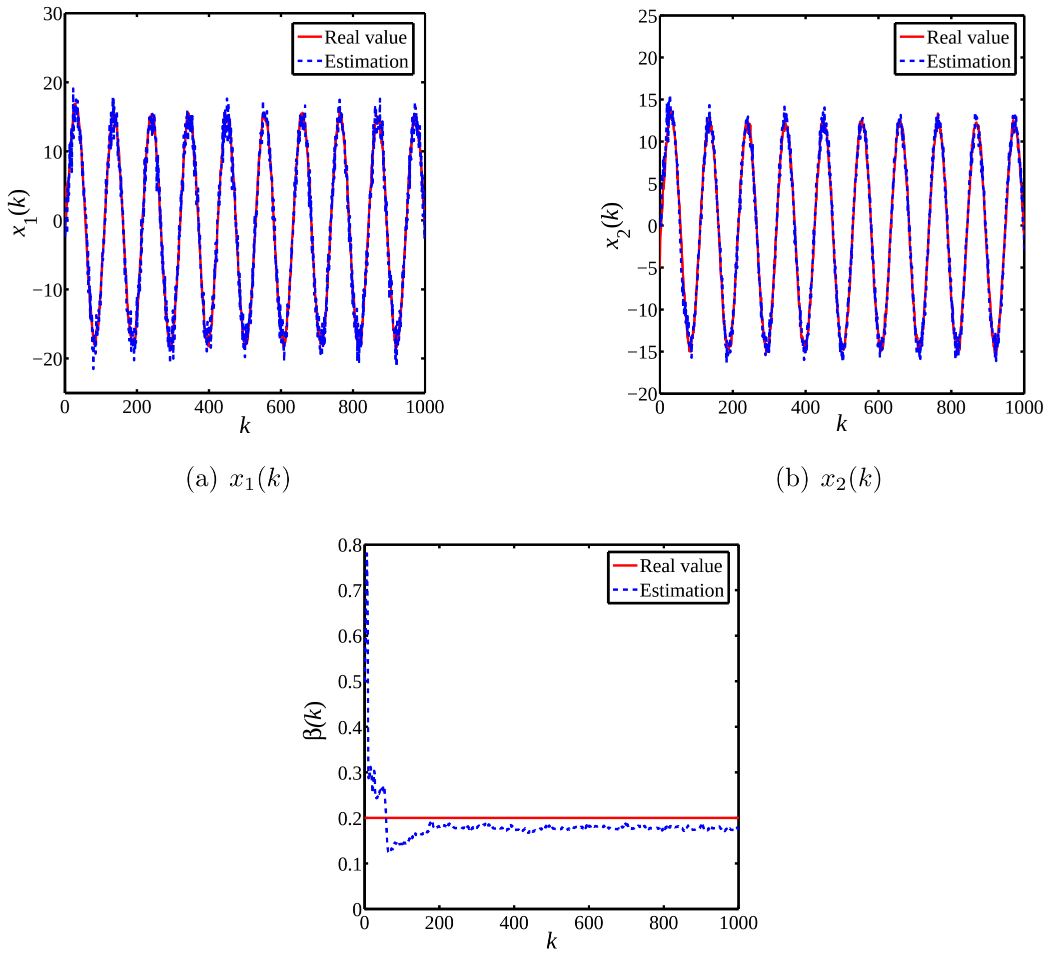


(a) $x_1(k)$

(b) $x_2(k)$

(c) $\beta(k)$

Figure 2. State, fractional-order and its estimations via Theorem 3.1



(a) $x_1(k)$ (b) $x_2(k)$

(c) $\beta(k)$

Figure 3. State, fractional-order and its estimations via Theorem 3.2

fractional-order estimation is more effective using the extended Kalman filter via Theorem 3.2 with the initial value compensation.

In order to obtain the comparison results of state estimation error under Theorem 3.1 and Theorem 3.2 for different fractional-orders β, the error index is defined as follows

$$E = \frac{1}{F_m + 1} \sum_{k=0}^{F_m} \sqrt{\frac{\sum_{s=1}^{3} \left(\kappa_s(k) - \hat{\kappa}_s(k|k)\right)^2}{\sum_{s=1}^{3} (\kappa_s(k))^2}}, \tag{24}$$

where $\kappa_1(k) = x_1(k)$, $\kappa_2(k) = x_2(k)$, $\kappa_3(k) = \beta(k)$, $F_m + 1$ is the number of input and output sampling data and $F_m = 1000$ for this example. Then, the comparison results of state estimation error via Theorem 3.1 and Theorem 3.2 for different fractional-orders β are provided in Table 1.

From Table 1, it is obvious that the state estimation based on extended Kalman filter proposed by Theorem 3.2 is more accurate for different fractional-orders $\beta = 0.2, 0.3, \ldots,$ 0.9, but more time is taken for computing. Besides, it can be seen that the difference $E_1 - E_2$ of state estimation error between Theorem 3.1 and Theorem 3.2 decreases with increase of fractional-order β, which indicates that the effect of state estimation based on extended Kalman filter via Theorem 3.2 with initial value compensation is more obvious in terms of a relatively small unknown fractional-order. Therefore, the extended Kalman filter based on Theorem 3.2 can be applied to estimate the state and fractional-order for continuous-time linear fractional-order systems with a relatively small fractional-order,

Table 1. Comparison results of state estimation error for different fractional-orders β

β	E_1 for Theorem 3.1	E_2 for Theorem 3.2	$E_1 - E_2$	Time for Theorem 3.1/s	Time for Theorem 3.2/s
0.2	0.5043	0.3880	0.1163	1.2186	1.3925
0.3	0.3090	0.2545	0.0545	1.2458	1.3713
0.4	0.2141	0.1830	0.0311	1.2209	1.4259
0.5	0.1698	0.1474	0.0224	1.2513	1.4012
0.6	0.1329	0.1164	0.0165	1.2858	1.4346
0.7	0.1146	0.0997	0.0149	1.2778	1.3570
0.8	0.1008	0.0871	0.0137	1.2429	1.3606
0.9	0.0944	0.0813	0.0131	1.2558	1.3913

and the extended Kalman filter based on Theorem 3.1 is more suitable for the system with a relatively large fractional-order if we consider running time of state estimation and fractional-order identification.

5. **Conclusions.** In this paper, two kinds of extended Kalman filters based on G-L difference are studied to achieve the estimations of state and fractional-order for continuous-time linear fractional-order systems. Firstly, the Sigmoid function is adopted to make the estimation of fractional-order in a suitable range. Then, the first-order Taylor expression is used to linearize the nonlinear function containing the unknown fractional-order. In fact, the extended Kalman filters proposed by Theorem 3.1 and Theorem 3.2 are effective to estimate the state and fractional-order for the investigated system. For a relatively small fractional-order, the estimations of state and fractional-order are more accurate using extended Kalman filter via Theorem 3.2 with the initial compensation. However, if the running time is concerned, the extended Kalman filter via Theorem 3.1 can be more suitable for a relatively large fractional-order. Therefore, the extended Kalman algorithms proposed in this paper are applicable for the practical engineering in terms of different conditions. This paper focuses on linear fractional-order systems, and the further step of this research will investigate the estimations of state and fractional-order for nonlinear fractional-order systems with unknown covariance matrices of process and measurement noises.

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