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APPROXIMATE STATE FEEDBACK LINEARIZATION FOR MIMO SYSTEMS

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Abstract. *Recently, we have proposed a design method of approximate state feedback linearization making the approximation term small. The aim of this paper is to expand the scope of that method to a wide class of nonlinear systems, namely multivariable nonlinear systems. In our procedure, we show first the problem formulation in order to specify the class of nonlinear systems under consideration. Then, we settle a state transformation matrix in order to transform the nonlinear system into an approximately controllable canonical form of Multiple-Input/Multiple-Output (MIMO) systems. After that, we explain the state feedback gain design and use a standard nonlinear linearization method. That makes the MIMO system become linear in the new coordinate. Then, for the resulting linear MIMO system, it becomes easy to apply the well established linear control techniques to stabilizing the transformed system.*

Keywords: Nonlinear control, Feedback linearization, Approximate linearization, Coordinate transformation, Multiple-input/multiple-output systems

1. **Introduction.** There exist a lot of tools to deal with linear systems. However, in the real world, most systems are nonlinear. That makes the linearization of nonlinear systems a paramount control problem that has attracted the attention of many scientists and engineers. So far, several linearization methods have been proposed for nonlinear systems. The exact feedback linearization [1, 2, 3] is based on coordinate transformation and nonlinear feedback control, and is adequate for nonlinear systems with a wide range of action. This method is effectively applied to robotics systems especially such as robotic manipulators [4], however, is limited to some kind of plants. In fact, it has been shown that for nonlinear systems of first and second order this method is straightforward [1], but for higher-order systems, it is tricky to apply this method. This motivates us to study the approximate linearization [5, 6] which can be applied to the higher-order nonlinear systems systematically. The common approach for the approximate linearization employs Taylor series expansions [1, 2, 3, 7]. This method is suitable for nonlinear systems with a narrow range of action. However, it is not quite appropriate for nonlinear systems having a wide range of action. In contrast, Yamada and Yuzawa [8] proposed an alternative approximate feedback linearization method by combining coordinate transformation and nonlinear feedback control. That method can be applied to various nonlinear systems to which the exact linearization method cannot be applied. Recently, we have expanded the method in [8] by proposing a design method of approximate state feedback linearization which makes the approximation term smaller [9]. However, that method is limited to single input systems.

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In this work, we expand the procedure in [9] considering an MIMO nonlinear system. In our approach, we settle first a state transformation matrix in order to transform approximately the nonlinear system into a controllable canonical form. After that, we explain the design method of the nonlinear state feedback control input making the approximation term small, and use a standard nonlinear linearization method to linearize the nonlinear MIMO system.

This paper is organized as follows. In Section 2, we show the problem formulation in order to specify the class of nonlinear systems under consideration. In Section 3, we explain the approximate feedback linearization method where we state first the coordinates transformation and then clarify the design of the stabilizing state feedback control. In Section 4, we give concluding remarks.

2. **Problem Formulation.**

Consider the nonlinear system of the form:

$$\dot{x} = A(x)x + B(x)u, \tag{1}$$

where $x \in R^n$ is the state variable, $u \in R^m$ is the control input, $A(x)$, and $B(x)$ are matrices depending on the state variable with proper dimensions. Without loss of generality, the origin $x = 0$ is the equilibrium point. It is assumed that all the elements of $A(0)$ and $B(0)$ are finite and

$$\text{rank}\begin{bmatrix} B(x) & A(x)B(x) & \cdots & A^{n-1}(x)B(x) \end{bmatrix} = n \tag{2}$$

holds in the neighborhood of $x = 0$.

In this paper, we expand the result in [9] by proposing a design method of an approximate feedback linearization for MIMO nonlinear systems.

3. **Approximate State Feedback Linearization.**

3.1. **Transformation into MIMO controllable canonical form.**

As it is easy to design a control law for canonical form, our focus in this section is to transform the MIMO nonlinear system into a controllable canonical form. First, a state transformation matrix is determined, and then, the nonlinear system is divided into the controllable canonical section and the remainder.

We adopt the coordinates transformation written as:

$$x = T(x)z. \tag{3}$$

$T(x)$ is designed as follows: From the assumption of (2), there exist n independent column vectors in

$$\begin{bmatrix} B(x) & A(x)B(x) & \cdots & A^{n-1}(x)B(x) \end{bmatrix}.$$
$$B(x) = \begin{bmatrix} b_1(x) & b_2(x) & \cdots & b_m(x) \end{bmatrix}.$$

Let integer ν_i $(i = 1, \ldots, m)$ be

$$\begin{aligned}\nu_i \equiv \min\big\{j : A^j(x)b_i(x) \in \text{span}\big[&B(x), A(x)B(x), \ldots, A^{j-1}(x)B(x),\\ &A^j(x)b_1(x), \ldots, A^j(x)b_{i-1}(x)\big]\big\}.\end{aligned} \tag{4}$$

Then we have

$$\begin{aligned}\text{rank}\big[\, &b_1(x) \quad A(x)b_1(x) \quad \cdots \quad A^{\nu_1-1}(x)b_1(x) \quad b_2(x) \quad \cdots \quad A^{\nu_2-1}(x)b_2(x) \quad \cdots\\ &A^{\nu_m-1}(x)b_m(x)\big] = n,\end{aligned} \tag{5}$$

since $\nu_1 + \nu_2 + \cdots + \nu_m = n$ is satisfied. From the definition of ν_i, there exists $\alpha^i_{j,k}(x)$ satisfying

$$\begin{aligned}A^{\nu_i}(x)b_i(x) = {} & \alpha^i_{0,1}(x)b_1(x) + \cdots + \alpha^i_{0,m}(x)b_m(x) + \alpha^i_{1,1}(x)A(x)b_1(x) + \cdots\\ & + \alpha^i_{1,m}(x)A(x)b_m(x) + \cdots + \alpha^i_{\nu_i-1,1}(x)A^{\nu_i-1}(x)b_1(x)\end{aligned}$$

$$\begin{aligned}&+\cdots+\alpha^i_{\nu_i-1,m}(x)A^{\nu_i-1}(x)b_m(x)\\&+\alpha^i_{\nu_i,1}(x)A^{\nu_i}(x)b_1(x)+\cdots+\alpha^i_{\nu_i,i-1}(x)A^{\nu_i}(x)b_{i-1}(x).\end{aligned}\tag{6}$$

We select

$$b_{i,j}(x)=\sum_{k=1}^{m}\alpha^i_{j,k}(x)b_k(x)\quad(j=0,\ldots,\nu_{i-1}),\tag{7}$$

$$\tilde{b}_i(x)=\sum_{k=1}^{i-1}\alpha^i_{\nu_i,k}(x)b_k(x)\quad(i=2,\ldots,m),\quad\tilde{b}_1(x)=0.\tag{8}$$

Equation (6) is rewritten by

$$A^{\nu_i}(x)b_i(x)=b_{i,0}(x)+A(x)b_{i,1}(x)+\cdots+A^{\nu_i-1}(x)b_{i,\nu_i-1}(x)+A^{\nu_i}(x)\tilde{b}_i(x).\tag{9}$$

$t_{i,j}(x)$ $(j=1,\ldots,\nu_i)$ is defined by

$$\begin{aligned}t_{i,j}(x)=&\,A^{\nu_i-j}(x)\left(b_i(x)-\tilde{b}_i(x)\right)-b_{i,j}(x)-A(x)b_{i,j+1}(x)-\cdots\\&-A^{\nu_i-j-1}(x)b_{i,\nu_i-1}(x).\end{aligned}\tag{10}$$

From the definition of ν_i,

$$\operatorname{rank}\left[\begin{array}{ccccccc}t_{1,1}(x)&\cdots&t_{1,\nu_i}(x)&\cdots&t_{m,1}(x)&\cdots&t_{m,\nu_m}(x)\end{array}\right]=n\tag{11}$$

is satisfied. By simple manipulations, we find that the following relationships hold:

$$A(x)t_{i,j}(x)=t_{i,j-1}(x)+b_{i,j-1}(x),\quad j=2,3,\ldots,\nu_i\tag{12}$$

$$A(x)t_{i,1}(x)=b_{i,0}(x).\tag{13}$$

On the other hand, since $t_{i,\nu_i}(x)=b_i(x)-\tilde{b}_i(x)$, using (8), there exist real numbers $\beta_{j,i}(x)$ satisfying the following equation:

$$b_i(x)=t_{i,\nu_i}(x)+\sum_{j=1}^{i-1}\beta_{j,i}(x)t_{j,\nu_j}(x),\quad i=2,\ldots,m,\quad b_1(x)=t_{1,\nu_1}(x).\tag{14}$$

$\beta_{j,i}(x)$ is determined by $\alpha^i_{\nu_i,k}(x)$. Substituting this relationship into (12), Equations (12) and (13) can be written as follows:

$$A(x)t_{i,j}(x)=t_{i,j-1}(x)+\sum_{k=1}^{m}\gamma_{k,i,j}(x)t_{k,\nu_k}(x),\quad j=2,3,\ldots,\nu_i,\tag{15}$$

$$A(x)t_{i,1}(x)=\sum_{k=1}^{m}\gamma_{k,i,1}(x)t_{k,\nu_k}(x).\tag{16}$$

Here, $\gamma_{k,i,j}(x)$ is a real number determined by $\alpha^i_{j,k}(x)$.

We settle $T(x)$ as:

$$T(x)=\left[\begin{array}{cccc}T_1(x)&T_2(x)&\cdots&T_m(x)\end{array}\right],\tag{17}$$

where

$$T_i(x)=\left[\begin{array}{cccc}t_{i,1}(x)&t_{i,2}(x)&\cdots&t_{i,\nu_i}(x)\end{array}\right],\quad i=1,\ldots,m.\tag{18}$$

We suppose that $T(x)$ satifies the following condition:

$$\det\left(T\left(x\right)\right)\neq 0.\tag{19}$$

Using $T(x)$ in (17) and applying (3) to the system in (1), we have

$$\begin{aligned}\dot{z}&=T^{-1}(x)A(x)T(x)z+T^{-1}(x)B(x)u-T^{-1}(x)\dot{T}(x)z\\&\equiv A_z(x)z+B_z(x)u-T^{-1}(x)\dot{T}(x)z,\end{aligned}\tag{20}$$

where

$$A_z(x) = T^{-1}(x)A(x)T(x) = \begin{bmatrix} A_{1,1}(x) & A_{1,2}(x) & \cdots & A_{1,m}(x) \\ A_{2,1}(x) & A_{2,2}(x) & \cdots & A_{2,m}(x) \\ & \cdots & \cdots & \\ A_{m,1}(x) & A_{m,2}(x) & \cdots & A_{m,m}(x) \end{bmatrix}, \tag{21}$$

$$A_{i,i}(x) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ \gamma_{i,i,1}(x) & \gamma_{i,i,2}(x) & \gamma_{i,i,3}(x) & \cdots & \gamma_{i,i,\nu_i}(x) \end{bmatrix} \in R^{\nu_i \times \nu_i}, \tag{22}$$

$$A_{i,j}(x) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ & \cdots & \cdots & \\ 0 & 0 & \cdots & 0 \\ \gamma_{i,j,1}(x) & \gamma_{i,j,2}(x) & \cdots & \gamma_{i,j,\nu_j}(x) \end{bmatrix} \in R^{\nu_i \times \nu_j}, \tag{23}$$

$$B_z(x) = T^{-1}(x)B(x) = \begin{bmatrix} B_{1,1}(x) & B_{1,2}(x) & \cdots & B_{1,m}(x) \\ B_{2,1}(x) & B_{2,2}(x) & \cdots & B_{2,m}(x) \\ & \cdots & \cdots & \\ B_{m,1}(x) & B_{m,2}(x) & \cdots & B_{m,m}(x) \end{bmatrix}, \tag{24}$$

$$B_{i,i}(x) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in R^{\nu_i}, \quad B_{i,j}(x) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \beta_{i,j}(x) \end{bmatrix} \in R^{\nu_i} \quad (i < j), \text{ and}$$

$$B_{i,j}(x) = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \in R^{\nu_i} \quad (i > j). \tag{25}$$

From (20), we find that the nonlinear system is divided into the controllable canonical section and the remainder $T^{-1}(x)\dot{T}(x)z$. Let us write $T^{-1}(x)\dot{T}(x)$ as:

$$T^{-1}(x)\dot{T}(x) = \begin{bmatrix} \omega_{11}(x) & \cdots & \omega_{1n}(x) \\ \vdots & \ddots & \vdots \\ \omega_{n1}(x) & \cdots & \omega_{nn}(x) \end{bmatrix} = \Omega(x) = \tilde{\Omega}(x) + \bar{\Omega}(x). \tag{26}$$

$\tilde{\Omega}(x)$ is the matrix extracting the $(\nu_1 + \cdots + \nu_i)$-th rows of $\Omega(x)$ and $\bar{\Omega}(x)$ is the matrix having 0 at the $(\nu_1 + \cdots + \nu_i)$-th rows, and the same elements as $\Omega(x)$ elsewhere, $i = 1, \ldots, m$.

3.2. **Nonlinear feedback linearization.** In this section, a standard nonlinear feedback linearization is used to transform the controllable canonical form into a linear system.

The control input u in (1) is settled as

$$u = -F(x)z + G(x)v. \tag{27}$$

Now, from (26) and (27), the system in (20) is written as:

$$\dot{z} = \left\{ A_z(x) - B_z(x)F(x) - \tilde{\Omega}(x) \right\} z + B_z(x)G(x)v - \bar{\Omega}(x)z, \tag{28}$$

where $F(x)$ acts as the linearization feedback law to make $A_z(x) - B_z(x)F(x) - \tilde{\Omega}(x)$ a linear matrix and v is an external signal. $F(x)$ is designed by

$$F(x) = -L(x) + q(x)k, \tag{29}$$

where

$$k = \begin{bmatrix} \xi_0 & \cdots & \xi_{n-1} \end{bmatrix} \in R^{1\times n}, \quad (i = 1, \ldots, n-1) \tag{30}$$

$$L(x) = \begin{bmatrix} l_1(x) \\ \vdots \\ l_m(x) \end{bmatrix} \in R^{m\times n}, \quad q(x) = \begin{bmatrix} q_1(x) \\ \vdots \\ q_m(x) \end{bmatrix} \in R^{m\times 1}, \tag{31}$$

$$\begin{cases} l_i(x) = \tilde{e}_i - \tilde{a}_i(x) + \tilde{\omega}_i(x) - \displaystyle\sum_{j=i+1}^{m} \beta_{i,j}(x) l_j(x) \quad (i = 1, 2, \ldots, m-1) \\ l_m(x) = -\tilde{a}_m(x) + \tilde{\omega}_m(x) \\ q_i(x) = -\displaystyle\sum_{j=i+1}^{m} \beta_{i,j}(x) q_j(x) \quad (i = 1, 2, \ldots, m-1) \\ q_m(x) = 1 \end{cases}, \tag{32}$$

$$\tilde{e}_i(x) = \begin{bmatrix} 0 & \cdots & 0 & 1 & \underset{\substack{\uparrow \\ \nu_1 + \cdots + \nu_i + 1}}{0} & \cdots & 0 \end{bmatrix} \in R^{1\times n}. \tag{33}$$

$\tilde{e}_i(x)$ is a row matrix having 1 at the $(\nu_1 + \cdots + \nu_i + 1)$-th column and 0 elsewhere. $\tilde{a}_i(x)$ and $\tilde{\omega}_i(x)$ are the $(\nu_1 + \cdots + \nu_i)$-th row, $(i = 1, \ldots, m)$, of respectively the matrix $A_z(x)$ in (20) and the matrix $\tilde{\Omega}(x)$ in (26). From (29), the system in (28) gets the following form:

$$\dot{z} = \left\{ A_z(x) + B_z(x)L(x) - B_z(x)q(x)k - \tilde{\Omega}(x) \right\} z + B_z(x)G(x)v - \bar{\Omega}(x)z, \tag{34}$$

where

$$B_z(x)q(x) = \begin{bmatrix} 0 & \cdots & 1 \end{bmatrix}^T \text{ and } B_z(x)q(x)k = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ 0 & & & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \cdots & 0 \\ \xi_0 & \xi_1 & \xi_2 & \cdots & \xi_{n-1} \end{bmatrix}. \tag{35}$$

Next, we give a design method of $G(x)$ to make $B_z(x)G(x)$ be a linear matrix. $G(x)$ is rewritten in the block matrix form with the same form as $B_z(x)$ by

$$G(x) = \begin{bmatrix} G_{1,1}(x) & G_{1,2}(x) & \cdots & G_{1,m}(x) \\ G_{2,1}(x) & G_{2,2}(x) & \cdots & G_{2,m}(x) \\ & \cdots & \cdots & \\ G_{m,1}(x) & G_{m,2}(x) & \cdots & G_{m,m}(x) \end{bmatrix}. \tag{36}$$

This yields the i-th colum and j-th row block matrix of $B_z(x)G(x)$, which is written as

$$\sum_{k=i}^{m} B_{i,k}(x) G_{k,j}(x), \quad (i, j = 1, 2, \ldots, m). \tag{37}$$

$G_{i,j}$ is selected sequentially by starting from the last row m to the first row 1 according to the formulas below:

$$\begin{aligned} &G_{m,j}(x) = \delta_{m,j} \quad (j = 1, 2, \ldots, m) \\ &G_{i,j}(x) = -\sum_{k=i+1}^{m} \beta_{i,k}(x) G_{k,j}(x) + \delta_{i,j} \quad (i = 1, 2, \ldots, m-1;\ j = 1, 2, \ldots, m) \end{aligned} \tag{38}$$

where $\delta_{i,j}$ are arbitrary real numbers ($i, j = 1, 2, \ldots, m$). Applying (27) into (20), we have

$$\dot{z} = \bar{A}_z z + \bar{B}_z v - \bar{\Omega}(x)z, \tag{39}$$

where

$$\begin{aligned} \bar{A}_z &= A_z(x) - B_z(x)\left(-L(x) + q(x)k\right) - \tilde{\Omega}(x) \\ &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ -\xi_0 & -\xi_1 & -\xi_2 & \cdots & -\xi_{n-1} \end{bmatrix} \in R^{n\times n}, \end{aligned} \tag{40}$$

$$\bar{B}_z = B_z(x)G(x) = \begin{bmatrix} H_{1,1} & H_{1,2} & \cdots & H_{1,m} \\ H_{2,1} & H_{2,2} & \cdots & H_{2,m} \\ & \cdots & \cdots & \\ H_{m,1} & H_{m,2} & \cdots & H_{m,m} \end{bmatrix}, \quad H_{i,j} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \delta_{i,j} \end{bmatrix} \in R^{\nu_i}. \tag{41}$$

We adopt the approximation as $\bar{\Omega}(x) \simeq 0$, $\forall t, x$. Thus, the nonlinear system becomes

$$\dot{z} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ & & & & 0 \\ & & & & 1 \\ -\xi_0 & -\xi_1 & \cdots & & -\xi_{n-1} \end{bmatrix} z + \begin{bmatrix} H_{1,1} & H_{1,2} & \cdots & H_{1,m} \\ H_{2,1} & H_{2,2} & \cdots & H_{2,m} \\ & \cdots & \cdots & \\ H_{m,1} & H_{m,2} & \cdots & H_{m,m} \end{bmatrix} v. \tag{42}$$

The system obtained is obviously a linear system. Now, we can settle the numbers ξ_i ($i = 0, \ldots, n-1$) in order to have all the roots of the characteristic polynomial

$$s^n + \xi_{n-1}s^{n-1} + \cdots + \xi_0 = 0 \tag{43}$$

of the obtained linear system to be in the open left half plane. In this way, the obtained linear system in (42) will be asymptotically stable.

4. **Conclusion.** We have proposed through this paper a design method of approximate linearization for multi-input nonlinear systems. In our procedure, first a state transformation matrix is determined such that the nominal nonlinear system is transformed into the controllable canonical form and the remainder. Second, a design method of nonlinear state feedback has been proposed. After making the approximation on the remainder which has been reduced by the state feedback law, the nonlinear system is transformed into a linear system in the controllable canonical form. This method can be applied to the nonlinear systems to which the exact linearization method cannot be applied. Such potential applications include inverted pendulums [10], ball-and-beam systems [11], fuel cells [12], and bio-reactors [13]. As the present paper proposed the approximate method, future study may establish methods of estimating nonlinear stability [14, 15], which predetermine that if the linearized system is stable, then the original nonlinear system is also stable.

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