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ESSENTIAL (m,n)-IDEAL AND ESSENTIAL FUZZY (m,n)-IDEALS IN SEMIGROUPS

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Abstract. *Ideal theory plays an important role in studying in semigroups. The theory of (m,n)-ideals in semigroups studied by Lajos in 1963. The notion of (m,n)-ideals generalized the notions of left ideals and right ideals of semigroups. In 2021, Baupradist et al. studied essential ideals and essential fuzzy ideals in semigroups. In this paper, we shall extend concepts of essential ideals and essential fuzzy ideals to (m,n)-ideals in semigroups. We investigate properties of essential (m,n)-ideals and essential fuzzy (m,n)-ideals in semigroups. Moreover, we show some relationship between essential (m,n)-ideals and essential fuzzy (m,n)-ideals.*
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1. **Introduction.** The theory of (m,n)-ideals in semigroups was studied by Lajos in 1963 [5]. The notion of (m,n)-ideals of semigroups generalized the notion of one sided ideals of semigroups. The classical fuzzy sets were proposed in 1965 by Zadeh [11]. These concepts were applied in many areas such as medical science, theoretical physics, robotics, computer science, control engineering, information science, measure theory, logic, set theory, and topology. In 1979, Kuroki [4] defined a fuzzy semigroup and various kinds of fuzzy ideals in semigroups and characterized them. Next, fuzzy set theory in semigroups was widely studied. Recently in 2020, Yiarayong [10] characterized semigroups by picture fuzzy sets. In 2019, Mahboob et al. [6] extended the notion of (m,n)-ideals in semigroups to the notion of fuzzy (m,n)-ideals in semigroups and they characterized (m,n)-regular semigroups by using fuzzy (m,n)-ideals. Essential fuzzy ideals of rings were studied by Medhi et al. in 2008 [7]. Later in 2013, Medhi and Saikia [8] studied concept of T-fuzzy essential ideals and proved properties of T-fuzzy essential ideals of rings. Recently in 2021, Baupradist et al. [1] studied essential ideals and essential fuzzy ideals in semigroups. Together, they studied 0-essential ideals and 0-essential fuzzy ideals in semigroups.

Our aim of this paper is to extend essential ideals and essential fuzzy ideals to (m,n)-ideals in semigroups. Our main results are divided in the following sections. In Section 2, we shall extend concepts of essential ideals and essential fuzzy ideals to (m,n)-ideals in semigroups. In Section 3, we study essential (m,n)-ideals and essential fuzzy (m,n)-ideals of semigroups. In Section 4, we study minimality of essential (m,n)-ideals and essential

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fuzzy (m,n)-ideals in semigroups. Then, we study 0-essential (m,n)-ideals and 0-essential fuzzy (m,n)-ideals of semigroups with zero. Finally, we conclude the paper with future work in Section 6.

2. **Preliminaries.** In this section, we review some basic concepts, necessary to understand our section.

A non-empty subset I of a semigroup S is called a *subsemigroup* of S if $I^2 \subseteq I$.

A non-empty subset I of a semigroup S is called a *left (right) ideal* of S if $SI \subseteq I$ $(IS \subseteq I)$.

An *ideal* I of S is a non-empty subset which is both a left ideal and a right ideal of S.

A subsemigroup I of a semigroup S is said to be an (m,n)-*ideal* of S if $I^m S I^n \subseteq I$ for any $m,n \in \mathbb{N}$.

A subsemigroup I of a semigroup S is said to be an $(m,0)$-*ideal* of S if $I^m S \subseteq I$ for any $m \in \mathbb{N}$.

A subsemigroup I of a semigroup S is said to be an $(0,n)$-*ideal* of S if $SI^n \subseteq I$ for any $n \in \mathbb{N}$.

For a non-empty subset I of a semigroup S and $m,n \in \mathbb{N}$, we denote the

$$[I](m,n) = \bigcup_{r=1}^{m+n} I^r \cap I^m S I^n \text{ is the principal } (m,n)\text{-ideal by } I,$$

$$[I](m,0) = \bigcup_{r=1}^{m} I^r \cap I^m S \text{ is the principal } (m,0)\text{-ideal by } I \text{ and}$$

$$[I](0,n) = \bigcup_{r=1}^{n} I^r \cap S I^n \text{ is the principal } (0,n)\text{-ideal by } I,$$

i.e., the smallest (m,n)-ideal, the smallest $(m,0)$-ideal and the smallest $(0,n)$-ideal of S containing I, respectively.

Lemma 2.1. [3] *Let S be a semigroup and m,n positive integers, $[u]_{(m,n)}$ the principal (m,n)-ideal generated by the element u. Then*

1) $\left([u]_{(m,0)}\right)^m S = u^m S$.
2) $S\left([u]_{(0,n)}\right)^n = Su^n$.
3) $\left([u]_{(m,0)}\right)^m S \left([u]_{(0,n)}\right)^n = u^m S u^n$.

For any $a,b \in [0,1]$, we have

$$a \vee b = \max\{a,b\} \text{ and } a \wedge b = \min\{a,b\}.$$

A *fuzzy set* of a non-empty set T is a function from T into unit closed interval $[0,1]$ of real numbers, i.e., $f : T \to [0,1]$.

For any two fuzzy sets of f and g of a non-empty of T, we define the *support* of f instead of $\operatorname{supp}(f) = \{u \in T \mid f(u) \neq 0\}$, $f \subseteq g$ if $f(u) \leq f(u)$ for all $u \in T$, for each $u \in T$, $(f \vee g)(u) = \max\{f(u), g(u)\} = f(u) \vee g(u)$ and for each $u \in T$, $(f \wedge g)(u) = \min\{f(u), g(u)\} = f(u) \wedge g(u)$.

For two fuzzy sets f and g in a semigroup S, define the product $f \circ g$ as follows: for all $u \in S$,

$$(f \circ g)(u) = \begin{cases} \bigvee\limits_{(y,z) \in F_u} \{\{f(y) \wedge g(z)\} \mid (y,z) \in F_u\} & \text{if } F_u \neq \emptyset, \\ 0 & \text{if } F_u = \emptyset, \end{cases}$$

where $F_u := \{(y,z) \in S \times S \mid u = yz\}$.

A *fuzzy subsemigroup* of a semigroup S if $f(uv) \geq f(u) \wedge f(v)$ for all $u,v \in S$.

A *fuzzy left (right) ideal* of a semigroup S if $f(uv) \geq f(v)$ $(f(uv) \geq f(u))$ for all $u,v \in S$.

A fuzzy subsemigroup f of a semigroup S is said to be a *fuzzy (m,n)-ideal* of S if

$$f(u_1u_2\ldots u_mzv_1v_2\ldots v_n)\geq f(u_1)\wedge f(u_2)\wedge\cdots\wedge f(u_m)\wedge f(v_1)\wedge f(v_2)\wedge\cdots\wedge f(v_n)$$

for all $u_1,u_2,\ldots,u_m,v_1,v_2,\ldots,v_n,z\in S$ and $m,n\in\mathbb{N}$.

The *characteristic fuzzy set* χ_I of a non-empty set I is defined as follows:

$$\chi_I:T\to[0,1],\quad u\mapsto\begin{cases}1 & \text{if } u\in I,\\ \emptyset & \text{if } u\notin I.\end{cases}$$

The following theorems are true.

Theorem 2.1. [6] *Let I be non-empty subset of semigroup S and $m,n\in\mathbb{N}$. Then I is an (m,n)-ideal of S if and only if characteristic function χ_I is an (m,n)-ideal of S.*

Theorem 2.2. [9] *Let I and J be subsets of a non-empty set S. Then $\chi_{I\cap J}=\chi_I\cap\chi_J$ and $\chi_I\circ\chi_J=\chi_{IJ}$.*

Theorem 2.3. *Let f be a nonzero fuzzy set of a semigroup S and $m,n\in\mathbb{N}$. Then f is a fuzzy (m,n)-ideal of S if and only if* supp(f) *is an (m,n)-ideal of S.*

Proof: Assume that f is a fuzzy (m,n)-ideal of S, and let $u_1u_2\ldots u_mzv_1v_2\ldots v_n\in\operatorname{supp}(f)^mS\operatorname{supp}(f)^n$. By assumption, we have

$$f(u_1u_2\ldots u_mzv_1v_2\ldots v_n)\geq f(u_1)\wedge f(u_2)\wedge\cdots\wedge f(u_m)\wedge f(v_1)\wedge f(v_2)\wedge\cdots\wedge f(v_n).$$

Therefore, $f(u_1u_2\ldots u_mv_1v_2)\neq 0$. Thus, $u_1u_2\ldots u_mv_1v_2\ldots v_n\in\operatorname{supp}(f)$. Hence, supp$(f)$ is an (m,n)-ideal of S.

Conversely, assume that supp(f) is an (m,n)-ideal of S and let $u_1,u_2,\ldots,u_m,v_1,v_2,\ldots,v_n,z\in S$.

Case 1. $u_1,u_2,\ldots,u_m,v_1,v_2,\ldots,v_n\in\operatorname{supp}(f)$. So $u_1u_2\ldots u_mzv_1v_2\ldots v_n\in\operatorname{supp}(f)$. Then $f(u_1u_2\ldots u_mzv_1v_2\ldots v_n)=1\geq f(u_1)\wedge f(u_2)\wedge\cdots\wedge f(u_m)\wedge f(v_1)\wedge f(v_2)\wedge\cdots\wedge f(v_n)$.

Case 2. $u_i\notin\operatorname{supp}(f)$ for some i or $v_j\notin\operatorname{supp}(f)$ for some j. We have that $f(u_1)\wedge f(u_2)\wedge\cdots\wedge f(u_m)\wedge f(v_1)\wedge f(v_2)\wedge\cdots\wedge f(v_n)=0\leq f(u_1u_2\ldots u_mzv_1v_2\ldots v_n)$.

Hence, f is a fuzzy (m,n)-ideal of S. □

Next, we will review essential ideals and fuzzy essential ideals in a semigroup and properties of those.

Definition 2.1. *An* essential left (right) ideal *I of a semigroup S if I is a left (right) ideal of S and $I\cap J\neq\emptyset$ for every left (right) ideal J of S.*

Definition 2.2. [1] *An* essential ideal *I of a semigroup S if I is an ideal of S and $I\cap J\neq\emptyset$ for every ideal J of S.*

Theorem 2.4. [1] *Let I be an essential ideal of a semigroup S. If I_1 is an ideal of S containing I, then I_1 is also an essential ideal of S.*

Theorem 2.5. [1] *Let I and J be essential ideals of a semigroup S. Then $I\cup J$ and $I\cap J$ are essential ideals of S.*

Definition 2.3. [1] *An* essential fuzzy ideal *f of a semigroup S if f is a nonzero fuzzy ideal of S and $f\cap g\neq\emptyset$ for every nonzero fuzzy ideal g of S.*

Theorem 2.6. [1] *Let I be an ideal of a semigroup S. Then I is an essential ideal of S if and only if χ_I is an essential fuzzy ideal of S.*

Theorem 2.7. [1] *Let f be a nonzero fuzzy ideal of a semigroup S. Then f is an essential fuzzy ideal of S if and only if* supp(f) *is an essential ideal of S.*

3. **Essential (m,n)-Ideals and Essential Fuzzy (m,n)-Ideals.** In this section, we will study concepts of essential (m,n)-ideals in a semigroup and essential fuzzy (m,n)-ideals in a semigroup and properties of those.

Definition 3.1. *An* essential (m,n)-ideal I *of a semigroup* S *if* I *is an* (m,n)*-ideal of* S *and* $I \cap J \neq \emptyset$ *for every* (m,n)*-ideal* J *of* S.

Example 3.1. *1) We have that a semigroup* S *is an essential* (m,n)*-ideal of* S *for all* $m,n \in \mathbb{N}$.

2) Let S *be a semigroup with zero. Then every ideal of* S *is an essential* (m,n)*-ideal of* S *for all* $m,n \in \mathbb{N}$.

Theorem 3.1. *Let* I *be an essential* (m,n)*-ideal of a semigroup* S. *If* I_1 *is an* (m,n)*-ideal of* S *containing* I, *then* I_1 *is also an essential* (m,n)*-ideal of* S.

Proof: Suppose that I_1 is an (m,n)-ideal of S such that $I_1 \subseteq I$ and let J be any (m,n)-ideal of S. Thus, $I \cap J \neq \emptyset$. Hence, $I_1 \cap J \neq \emptyset$. Therefore, I_1 is an essential (m,n)-ideal of S. $\square$

Theorem 3.2. *Let* I *and* J *be essential* (m,n)*-ideals of a semigroup* S. *Then* $I \cup J$ *and* $I \cap J$ *are essential* (m,n)*-ideals of* S.

Proof: Since $I \subseteq I \cup J$ and I is an essential (m,n)-ideal of S, we have $I \cup J$ is an essential (m,n)-ideal of S by Theorem 3.1.

Since I and J are essential (m,n)-ideals of S, we have I and J are (m,n)-ideals of S. Thus, $I \cap J$ is an (m,n)-ideal of S. Let K be an (m,n)-ideal of S. Then $I \cap K \neq \emptyset$. Thus, there exist $u, v \in I \cap K$. Let $u, v \in J$. Then $uv \in (I \cap J) \cap K$. Thus, $(I \cap J) \cap K \neq \emptyset$. Hence, $I \cap J$ is an essential (m,n)-ideal of S. $\square$

Theorem 3.3. *Let* $\{K_i \mid i \in \mathcal{A}\}$ *be a non-empty collection of essential* (m,n)*-ideals of a semigroup* S. *Then* $\bigcap_{i\in\mathcal{A}} K_i$ *is an essential* (m,n)*-ideal of* S.

Proof: By assumption, we have $\{K_i \mid i \in \mathcal{A}\}$ is an (m,n)-ideal of S. Let J be any (m,n)-ideal of S. Then $\bigcap_{i\in\mathcal{A}} K_i \cap J$ is an (m,n)-ideal of S and $\bigcap_{i\in\mathcal{J}} K_i \cap J \neq \emptyset$. Thus, $\bigcap_{i\in\mathcal{A}} K_i$ is an essential (m,n)-ideal of S. $\square$

Definition 3.2. *An* essential fuzzy (m,n)-ideal f *of a semigroup* S *if* f *is a nonzero fuzzy* (m,n)*-ideal of* S *and* $f \cap g \neq 0$ *for every nonzero fuzzy* (m,n)*-ideal* g *of* S.

Theorem 3.4. *Let* $\{f_i \mid i \in \mathcal{A}\}$ *be a non-empty collection of essential fuzzy* (m,n)*-ideals of a semigroup* S. *Then* $\bigwedge_{i\in\mathcal{A}} f_i$ *is an essential fuzzy* (m,n)*-ideal of* S.

Proof: By assumption we have $\{f_i \mid i \in \mathcal{A}\}$ is a fuzzy (m,n)-ideal of S. Let g be any fuzzy (m,n)-ideal of S. Then $\bigwedge_{i\in\mathcal{A}} f_i \cap g$ is a fuzzy (m,n)-ideal of S and $\bigwedge_{i\in\mathcal{A}} f_i \cap g \neq 0$. Thus, $\bigwedge_{i\in\mathcal{A}} f_i$ is an essential fuzzy (m,n)-ideal of S. $\square$

Theorem 3.5. *Let* I *be an* (m,n)*-ideal of a semigroup* S. *Then* I *is an essential* (m,n)*-ideal of* S *if and only if* χ_I *is an essential fuzzy* (m,n)*-ideal of* S.

Proof: Suppose that I is an essential (m,n)-ideal of S and let g be any nonzero fuzzy (m,n)-ideal of S. Then $\mathrm{supp}(g)$ is an (m,n)-ideal of S. By assumption we have I is an (m,n)-ideal of S. Then χ_I is an (m,n)-ideal of S. Thus, $I \cap \mathrm{supp}(g) \neq 0$. So there exist $u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n, z \in I \cap \mathrm{supp}(g)$. It implies that

$$(\chi_I \cap g)(u_1 u_2 \dots u_m z v_1 v_2 \dots v_n) \neq 0.$$

Hence, $\chi_I \cap g \neq \emptyset$. Therefore, χ_I is an essential fuzzy (m,n)-ideal of S.

Conversely, assume that χ_I is an essential fuzzy (m,n)-ideal of S and let J be any (m,n)-ideal of S. Then χ_J is a nonzero fuzzy (m,n)-ideal of S. Since χ_I is an essential fuzzy (m,n)-ideal of S, we have χ_I is a fuzzy (m,n)-ideal of S. Thus, $\chi_I \cap \chi_J \neq \emptyset$. So by Theorem 2.2, $\chi_{I\cap J} \neq 0$. Hence, $I \cap J \neq \emptyset$. Therefore, I is an essential (m,n)-ideal of S. □

Theorem 3.6. *Let f be a nonzero fuzzy (m,n)-ideal of a semigroup S. Then f is an essential fuzzy (m,n)-ideal of S if and only if* $\mathrm{supp}(f)$ *is an essential (m,n)-ideal of S.*

Proof: Assume that f is an essential fuzzy (m,n)-ideal of S. Then $\mathrm{supp}(f)$ is an (m,n)-ideal of S. Let I be any (m,n)-ideal of S. Then by Theorem 2.1, χ_I is an (m,n)-ideal of S. Since f is an essential fuzzy (m,n)-ideal of S, we have f is a fuzzy (m,n)-ideal of S. Thus, $f \cap \chi_I \neq \emptyset$. So there exist $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n, z \in S$ such that

$$(f \cap \chi_I)(u_1u_2\ldots u_mzv_1v_2\ldots v_n) \neq \emptyset.$$

It implies that $f(u_1u_2\ldots u_mzv_1v_2\ldots v_n) \neq 0$ and $\chi_I(u_1u_2\ldots u_mzv_1v_2\ldots v_n) \neq 0$. Hence, $u_1u_2\ldots u_mzv_1v_2\ldots v_n \in \mathrm{supp}(f) \cap I$ so, $\mathrm{supp}(f) \cap I \neq \emptyset$. It implies that $\mathrm{supp}(f)$ is an essential (m,n)-ideal of S.

Conversely, assume that $\mathrm{supp}(f)$ is an essential (m,n)-ideal of S and let g be a nonzero fuzzy (m,n)-ideal of S. Then $\mathrm{supp}(g)$ is an (m,n)-ideal of S. Thus, $\mathrm{supp}(f) \cap \mathrm{supp}(g) \neq \emptyset$. So there exist $u_1u_2\ldots u_mzv_1v_2\ldots v_n \in \mathrm{supp}(f) \cap \mathrm{supp}(g)$. This implies that

$$f(u_1u_2\ldots u_mzv_1v_2\ldots v_n) \neq 0 \text{ and } g(u_1u_2\ldots u_mzv_1v_2\ldots v_n) \neq 0$$

for all $u_1, u_2, \ldots, u_m, z, v_1, v_2, \ldots, v_n \in S$. Hence, $(f \cap g)(u_1u_2\ldots u_mzv_1v_2\ldots v_n) \neq 0$ for all $u_1, u_2, \ldots, u_m, z, v_1, v_2.\ldots, v_n \in S$. Therefore, $f \cap g \neq 0$. We conclude that f is an essential fuzzy (m,n)-ideal of S. □

Theorem 3.7. *Let f be an essential fuzzy (m,n)-ideal of a semigroup S. If f_1 is a fuzzy (m,n)-ideal of S such that $f \subseteq f_1$, then f_1 is also an essential fuzzy (m,n)-ideal of S.*

Proof: Let f_1 be a fuzzy (m,n)-ideal of S such that $f \subseteq f_1$ and let g be any fuzzy (m,n)-ideal of S. Thus, $f \cap g \neq 0$. So $f_1 \cap g \neq 0$. Hence f_1 is an essential fuzzy (m,n)-ideal of S. □

Theorem 3.8. *Let f_1 and f_2 be essential fuzzy (m,n)-ideals of a semigroup S. Then $f_1 \cup f_2$ and $f_1 \cap f_2$ are essential fuzzy (m,n)-ideals of S.*

Proof: By Theorem 3.7, $f_1 \cup f_2$ is an essential fuzzy (m,n)-ideal of S. Since f_1 and f_2 are essential fuzzy (m,n)-ideals of S, we have $f_1 \cap f_2$ is a fuzzy (m,n)-ideal of S. Let g be a nonzero fuzzy (m,n)-ideal of S. Then, $f_1 \cap g \neq 0$. Thus, there exist $u_1, u_2, \ldots, u_m \in S$ such that

$$f_1(u_1u_2\ldots u_m) \neq 0 \text{ and } g(u_1u_2\ldots u_m) \neq 0.$$

Since $f_2 \neq 0$, let $v_1, v_2, \ldots, v_n \in S$ such that $f_2(v_1v_2\ldots v_n) \neq 0$.

Since f_1 and f_2 are fuzzy (m,n)-ideals of S, we have

$$f_1(u_1u_2\ldots u_m) \geq f_1(u_1u_2\ldots u_m) \wedge f_1(v_1v_2\ldots v_n) > 0$$

and

$$f_2(u_1u_2\ldots u_mzv_1v_2\ldots v_n) \geq f_2(u_1u_2\ldots u_m) \wedge f_2(v_1v_2\ldots v_n) > 0,$$

for all $z \in S$. Thus,

$$\begin{aligned}&(f_1 \cap f_2)(u_1u_2\ldots u_mzv_1v_2\ldots v_n)\\ &= f_1(u_1u_2\ldots u_mzv_1v_2\ldots v_n) \wedge f_2(u_1u_2\ldots u_mzv_1v_2\ldots v_n) \neq 0.\end{aligned}$$

Since g is a fuzzy (m,n)-ideal of S and $g(u_1u_2\ldots u_m) \neq 0$, we have

$$g(u_1u_2\ldots u_mzv_1v_2\ldots v_n) \neq 0 \text{ for all } u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n, z \in S.$$

Thus, $[(f_1 \cap f_2) \cap g](u_1u_2\ldots u_mzv_1v_2\ldots v_n) \neq 0$. Hence, $[(f_1 \cap f_2) \cap g] \neq 0$. Therefore, $f_1 \cap f_2$ is an essential fuzzy (m,n)-ideal of S. □

4. **Minimality of Essential (m,n)-Ideals and Essential Fuzzy (m,n)-Ideals.** In this section, we will focus on minimality of essential (m,n)-ideals and essential fuzzy (m,n)-ideals.

Definition 4.1. *An essential (m,n)-ideal I of a semigroup S is called a* minimal essential (m,n)-ideal *if for every essential (m,n)-ideal J of S such that $J \subseteq I$, we have $J = I$.*

Definition 4.2. *An essential fuzzy (m,n)-ideal f of a semigroup S is called a* minimal essential fuzzy (m,n)-ideal *if for every essential fuzzy (m,n)-ideal g of S such that $g \subseteq f$, we have* $\mathrm{supp}(f) = \mathrm{supp}(g)$.

Theorem 4.1. *Let K be a non-empty subset of a semigroup S. Then K is a minimal essential (m,n)-ideal of S if and only if χ_K is a minimal essential fuzzy (m,n)-ideal of S.*

Proof: Let K be a minimal essential (m,n)-ideal of S. Then K is an essential (m,n)-ideal of S. Thus by Theorem 3.5, χ_K is an essential fuzzy (m,n)-ideal of S. Let f be an essential fuzzy (m,n)-ideal of S such that $f \subseteq \chi_K$. Then $\mathrm{supp}(f) \subseteq \mathrm{supp}(\chi_K) = K$. Thus, by Theorem 3.6, $\mathrm{supp}(f)$ is an essential (m,n)-ideal of S. Since K is a minimal essential (m,n)-ideal of S, we have $\mathrm{supp}(f) = K = \mathrm{supp}(\chi_K)$. Thus, χ_K is a minimal essential fuzzy (m,n)-ideal of S.

Conversely, assume that χ_K is a minimal essential fuzzy (m,n)-ideal of S and let J be an essential fuzzy (m,n)-ideal of S such that $J \subseteq K$. Then χ_J is an essential fuzzy (m,n)-ideal of S such that $\chi_J \subseteq \chi_K$. Thus, $J = \mathrm{supp}(\chi_J) = \mathrm{supp}(\chi_K) = K$. Hence, K is a minimal essential (m,n)-ideal of S. □

5. **0-Essential (m,n)-Ideals and 0-Essential Fuzzy (m,n)-Ideals.** In this section, let S be a semigroup with zero.

Definition 5.1. *Let S be a semigroup with zero. A nonzero (m,n)-ideal I of S is called a* 0-essential (m,n)-ideal *of S if $I \cap J \neq \{0\}$ for every nonzero (m,n)-ideal J of S.*

Theorem 5.1. *Let S be a semigroup with zero and K be a 0-essential (m,n)-ideal of S. If I_1 is an (m,n)-ideal of S containing I, then I_1 is also a 0-essential (m,n)-ideal of S.*

Proof: Let I be a 0-essential (m,n)-ideal of S and I_1 be an (m,n)-ideal of S such that $I \subseteq I_1$. Let J be any nonzero (m,n)-ideal of S. Thus, $I \cap J \neq \{0\}$. So, $I_1 \cap J \neq \{0\}$. Hence, I_1 is an essential (m,n)-ideal of S. □

Theorem 5.2. *Let S be a semigroup with zero. Assume that I_1 and I_2 are 0-essential (m,n)-ideals of S. Then $I_1 \cup I_2$ and $I_1 \cap I_2$ are 0-essential (m,n)-ideals of S.*

Proof: By Theorem 5.1, $I_1 \cup I_2$ is a 0-essential (m,n)-ideal of S. Since I_1 and I_2 are 0-essential (m,n)-ideals of S, we have I_1 and I_2 are (m,n)-ideals of S. Thus, $I_1 \cap I_2$ is an (m,n)-ideal of S. Let J be any nonzero (m,n)-ideal of S. Then $I_1 \cap J \neq \{0\}$. Thus, there exists a nonzero $u \in I \cap J$. Let $(u)_i$ be an (m,n)-ideal of S generated by u. Then $(u)_i \neq 0$. Thus, there exists a nonzero element $v \in (u)_i \cap I_2$. So, $y \in (I_1 \cap I_2) \cap J$. Hence, $I_1 \cap I_2$ is a 0-essential (m,n)-ideal of S. □

Definition 5.2. *Let S be a semigroup with zero. A fuzzy (m,n)-ideal f of S is called a* 0-essential fuzzy (m,n)-ideal *of S if* $\mathrm{supp}(f \cap g) \neq \{0\}$ *for every fuzzy (m,n)-ideal g of S.*

Theorem 5.3. *Let S be a semigroup with zero and K be a nonzero (m,n)-ideal of S. Then K is a 0-essential (m,n)-ideal of S if and only if χ_K is a 0-essential fuzzy (m,n)-ideal of S.*

Proof: Assume that K is a 0-essential (m,n)-ideal of S and let g be a fuzzy (m,n)-ideal of S. Then $\mathrm{supp}(g)$ is a nonzero (m,n)-ideal of S. Thus, $K \cap \mathrm{supp}(g) \neq \{0\}$. So there exists a nonzero element $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n, z \in S$ such that $u_1u_2 \ldots u_m z v_1 v_2 \ldots v_n \in$

$K \cap \mathrm{supp}(g)$. Since K is a 0-essential (m,n)-ideal of S, we have K is an essential (m,n)-ideal of S. Thus, χ_K is an essential (m,n)-ideal of S. So, $(\chi_K \cap g)(u_1u_2 \dots u_m z v_1 v_2 \dots v_n) \neq 0$. Hence, $u_1u_2 \dots u_m z v_1 v_2 \dots v_n \in \mathrm{supp}(\chi_K \cap g)$. Therefore, χ_K is a 0-essential fuzzy (m,n)-ideal of S.

Conversely, assume that χ_K is a 0-essential fuzzy (m,n)-ideal of S and let J be a nonzero (m,n)-ideal of S. Then χ_J is a fuzzy (m,n)-ideal of S. Thus, $\mathrm{supp}(\chi_K \cap \chi_J) \neq \{0\}$ so by Theorem 2.2, $\chi_K \cap \chi_J = \chi_{K\cap J} \neq \{0\}$. Hence, $K \cap J \neq \emptyset$. □

Theorem 5.4. *Let S be a semigroup with zero and f be a fuzzy (m,n)-ideal of S. Then f is a 0-essential fuzzy (m,n)-ideal of S if and only if* $\mathrm{supp}(f)$ *is a 0-essential (m,n)-ideal of S.*

Proof: Assume that K is a 0-essential (m,n)-ideal of S and let g be a fuzzy (m,n)-ideal of S. Then $\mathrm{supp}(g)$ is an (m,n)-ideal of S. So, $K \cap \mathrm{supp}(g) \neq \{0\}$. Thus, there exists a nonzero element $u_1u_2 \dots u_m z v_1 v_2 \dots v_n \in K \cap \mathrm{supp}(g)$. So, $(\chi_K \cap g)(u_1u_2 \dots u_m z v_1 v_2 \dots v_n) \neq 0$. Hence, $\mathrm{supp}(\chi_K \cap g) \neq 0$. Therefore, χ_K is a 0-essential fuzzy (m,n)-ideal of S.

Conversely, assume that χ_K is a 0-essential fuzzy (m,n)-ideal of S and let J be an (m,n)-ideal of S. Then χ_J is a fuzzy (m,n)-ideal of S. Thus, $\mathrm{supp}(\chi_K \cap \chi_J) \neq \{0\}$. Hence, $K \cap J \neq \{0\}$. Therefore, K is a 0-essential (m,n)-ideal of S. □

Theorem 5.5. *Let S be a semigroup with zero and f be a 0-essential fuzzy (m,n)-ideal of S. If f_1 is a fuzzy (m,n)-ideal of S such that $f \subseteq f_1$, then f_1 is also a 0-essential fuzzy (m,n)-ideal of S.*

Proof: Assume that f is a 0-essential fuzzy (m,n)-ideal of S and f is a fuzzy (m,n)-ideal of S such that $f \subseteq f_1$. Let g be any fuzzy (m,n)-ideal of S. Then $\mathrm{supp}(f \cap g) \neq \{0\}$. Thus, $\mathrm{supp}(f_1 \cap g) \neq \{0\}$. Hence, f_1 is a 0-essential fuzzy (m,n)-ideal of S. □

Theorem 5.6. *Let S be a semigroup with zero. Assume that f_1 and f_2 are 0-essential fuzzy (m,n)-ideals of S. Then $f_1 \cup f_2$ and $f_1 \cap f_2$ are 0-essential fuzzy (m,n)-ideals of S.*

Proof: By Theorem 5.5, $f_1 \cup f_2$ is a 0-essential fuzzy (m,n)-ideal of S. Since f_1 and f_2 are 0-essential fuzzy (m,n)-ideals of S, we have f_1 and f_2 are fuzzy (m,n)-ideal of S. Thus, $f_1 \cap f_2$ is a fuzzy (m,n)-ideal of S. Let g be any fuzzy (m,n)-ideal of S. Then $\mathrm{supp}(f_1 \cap g) \neq \{0\}$. Thus, there exists a nonzero element $u_1, u_2, \dots, u_m, z, v_1, v_2, \dots, v_n \in S$ such that

$$(f_1 \cap g)(u_1u_2 \dots u_m z v_1 v_2 \dots v_n) \neq 0.$$

Since f_2 is a 0-essential fuzzy (m,n)-ideal of S, we have $\mathrm{supp}(f_2)$ is a 0-essential fuzzy (m,n)-ideal of S. Thus, $\mathrm{supp}(f_2 \cap x) \neq \{0\}$. So there exist $u_1, u_2, \dots, u_m, z, v_1, v_2, \dots,$ $v_n \in \mathrm{supp}(f_2) \cap x$. Hence, $f_2(u_1u_2 \dots u_m z v_1 v_2 \dots v_n) \neq 0$. Since f_1 and g are fuzzy (m,n)-ideals of S, we have $((f_1 \cap f_2) \cap g)(u_1u_2 \dots u_m z v_1 v_2 \dots v_n) \neq 0$. Thus, $\mathrm{sup}[(f_1 \cap f_2) \cap g] \neq \{0\}$. Hence, $f_1 \cap f_2$ is a 0-essential fuzzy (m,n)-ideal of S. □

6. **Conclusion.** In Sections 2-4, we define essential (m,n)-ideals and essential fuzzy (m,n)-ideals in semigroups. We show that the union and intersection of essential (m,n)-ideals (essential fuzzy (m,n)-ideals) of semigroup S are also an essential (m,n)-ideal (essential fuzzy (m,n)-ideal) of S. Moreover, we show some relationship between essential (m,n)-ideals and essential fuzzy (m,n)-ideals. In Section 5, we define 0-essential (m,n)-ideals and 0-essential fuzzy (m,n)-ideals in semigroups with zero.

In the future work, we can study essential (m,n)-ideals and essential fuzzy (m,n)-ideals in generalizations of semigroups. For example, we can define essential (m,n)-hyperideals and essential fuzzy (m,n)-hyperideals in semihypergroups.

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