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ADMISSIBILITY ANALYSIS OF SINGULAR SYSTEMS WITH TIME-VARYING DELAY BASED ON THE NEUTRAL SYSTEM APPROACH

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Abstract. *In this paper, Lyapunov's second method is adopted to study the admissibility of a class of singular systems with time-varying delay. Firstly, the neutral system approach is used to transform singular time-varying delay systems into neutral time-varying delay systems on the basis of ensuring that singular time-varying delay systems are regular and impulse free. Secondly, an augmented Lyapunov-Krasovskii functional (LKF) is proposed, and a quadruple integral term is added to the LKF to obtain more time delay information. Thirdly, the fourth-order Bessel-Legendre integral inequality and the double third-order Bessel-Legendre integral inequality are used to deal with the integral terms of LKF after derivation, and the stability condition for neutral time-varying delay systems is obtained. It is worth noting that the stability criterion is a new improved admissibility condition of singular time-varying delay systems. Finally, a numerical example is employed to illustrate the feasibility and effectiveness of the proposed result.*

Keywords: Singular time-varying delay systems, Neutral time-varying delay systems, Admissibility, Lyapunov-Krasovskii functional (LKF), Bessel-Legendre (B-L) integral inequality

1. **Introduction.** Singular systems are also referred to as descriptor systems, implicit systems, semi-state systems, or generalized state-space systems. Compared to normal systems, singular systems are a more extensive type of dynamic systems, which can more accurately describe the actual dynamic systems, due to the fact that singular system models include not only dynamic equations but also algebraic constraints [1].

Delay is often encountered in various engineering systems, such as digital control systems, manufacturing processes and remote control systems, and the existence of delay is frequently a source of instability and poor performance [2,3]. The conservativeness of the sufficient condition for the stability of the time-delay systems is related to the selection of LKF and the treatment of its derivative terms [4].

In order to reduce the conservativeness of the stability conditions of the singular time-delay systems, control scholars have done a lot of work in the study of LKF construction. In 2011, Ding et al. introduced more time-delay information by augmenting sereval terms of the simple LKF [5]. In 2012, Balasubramaniam et al. adopted the LKF with triple integrals for the singular interval time-varying delay systems [6]. In 2014 and 2016, Liu et al. proposed a method to transform a singular time-delay system into a neutral time-delay system [7,8].

In order to make the integral terms of LKF after derivation closer to the true value, an appropriate inequality method can be selected. Since 2001, Jensen inequality [9] has been widely used to deal with the integral terms of LKF after derivation. Seuret and Gouaisbaut proposed a new Wirtinger inequality based on Fourier theory in 2013 [10],

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which gives a more accurate boundary estimate than Jensen integral inequality. After that, Seuret and Gouaisbaut proposed Bessel-Legendre (B-L) inequality in 2015 [11]. Compared with Jensen inequality and Wirtinger inequality, B-L inequality enlarges the integral term less [12].

Under certain conditions, the admissibility of singular time-varying delay systems is equivalent to the stability of neutral time-varying delay systems by transforming singular time-varying delay systems into neutral time-varying delay systems. It is proved that it is efficient to obtain the admissibility condition for singular time-varying delay systems and it is found in [7,8]. Moreover, constructing an appropriate LKF could effectively reduce the conservatism of admissibility condition for singular time-varying delay systems. Currently, there are two newly developed techniques to LKF. One technique is to augment LKF in [5], and the other is to add triple integral terms into LKF in [6].

The admissibility condition for a class of singular systems with time-varying delays is discussed in this paper. Firstly, the singular time-varying delay system is equivalently transformed into the neutral time-varying delay system. Secondly, the LKF is augmented and the quadruple integral term is added into LKF to obtain more delay information. Thirdly, B-L integral inequalities are used to amplify the integral term generated by the derivation of LKF and the amplification result is closer to the true value of the integral term. Fourthly, the less conservative stability condition of the neutral time-varying delay system is deduced and equivalently, a new admissibility condition of the singular time-varying delay system is obtained. Finally, a numerical example is included to illustrate the effectiveness and superiority of the proposed method.

2. **Problem Statement and Preliminaries.** Consider the following linear singular system with time-varying delay:

$$\begin{cases} E\dot{x}(t) = Ax(t) + A_d x(t-h(t)) \\ x(t) = \phi(t), \quad t \in [-h, 0] \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the system state, and $\mathbb{R}^n$ is an n-dimensional vector space, defined in the real number field. $E, A, A_d \in \mathbb{R}^{n\times n}$ are known real constant matrices, where E may be singular, $rank(E) = r \leq n$, $\phi(t)$ is a differentiable vector-valued initial continuous function and $h(t)$ is a time-varying delay with the differentiable function satisfying

$$0 \leq h(t) \leq h, \quad \dot{h}(t) \leq d < 1, \quad \forall t \geq 0$$

where h is the upper bound of time delay, and d are given bounds.

Definition 2.1. [13] *The pair (E, A) is said to be regular if* $\det(sE - A)$ *is not identically zero. The pair (E, A) is said to be impulse free if $deg(\det(sE - A)) = rank(E)$.*

Lemma 2.1. [1] *If the pair (E, A) is regular and impulse free, then the solution to the singular time-delay system (1) exists and is impulse free and unique on $[0, \infty)$.*

Definition 2.2. [1] *The singular time-delay system (1) is said to be regular and impulse free, if the pair (E, A) is regular and impulse free. The singular time-delay system (1) is said to be admissible, if it is regular, impulse free and stable.*

If the pair (E, A) is regular and impulse free, invertible matrices $M, N \in \mathbb{R}^{n\times n}$ can always be found such that

$$\bar{E} := MEN = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{A} := MAN = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix}$$

where $A_1 \in \mathbb{R}^{r\times r}$. Let

$$\bar{A}_d = MA_dN = \begin{bmatrix} A_{d_1} & A_{d_2} \\ A_{d_3} & A_{d_4} \end{bmatrix}, \quad \mu(t) = N^{-1}x(t) = \begin{bmatrix} \mu_1(t) \\ \mu_2(t) \end{bmatrix}$$

where the partitions of $\bar{A}_d$ are compatible with the structure of $\bar{E}$, $\mu_1(t)$ is a vector with dimension r, and $\mu_2(t)$ is a vector with dimension $n-r$.

Thus, the first equation in system (1) is equivalent to

$$\bar{E}\dot{\mu}(t) = \bar{A}\mu(t) + \bar{A}_d\mu(t-h(t)) \tag{2}$$

which is with the form of

$$\dot{\mu}_1 = A_1\mu_1(t) + A_{d_1}\mu_1(t-h(t)) + A_{d_2}\mu_2(t-h(t)) \tag{3}$$

$$0 = \mu_2(t) + A_{d_3}\mu_1(t-h(t)) + A_{d_4}\mu_2(t-h(t)) \tag{4}$$

where $\mu_1(t-h(t))$ is an r-dimensional vector, and $\mu_2(t-h(t))$ is an $n-r$-dimensional vector.

Calculating derivative of Equation (4) with respect to t, we have

$$\dot{\mu}_2(t) + \left(1-\dot{h}(t)\right)A_{d_3}\dot{\mu}_1(t-h(t)) + \left(1-\dot{h}(t)\right)A_{d_4}\dot{\mu}_2(t-h(t)) = 0 \tag{5}$$

The right side of Equation (4) is equal to zero, and the left side of Equation (5) is equal to zero, so the right side of Equation (4) is equal to the left side of Equation (5). We have

$$\begin{aligned}\dot{\mu}_2(t) = & -\mu_2(t) - A_{d_3}\mu_1(t-h(t)) - A_{d_4}\mu_2(t-h(t)) \\ & -\left(1-\dot{h}(t)\right)A_{d_3}\dot{\mu}_1(t-h(t)) - \left(1-\dot{h}(t)\right)A_{d_4}\dot{\mu}_2(t-h(t))\end{aligned} \tag{6}$$

According to (3) and (6), combining $\dot{\mu}_1(t)$ and $\dot{\mu}_2(t)$ into a vector can form a new system

$$\begin{aligned}\begin{bmatrix}\dot{\mu}_1(t)\\ \dot{\mu}_2(t)\end{bmatrix} = & \begin{bmatrix}A_1 & 0\\ 0 & -I_{n-r}\end{bmatrix}\begin{bmatrix}\mu_1(t)\\ \mu_2(t)\end{bmatrix} + \begin{bmatrix}A_{d_1} & A_{d_2}\\ -A_{d_3} & -A_{d_4}\end{bmatrix}\begin{bmatrix}\mu_1(t-h(t))\\ \mu_2(t-h(t))\end{bmatrix} \\ & + \left(1-\dot{h}(t)\right)\begin{bmatrix}0 & 0\\ -A_{d_3} & -A_{d_4}\end{bmatrix}\begin{bmatrix}\dot{\mu}_1(t-h(t))\\ \dot{\mu}_2(t-h(t))\end{bmatrix}\end{aligned} \tag{7}$$

Let

$$\hat{A} = \begin{bmatrix}A_1 & 0\\ 0 & -I_{n-r}\end{bmatrix},\ \hat{A}_d = \begin{bmatrix}A_{d_1} & A_{d_2}\\ -A_{d_3} & -A_{d_4}\end{bmatrix},\ \hat{C}(t) = \left(1-\dot{h}(t)\right)C,\ C = \begin{bmatrix}0 & 0\\ -A_{d_3} & -A_{d_4}\end{bmatrix}$$

Then the singular time-varying delay system (1) can be transformed into the following neutral time-varying delay system

$$\begin{cases}\dot{\mu}(t) - \hat{C}(t)\dot{\mu}(t-h(t)) = \hat{A}\mu(t) + \hat{A}_d\mu(t-h(t))\\ \mu(t) = \psi(t), \quad t \in [-h, 0]\end{cases} \tag{8}$$

It is noted that on the assumption that the singular time-varying delay system (1) is regular and impulse free, the asymptotic stability of the neutral time-varying delay system (8) will ensure the admissibility of the singular time-varying delay system (1) [8].

Lemma 2.2. *Given a symmetric matrix* $S = \begin{bmatrix}S_{11} & S_{12}\\ * & S_{22}\end{bmatrix}$, *where* $S \in \mathbb{R}^{n\times n}$, $S_{11} \in \mathbb{R}^{r\times r}$. *The following three conditions are equivalent:*
1) $S < 0$*;*
2) $S_{11} < 0$, $S_{22} - S_{12}^{\mathrm{T}}S_{11}^{-1}S_{12} < 0$*;*
3) $S_{22} < 0$, $S_{11} - S_{12}S_{22}^{-1}S_{12}^{\mathrm{T}} < 0$.

Lemma 2.3. [12] *(B-L integral inequlity) For a matrix* $R > 0$, $R \in \mathbb{R}^{n\times n}$, *scalars* a *and* b *with* $a < b$, *a vector function* $\dot{x} : [a, b] \to \mathbb{R}^n$, *the following integral inequality holds*

$$(b-a)\int_a^b \dot{x}^{\mathrm{T}}(s)R\dot{x}(s)ds \geq \Lambda_1^{\mathrm{T}}R\Lambda_1 + 3\Lambda_2^{\mathrm{T}}R\Lambda_2 + 5\Lambda_3^{\mathrm{T}}R\Lambda_3 + 7\Lambda_4^{\mathrm{T}}R\Lambda_4 + 9\Lambda_5^{\mathrm{T}}R\Lambda_5$$

where

$$\Lambda_1 = x(b) - x(a); \quad \Lambda_2 = x(b) + x(a) - \frac{2}{b-a}\int_a^b x(s)ds;$$
$$\Lambda_3 = x(b) - x(a) + \frac{6}{b-a}\int_a^b x(s)ds - \frac{12}{(b-a)^2}\int_a^b\int_\theta^b x(s)dsd\theta;$$
$$\Lambda_4 = x(b) + x(a) - \frac{12}{b-a}\int_a^b x(s)ds + \frac{60}{(b-a)^2}\int_a^b\int_\theta^b x(s)dsd\theta - \frac{120}{(b-a)^3}\int_a^b\int_u^b\int_\theta^b x(s)dsd\theta du;$$
$$\Lambda_5 = x(b) - x(a) + \frac{20}{b-a}\int_a^b x(s)ds - \frac{180}{(b-a)^2}\int_a^b\int_\theta^b x(s)dsd\theta + \frac{840}{(b-a)^3}\int_a^b\int_u^b\int_\theta^b x(s)dsd\theta du - \frac{1680}{(b-a)^3}\int_a^b\int_u^b\int_\theta^b\int_\lambda^b x(s)dsd\lambda d\theta du.$$

Lemma 2.4. [12] *(B-L integral inequlity) For a matrix $R > 0$, $R \in \mathbb{R}^{n\times n}$, scalars a and b with $a < b$, a vector function $\dot{x} : [a, b] \to \mathbb{R}^n$, the following integral inequality holds*

$$(b-a)^2\int_a^b\int_u^b \dot{x}^{\mathrm{T}}(s)R\dot{x}(s)dsdu \geq 2{\Theta_1}^{\mathrm{T}}R\Theta_1 + 16{\Theta_2}^{\mathrm{T}}R\Theta_2 + 54{\Theta_3}^{\mathrm{T}}R\Theta_3 + 128{\Theta_4}^{\mathrm{T}}R\Theta_4$$

where

$$\Theta_1 = (b-a)x(b) - \int_a^b x(s)ds; \quad \Theta_2 = \frac{b-a}{2}x(b) + \int_a^b x(s)ds - \frac{3}{b-a}\int_a^b\int_\theta^b x(s)dsd\theta;$$
$$\Theta_3 = \frac{b-a}{3}x(b) - \int_a^b x(s)ds + \frac{8}{b-a}\int_a^b\int_\theta^b x(s)dsd\theta - \frac{20}{(b-a)^2}\int_a^b\int_u^b\int_\theta^b x(s)dsd\theta du;$$
$$\Theta_4 = \frac{b-a}{4}x(b) + \int_a^b x(s)ds - \frac{15}{b-a}\int_a^b\int_\theta^b x(s)dsd\theta + \frac{90}{(b-a)^2}\int_a^b\int_u^b\int_\theta^b x(s)dsd\theta du - \frac{210}{(b-a)^3}\int_a^b\int_u^b\int_\theta^b\int_\lambda^b x(s)dsd\lambda d\theta du.$$

3. Main Results.

Theorem 3.1. *Given scalars h and d $(0 \leq h, d < 1)$, the neutral time-varying delay system (8) is asymptotically stable, if there exist matrixs $P > 0$, $Q > 0$, $R > 0$, $S > 0$, $W > 0$, $P \in \mathbb{R}^{6n\times 6n}$, $Q = \begin{bmatrix} Q_1 & Q_2 \\ * & Q_3 \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$, $Q_1, Q_2, Q_3, R, S, W \in \mathbb{R}^{n\times n}$ such that the following LMI holds*

$$\begin{bmatrix} \Phi(h,d) & \left(\hat{A}e_1 + \hat{A}_d e_2 - (1-d)Ce_3\right)^{\mathrm{T}}\left(Q_3 + h^2 S + \frac{h^4}{4}W\right) \\ * & -Q_3 - h^2 S - \frac{h^4}{4}W \end{bmatrix} < 0 \tag{9}$$

where

$$\begin{aligned}\Phi(h,d) = {} & sym\left\{{\Gamma_1}^{\mathrm{T}}P\Gamma_2\right\} + {e_1}^{\mathrm{T}}Q_1e_1 + sym\left\{{e_1}^{\mathrm{T}}Q_2e_0\right\} - (1-d){e_2}^{\mathrm{T}}Q_1e_2 \\ & - sym\left\{(1-d){e_2}^{\mathrm{T}}Q_2e_3\right\} - (1-d){e_3}^{\mathrm{T}}Q_3e_3 + {e_1}^{\mathrm{T}}Re_1 - {e_4}^{\mathrm{T}}Re_4 \\ & - {\Gamma_3}^{\mathrm{T}}S\Gamma_3 - 3{\Gamma_4}^{\mathrm{T}}S\Gamma_4 - 5{\Gamma_5}^{\mathrm{T}}S\Gamma_5 - 7{\Gamma_6}^{\mathrm{T}}S\Gamma_6 - 9{\Gamma_7}^{\mathrm{T}}S\Gamma_7 \\ & - {\Gamma_8}^{\mathrm{T}}W\Gamma_8 - 8{\Gamma_9}^{\mathrm{T}}W\Gamma_9 - 27{\Gamma_{10}}^{\mathrm{T}}W\Gamma_{10} - 64{\Gamma_{11}}^{\mathrm{T}}W\Gamma_{11}\end{aligned}$$

$e_0 = \hat{A}e_1 + \hat{A}_d e_2 - (1-d)Ce_3$; $\quad e_i = [0_{n\times(i-1)n}, I_n, 0_{n\times(8-i)n}]$, $i = 1, 2, \ldots, 8$;

$\Gamma_1 = [e_1, e_2, e_5, e_6, e_7, e_8]^{\mathrm{T}}$; $\Gamma_2 = \left[e_0, (1-d)e_3, e_1 - e_4, he_1 - e_5, \frac{h^2}{2}e_1 - e_6, \frac{h^3}{6}e_1 - e_7\right]^{\mathrm{T}}$;
$\Gamma_3 = e_1 - e_4$; $\Gamma_4 = e_1 + e_4 - \frac{2}{h}e_5$; $\Gamma_5 = e_1 - e_4 + \frac{6}{h}e_5 - \frac{12}{h^2}e_6$;
$\Gamma_6 = e_1 + e_4 - \frac{12}{h}e_5 + \frac{60}{h^2}e_6 - \frac{120}{h^3}e_7$; $\Gamma_7 = e_1 - e_4 + \frac{20}{h}e_5 - \frac{180}{h^2}e_6 + \frac{840}{h^3}e_7 - \frac{1680}{h^4}e_8$;
$\Gamma_8 = he_1 - e_5$; $\Gamma_9 = \frac{h}{2}e_1 + e_5 - \frac{3}{h}e_6$; $\Gamma_{10} = \frac{h}{3}e_1 - e_5 + \frac{8}{h}e_6 - \frac{20}{h^2}e_7$;
$\Gamma_{11} = \frac{h}{4}e_1 + e_5 - \frac{15}{h}e_6 + \frac{90}{h^2}e_7 - \frac{210}{h^3}e_8$.

Proof: At first, we define

$$\begin{aligned}\eta(t) := col\Big\{&\mu(t), \mu(t-h(t)), \int_{t-h}^{t}\mu(s)ds, \int_{t-h}^{t}\int_{\theta}^{t}\mu(s)dsd\theta,\\ &\int_{t-h}^{t}\int_{u}^{t}\int_{\theta}^{t}\mu(s)dsd\theta du, \int_{t-h}^{t}\int_{u}^{t}\int_{\theta}^{t}\int_{\lambda}^{t}\mu(s)dsd\lambda d\theta du\Big\}\\ \xi(t) := col\Big\{&\mu(t), \mu(t-h(t)), \dot{\mu}(t-h(t)), \mu(t-h), \int_{t-h}^{t}\mu(s)ds, \int_{t-h}^{t}\int_{\theta}^{t}\mu(s)dsd\theta,\\ &\int_{t-h}^{t}\int_{u}^{t}\int_{\theta}^{t}\mu(s)dsd\theta du, \int_{t-h}^{t}\int_{u}^{t}\int_{\theta}^{t}\int_{\lambda}^{t}\mu(s)dsd\lambda d\theta du\Big\}.\end{aligned}$$

Then choose an LKF candidate as

$$\begin{aligned}&V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t) + V_5(t); \quad V_1(t) = \eta^{\mathrm{T}}(t)P\eta(t);\\ &V_2(t) = \int_{t-h(t)}^{t}\begin{bmatrix}\mu(s)\\ \dot{\mu}(s)\end{bmatrix}^{\mathrm{T}} Q \begin{bmatrix}\mu(s)\\ \dot{\mu}(s)\end{bmatrix}ds; \quad V_3(t) = \int_{t-h}^{t}\mu^{\mathrm{T}}(s)R\mu(s)ds;\\ &V_4(t) = h\int_{t-h}^{t}\int_{\theta}^{t}\dot{\mu}^{\mathrm{T}}(s)S\dot{\mu}(s)dsd\theta; \quad V_5(t) = \frac{h^2}{2}\int_{t-h}^{t}\int_{u}^{t}\int_{\theta}^{t}\dot{\mu}^{\mathrm{T}}(s)W\dot{\mu}(s)dsd\theta du.\end{aligned}$$

The derivative of $V(t)$ with respect to t is calculated as

$$\dot{V}_1(t) = 2\eta^{\mathrm{T}}(t)P\dot{\eta}(t) = 2\xi^{\mathrm{T}}(t)\begin{bmatrix}e_1\\ e_2\\ e_5\\ e_6\\ e_7\\ e_8\end{bmatrix}^{\mathrm{T}} P \begin{bmatrix}\hat{A}e_1 + \hat{A}_d e_2 - \left(1-\dot{h}(t)\right)Ce_3\\ \left(1-\dot{h}(t)\right)e_3\\ e_1 - e_4\\ he_1 - e_5\\ \frac{h^2}{2}e_1 - e_6\\ \frac{h^3}{6}e_1 - e_7\end{bmatrix}\xi(t)$$

$$\begin{aligned}\dot{V}_2(t) =&\xi^{\mathrm{T}}(t)\Big\{e_1^{\mathrm{T}}Q_1e_1 + 2e_1^{\mathrm{T}}Q_2\left(\hat{A}e_1 + \hat{A}_de_2 - \left(1-\dot{h}(t)\right)Ce_3\right)\\ &+\left(\hat{A}e_1 + \hat{A}_de_2 - \left(1-\dot{h}(t)\right)Ce_3\right)^{\mathrm{T}}Q_3\left(\hat{A}e_1 + \hat{A}_de_2 - \left(1-\dot{h}(t)\right)Ce_3\right)\\ &-\left(1-\dot{h}(t)\right)e_2^{\mathrm{T}}Q_1e_2 - 2\left(1-\dot{h}(t)\right)e_2^{\mathrm{T}}Q_2e_3 - \left(1-\dot{h}(t)\right)e_3^{\mathrm{T}}Q_3e_3\Big\}\xi(t)\end{aligned}$$

$$\dot{V}_3(t) = \mu^{\mathrm{T}}(t)R\mu(t) - \mu^{\mathrm{T}}(t-h)R\mu(t-h) = \xi^{\mathrm{T}}(t)\left\{e_1^{\mathrm{T}}Re_1 - e_4^{\mathrm{T}}Re_4\right\}\xi(t)$$

$$\begin{aligned}\dot{V}_4(t) =& h^2\dot{\mu}^{\mathrm{T}}(t)S\dot{\mu}(t) - h\int_{t-h}^{t}\dot{\mu}^{\mathrm{T}}(s)S\dot{\mu}(s)ds = -h\int_{t-h}^{t}\dot{\mu}^{\mathrm{T}}(s)S\dot{\mu}(s)ds \\ & + h^2\xi^{\mathrm{T}}(t)\Big(\hat{A}e_1 + \hat{A}_d e_2 - \Big(1-\dot{h}(t)\Big)Ce_3\Big)^{\mathrm{T}} S\Big(\hat{A}e_1 + \hat{A}_d e_2 \\ & - \Big(1-\dot{h}(t)\Big)Ce_3\Big)\xi(t)\end{aligned} \tag{10}$$

$$\begin{aligned}\dot{V}_5(t) =& \frac{h^4}{4}\dot{\mu}^{\mathrm{T}}(t)W\dot{\mu}(t) - \frac{h^2}{2}\int_{t-h}^{t}\int_{\theta}^{t}\dot{\mu}^{\mathrm{T}}(s)W\dot{\mu}(s)dsd\theta \\ =& -\frac{h^2}{2}\int_{t-h}^{t}\int_{\theta}^{t}\dot{\mu}^{\mathrm{T}}(s)W\dot{\mu}(s)dsd\theta + \frac{h^4}{4}\xi^{\mathrm{T}}(t)\Big(\hat{A}e_1 + \hat{A}_d e_2 \\ & - \Big(1-\dot{h}(t)\Big)Ce_3\Big)^{\mathrm{T}} W\Big(\hat{A}e_1 + \hat{A}_d e_2 - \Big(1-\dot{h}(t)\Big)Ce_3\Big)\xi(t).\end{aligned} \tag{11}$$

For the term of $-h\int_{t-h}^{t}\dot{\mu}^{\mathrm{T}}(s)S\dot{\mu}(s)ds$ in (10), according to Lemma 2.3, we have

$$\begin{aligned}& -h\int_{t-h}^{t}\dot{\mu}^{\mathrm{T}}(s)S\dot{\mu}(s)ds \\ \le& -\xi^{\mathrm{T}}(t)\Bigg\{(e_1-e_4)^{\mathrm{T}}S(e_1-e_4) + 3\Big(e_1+e_4-\frac{2}{h}e_5\Big)^{\mathrm{T}}S\Big(e_1+e_4-\frac{2}{h}e_5\Big) \\ & + 5\Big(e_1-e_4+\frac{6}{h}e_5-\frac{12}{h^2}e_6\Big)^{\mathrm{T}}S\Big(e_1-e_4+\frac{6}{h}e_5-\frac{12}{h^2}e_6\Big) \\ & + 7\Big(e_1+e_4-\frac{12}{h}e_5+\frac{60}{h^2}e_6-\frac{120}{h^3}e_7\Big)^{\mathrm{T}}S\Big(e_1+e_4-\frac{12}{h}e_5+\frac{60}{h^2}e_6-\frac{120}{h^3}e_7\Big) \\ & + 9\Big(e_1-e_4+\frac{20}{h}e_5-\frac{180}{h^2}e_6+\frac{840}{h^3}e_7-\frac{1680}{h^4}e_8\Big)^{\mathrm{T}}S\Big(e_1-e_4+\frac{20}{h}e_5-\frac{180}{h^2}e_6 \\ & + \frac{840}{h^3}e_7-\frac{1680}{h^4}e_8\Big)\Bigg\}\xi(t).\end{aligned}$$

For the term of $-\frac{h^2}{2}\int_{t-h}^{t}\int_{\theta}^{t}\dot{\mu}^{\mathrm{T}}(s)W\dot{\mu}(s)dsd\theta$ in (11), according to Lemma 2.4, we have

$$\begin{aligned}& -\frac{h^2}{2}\int_{t-h}^{t}\int_{\theta}^{t}\dot{\mu}^{\mathrm{T}}(s)W\dot{\mu}(s)dsd\theta \\ \le& -\xi^{\mathrm{T}}(t)\Bigg\{(he_1-e_5)^{\mathrm{T}}W(he_1-e_5) + 8\Big(\frac{h}{2}e_1+e_5-\frac{3}{h}e_6\Big)^{\mathrm{T}}W\Big(\frac{h}{2}e_1+e_5-\frac{3}{h}e_6\Big) \\ & + 27\Big(\frac{h}{3}e_1-e_5+\frac{8}{h}e_6-\frac{20}{h^2}e_7\Big)^{\mathrm{T}}W\Big(\frac{h}{3}e_1-e_5+\frac{8}{h}e_6-\frac{20}{h^2}e_7\Big) \\ & + 64\Big(\frac{h}{4}e_1+e_5-\frac{15}{h}e_6+\frac{90}{h^2}e_7-\frac{210}{h^3}e_8\Big)^{\mathrm{T}}W\Big(\frac{h}{4}e_1+e_5-\frac{15}{h}e_6+\frac{90}{h^2}e_7 \\ & - \frac{210}{h^3}e_8\Big)\Bigg\}\xi(t).\end{aligned}$$

Let

$$\tilde{e}_0 = \hat{A}e_1 + \hat{A}_d e_2 - \Big(1-\dot{h}(t)\Big)Ce_3;$$

$$\tilde{\Gamma}_2 = \Big[\tilde{e}_0, \Big(1-\dot{h}(t)\Big)e_3, e_1-e_4, he_1-e_5, \frac{h^2}{2}e_1-e_6, \frac{h^3}{6}e_1-e_7\Big]^{\mathrm{T}};$$

$$\begin{aligned}\Phi\left(h(t),\dot{h}(t)\right) =& sym\left\{{\Gamma_1}^{\mathrm{T}}P\tilde{\Gamma}_2\right\}+{e_1}^{\mathrm{T}}Q_1e_1+sym\left\{{e_1}^{\mathrm{T}}Q_2\tilde{e}_0\right\}-\left(1-\dot{h}(t)\right){e_2}^{\mathrm{T}}Q_1e_2\\ &-sym\left\{\left(1-\dot{h}(t)\right){e_2}^{\mathrm{T}}Q_2e_3\right\}-\left(1-\dot{h}(t)\right){e_3}^{\mathrm{T}}Q_3e_3+{e_1}^{\mathrm{T}}Re_1\\ &-{e_4}^{\mathrm{T}}Re_4-{\Gamma_3}^{\mathrm{T}}S\Gamma_3-3{\Gamma_4}^{\mathrm{T}}S\Gamma_4-5{\Gamma_5}^{\mathrm{T}}S\Gamma_5-7{\Gamma_6}^{\mathrm{T}}S\Gamma_6-9{\Gamma_7}^{\mathrm{T}}S\Gamma_7\\ &-{\Gamma_8}^{\mathrm{T}}W\Gamma_8-8{\Gamma_9}^{\mathrm{T}}W\Gamma_9-27{\Gamma_{10}}^{\mathrm{T}}W\Gamma_{10}-64{\Gamma_{11}}^{\mathrm{T}}W\Gamma_{11}.\end{aligned}$$

Then, we get

$$\dot{V}(t)\le\xi^{\mathrm{T}}(t)\left\{\Phi\left(h(t),\dot{h}(t)\right)+\tilde{e}_0^{\mathrm{T}}\left(Q_3+h^2S+\frac{h^4}{4}W\right)\tilde{e}_0\right\}\xi(t).$$

Therefore, the system (8) is asymptotically stable if the following inequality holds

$$\begin{aligned}&\xi^{\mathrm{T}}(t)\left\{\Phi\left(h(t),\dot{h}(t)\right)+\left(\hat{A}e_1+\hat{A}_de_2-\left(1-\dot{h}(t)\right)Ce_3\right)^{\mathrm{T}}\left(Q_3+h^2S+\frac{h^4}{4}W\right)\right.\\ &\left.\left(\hat{A}e_1+\hat{A}_de_2-\left(1-\dot{h}(t)\right)Ce_3\right)\right\}\xi(t)<0\end{aligned}\tag{12}$$

Note that $P>0$, $Q>0$, $R>0$, $S>0$, $W>0$, $h(t)$ and $\dot{h}(t)$ are the two uncertain parameters in the above formula, and $h(t)\le h$, $\dot{h}(t)\le d<1$. So a sufficient condition to ensure that Inequality (12) holds is

$$\begin{aligned}&\Phi(h(t),d)+\left(\hat{A}e_1+\hat{A}_de_2-(1-d)Ce_3\right)^{\mathrm{T}}\left(Q_3+h^2S+\frac{h^4}{4}W\right)\left(\hat{A}e_1\right.\\ &\left.+\hat{A}_de_2-(1-d)Ce_3\right)<0\end{aligned}\tag{13}$$

According to Lemma 2.2, Inequality (13) is equivalent to Inequality (9), that is

$$\begin{bmatrix}\Phi(h,d) & \left(\hat{A}e_1+\hat{A}_de_2-(1-d)Ce_3\right)^{\mathrm{T}}\left(Q_3+h^2S+\frac{h^4}{4}W\right)\\ * & -Q_3-h^2S-\frac{h^4}{4}W\end{bmatrix}<0.$$

Therefore, we know that if the matrix inequality (9) is feasible, the neutral time-varying delay system (8) is asymptotically stable. So far, the proof is completed.

Remark 3.1. *It is noted that under the condition that the singular time-varying delay system (1) is regular and impulse free, the asymptotic stability of the neutral time-varying delay system (8) will ensure the admissibility of the singular time-varying delay system (1). In particular, Theorem 3.1 proposes a new admissibility condition of singular time-varying delay system (1) based on the neutral system approach.*

Remark 3.2. *Additionally, an extended LKF is selected, which contains more delay information. Moreover, B-L integral inequality plays an important role in dealing with the integral term after derivation on LKF, and its application also helps to reduce the conservativeness of the results.*

4. **Numerical Example.**

Example 4.1. *Consider the following singular time-varying delay system*

$$E\dot{x}(t)=Ax(t)+A_dx(t-h(t))$$

where

$$E=\begin{bmatrix}1&0\\0&0\end{bmatrix},\quad A=\begin{bmatrix}-0.5&0\\0&-1\end{bmatrix},\quad A_d=\begin{bmatrix}-1&-1\\0&0.5\end{bmatrix}.$$

By using the neutral system approach mentioned in this paper, the following neutral time-varying delay system can be obtained

$$\begin{cases} \dot{\mu}(t) - \hat{C}(t)\dot{\mu}(t-h(t)) = \hat{A}\mu(t) + \hat{A}_d\mu(t-h(t)) \\ \mu(t) = \psi(t), \quad t \in [-h, 0] \end{cases}$$

where

$$\hat{A} = \begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix}, \quad \hat{A}_d = \begin{bmatrix} -1 & 1 \\ 0 & 0.5 \end{bmatrix}, \quad \hat{C}(t) = \left(1 - \dot{h}(t)\right) \begin{bmatrix} 0 & 0 \\ 0 & 0.5 \end{bmatrix}.$$

By Theorem 3.1, for given d $\left(\dot{h}(t) \le d < 1\right)$, the maximum allowable delay bounds h are calculated and shown in Table 1.

Table 1. The maximum allowable delay bounds h for given d in Example 4.1

Method	$d = 0$	$d = 0.1$	$d = 0.3$	$d = 0.5$
Corollary 1 [14]	2.000	1.902	1.725	1.583
Theorem 1 [15]	2.187	1.905	1.798	1.751
Theorem 1 [8]	2.402	2.253	2.032	1.906
Theorem 3.1	2.555	2.550	2.533	1.920

From Table 1, we can see that compared with the results in literature, Theorem 3.1 obtains a larger upper bound of delay which is allowable for the admissibility of the singular time-varying delay system.

5. Conclusions.

This paper has established an improved vision of time-delay admissibility criterion for a singular time-delay system through the neutral system approach in terms of LMI. The augmented LKF is established, and the fourth-order B-L integral inequality and the double third-order B-L integral inequality are used to deal with the integral term of the derivative of LKF. Compared with existing results, the proposed method is less conservative which is shown by a numerical example. In future, state feedback controller or output feedback controller could be considered in order to control the singular time-varying delay system.

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